Gravity Simulator

Dr Andrew French.
Winchester, Hampshire. UK

April 18, 2013

Abstract

This document describes a mathematical model of gravitational attraction sufficient to construct a dynamic computer simulation of interacting astronomical bodies. Exact results for a two-body system are derived, including Kepler's Laws. These solutions will form the inputs to a multi-body system comprised of pairs of point masses in elliptical orbits plus rings and clusters of essentially massless objects. The planes of orbit for both masses and rings can be specified in any orientation. A recipe for achieving the associated coordinate rotation (via *Rodrigues' formula*) is defined and derived.

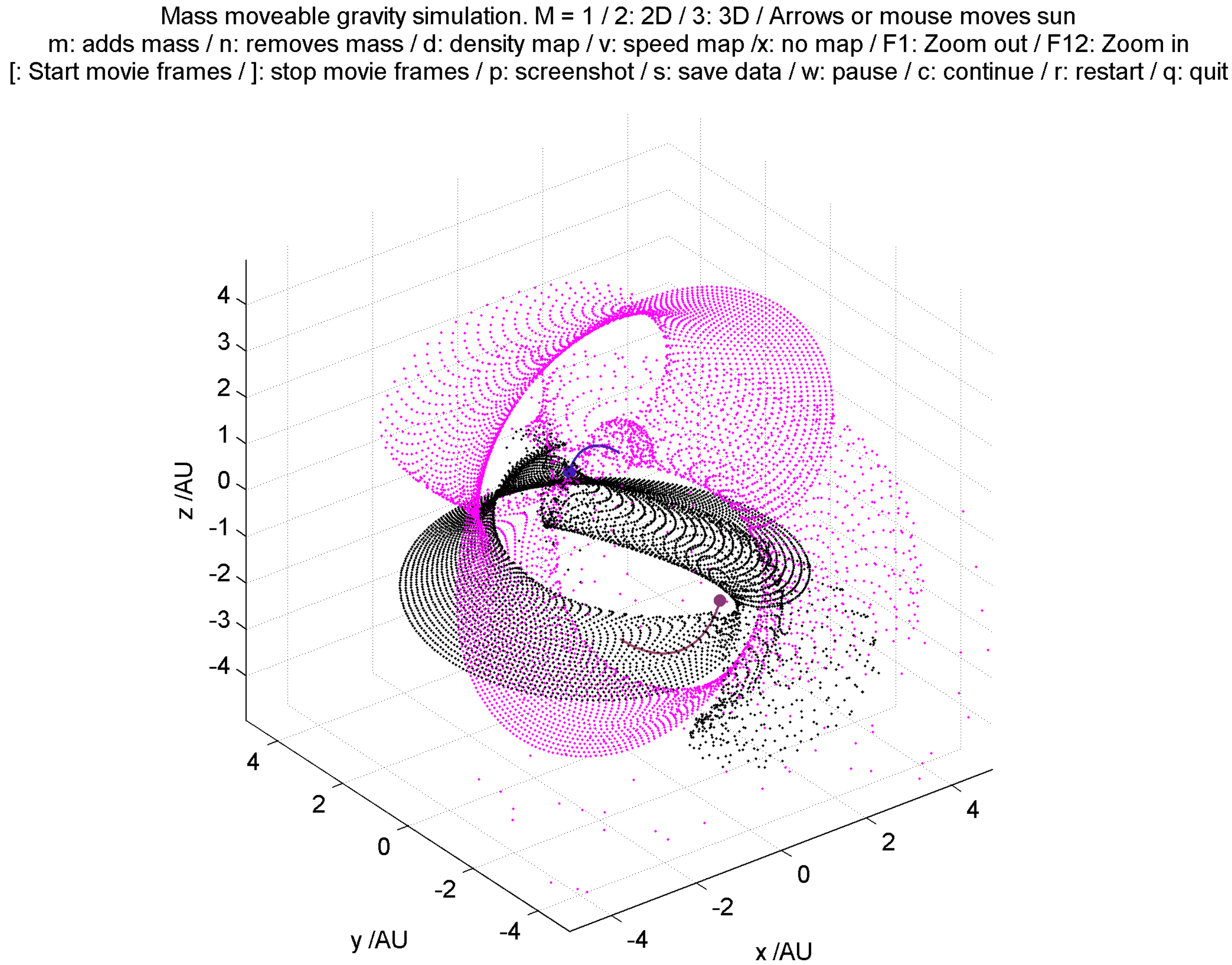


Figure 1: Screen shot from the MATLAB simulation `gravity_sim`, written by the author. In this scenario, two suns in the mass ratio 1:2 are set to orbit about each other in an elliptical fashion with eccentricity $\varepsilon = 0.2$. Initially circular rings of massless planetlets are set up about the masses. The plane of rotation of the rings are 90 degrees apart. `gravity_sim` enables dynamic changes resulting from key presses. The first mass can be changed by pressing the 'm' or 'n' buttons, and its position via the arrow keys or mouse.

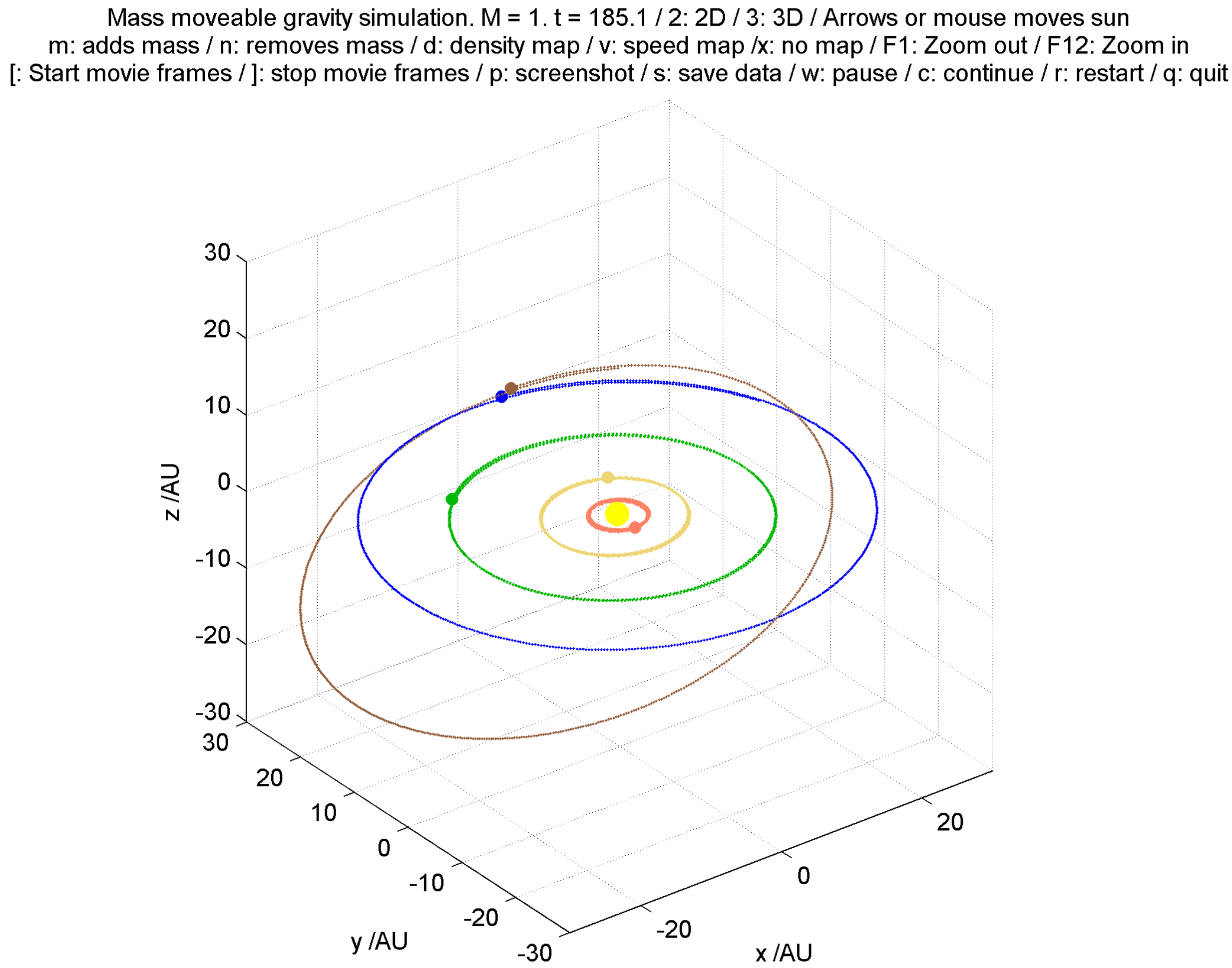


Figure 2: `gravity_sim` screenshot illustrating a 3D visualization of the solar system. Timescales are years and all distances are in Astronomical units (AU). The x,y plane is the ecliptic i.e. the orbital plane of the Sun and Earth.

0.1 Multi-body equation of motion

0.2 Definition of quantities

Define the following properties:

m_i	Mass of i^{th} object in units of solar masses $M_\odot = 1.99 \times 10^{30}$ kg
t	Time in Earth years since the start of the simulation. 1 Yr $= 3.1536 \times 10^7$ s
$\mathbf{r}_i$	Displacement of i^{th} object from origin of coordinate system in Astronomical Units AU. 1 AU $= 1.496 \times 10^{11}$ m A Cartesian coordinate system (with a right handed set of orthonormal basis vectors $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ shall be used so $\mathbf{r}_i = x_i\hat{\mathbf{x}} + y_i\hat{\mathbf{y}} + z_i\hat{\mathbf{z}}$
$\mathbf{v}_i$	Velocity of i^{th} object. $\mathbf{v}_i = v_{xi}\hat{\mathbf{x}} + v_{yi}\hat{\mathbf{y}} + v_{zi}\hat{\mathbf{z}}$
R_i	Radius in solar radii of the i^{th} object. $R_\odot = 6.960 \times 10^8$ m

0.3 Defining the equation of motion

We shall adopt the following equation of motion, based on a classical usage of Newton's Second Law, for all objects. A typical simulation will consist of massive objects and essentially massless objects. The gravitational interactions *between* essentially massless objects shall be ignored. They will typically form the large majority of objects and will be used as markers to illustrate the dynamic effect of the gravitational field resulting from the massive objects. An inverse law of universal gravitation shall be used, coupled with a repulsion term to model collisions. The strength of the latter shall be proportional to the object radius.

$$m_i \frac{d\mathbf{v}_i}{dt} = A m_i \sum_{j \neq i} \frac{m_j \left(\mathbf{r}_j - \mathbf{r}_i\right)}{\left|\mathbf{r}_j - \mathbf{r}_i\right|^{P+1}} - B m_i \sum_{j \neq i} R_j \frac{m_j \left(\mathbf{r}_j - \mathbf{r}_i\right)}{\left|\mathbf{r}_j - \mathbf{r}_i\right|^{Q+1}} \tag{1}$$

Where A and B are constants.

Note $P = 2$ and $B = 0$ correspond to Newton's Law of Gravitation. In this case $A =$ the universal gravitational constant $G = 6.67 \times 10^{-11}$ m^{3}kg^{-1}s^{-2} .

0.4 Scaling to make all variables dimensionless

All the quantities in the simulation shall be defined in dimensionless terms, i.e. as pure numbers. Hence:

$$\begin{aligned} m_i &\rightarrow m_i M_\odot \\ \mathbf{r}_i &\rightarrow \mathbf{r}_i \text{AU} \\ \mathbf{v}_i &\rightarrow \mathbf{v}_i \frac{\text{AU}}{\text{Yr}} \\ t &\rightarrow t\text{Yr} \\ R_i &= R_i R_\odot \end{aligned} \tag{2}$$

The equation of motion 1 can therefore be re-written as

$$m_i M_\odot \frac{\text{AU}}{\text{Yr}^2} \frac{d\mathbf{v}_i}{dt} = A m_i M_\odot \sum_{j \neq i} \frac{m_j M_\odot \left(\mathbf{r}_j - \mathbf{r}_i\right)}{\left|\mathbf{r}_j - \mathbf{r}_i\right|^{P+1} \text{AU}^{P+1}} - B m_i M_\odot \sum_{j \neq i} R_j R_\odot \frac{m_j M_\odot \left(\mathbf{r}_j - \mathbf{r}_i\right) \text{AU}}{\left|\mathbf{r}_j - \mathbf{r}_i\right|^{Q+1} \text{AU}^{Q+1}} \tag{3}$$

Hence

$$\begin{aligned} \frac{\text{AU}}{\text{Yr}^2} \frac{d\mathbf{v}_i}{dt} &= \frac{A M_\odot}{\text{AU}^P} \sum_{j \neq i} \frac{m_j \left(\mathbf{r}_j - \mathbf{r}_i\right)}{\left|\mathbf{r}_j - \mathbf{r}_i\right|^{P+1}} - \frac{B R_\odot M_\odot}{\text{AU}^Q} \sum_{j \neq i} R_j \frac{m_j \left(\mathbf{r}_j - \mathbf{r}_i\right)}{\left|\mathbf{r}_j - \mathbf{r}_i\right|^{Q+1}} \\ \frac{d\mathbf{v}_i}{dt} &= \frac{A M_\odot \text{Yr}^2}{\text{AU}^{P+1}} \sum_{j \neq i} \frac{m_j \left(\mathbf{r}_j - \mathbf{r}_i\right)}{\left|\mathbf{r}_j - \mathbf{r}_i\right|^{P+1}} - \frac{B R_\odot M_\odot \text{Yr}^2}{\text{AU}^{Q+1}} \sum_{j \neq i} R_j \frac{m_j \left(\mathbf{r}_j - \mathbf{r}_i\right)}{\left|\mathbf{r}_j - \mathbf{r}_i\right|^{Q+1}} \end{aligned} \tag{4}$$

$$\frac{d\mathbf{v}_i}{dt} = \Lambda \sum_{j \neq i} \frac{m_j \left(\mathbf{r}_j - \mathbf{r}_i\right)}{\left|\mathbf{r}_j - \mathbf{r}_i\right|^{P+1}} - \Theta \sum_{j \neq i} R_j \frac{m_j \left(\mathbf{r}_j - \mathbf{r}_i\right)}{\left|\mathbf{r}_j - \mathbf{r}_i\right|^{Q+1}} \tag{5}$$

where pure numbers Λ and Θ are defined

$$\begin{aligned}\Lambda &= \frac{AM_{\odot}\mathrm{Yr}^2}{\mathrm{AU}^{P+1}} \\ \Theta &= \frac{BR_{\odot}M_{\odot}\mathrm{Yr}^2}{\mathrm{AU}^{Q+1}}\end{aligned} \tag{6}$$

Let the repulsive scaling constant B also be defined via a new dimensionless quantity $\chi = \frac{\Lambda}{\Theta}$

$$\begin{aligned}\chi &= \frac{\Lambda}{\Theta} \\ &= \frac{AM_{\odot}\mathrm{Yr}^2}{\mathrm{AU}^{P+1}} \times \frac{\mathrm{AU}^{Q+1}}{BR_{\odot}M_{\odot}\mathrm{Yr}^2} \\ &= \frac{A}{\mathrm{AU}^{P+1}} \times \frac{\mathrm{AU}^{Q+1}}{BR_{\odot}} \\ &= \frac{A\mathrm{AU}^{Q-P}}{BR_{\odot}}\end{aligned} \tag{7}$$

Hence

$$B = \frac{A\mathrm{AU}^{Q-P}}{\chi R_{\odot}} \tag{8}$$

which means

$$\Theta = \frac{A\mathrm{AU}^{Q-P}}{\chi R_{\odot}} \frac{R_{\odot}M_{\odot}\mathrm{Yr}^2}{\mathrm{AU}^{Q+1}} = \frac{AM_{\odot}\mathrm{Yr}^2}{\chi \mathrm{AU}^{P+1}} = \frac{\Lambda}{\chi} \tag{9}$$

So in summary

$$\begin{aligned}\frac{d\mathbf{v}_i}{dt} &= \Lambda \sum_{j\neq i} \frac{m_j\left(\mathbf{r}_j - \mathbf{r}_i\right)}{\left|\mathbf{r}_j - \mathbf{r}_i\right|^{P+1}} - \frac{\Lambda}{\chi} \sum_{j\neq i} R_j \frac{m_j\left(\mathbf{r}_j - \mathbf{r}_i\right)}{\left|\mathbf{r}_j - \mathbf{r}_i\right|^{Q+1}} \\ \Lambda &= \frac{AM_{\odot}\mathrm{Yr}^2}{\mathrm{AU}^{P+1}}\end{aligned} \tag{10}$$

If $P = 2$, $A = G$, $Q = 5$ and $\chi = 1$

$$\begin{aligned}\frac{d\mathbf{v}_i}{dt} &= \Lambda \sum_{j\neq i} \frac{m_j\left(\mathbf{r}_j - \mathbf{r}_i\right)}{\left|\mathbf{r}_j - \mathbf{r}_i\right|^{3}} - \Lambda \sum_{j\neq i} R_j \frac{m_j\left(\mathbf{r}_j - \mathbf{r}_i\right)}{\left|\mathbf{r}_j - \mathbf{r}_i\right|^{6}} \\ \Lambda &= \frac{GM_{\odot}\mathrm{Yr}^2}{\mathrm{AU}^3} \approx 39.43\end{aligned} \tag{11}$$

Note the simulation could be adapted for a planetary or indeed galactic scale by a redefinition of the mass, length and time scalings. In general if these are M, L and T (defined in kg, metres and seconds)

$$\Lambda = \frac{AMT^2}{L^{P+1}} \tag{12}$$

For the Milky Way[1]
Let the time scaling be the time for one rotation $T = 100 \times 10^6$ Yr $= 3.2 \times 10^{15}$ s
Let the length scaling be the mean radius $L = 4.8 \times 10^{20}$ m
Let the mass scaling be the approximate mass of the milky way $M = 6 \times 10^{42}$ kg

Hence if $P = 2$, and if in this case $A = G = 6.67 \times 10^{-11}$ m^{3}kg^{-1}s^{-2}

$$\Lambda \approx 37 \tag{13}$$

Therefore we should expect the large scale structure of a galactic scale simulation to look similar to a star-based simulation with rings of planetlets. A milky way solution should be held together slightly more loosely given Λ is a bit lower than the solar system value.

[1] From Wolfram Alpha 13/4/2013

0.5 Numerical solver for equation of motion

The second-order *Verlet Algorithm* (essentially constant acceleration motion between time steps) shall be used to solve the equation of motion. A time-step of $\Delta t = 0.01$ shall be used.

$$\begin{aligned}
\mathbf{a}_i(t_n) &= \Lambda \sum_{j \neq i} \frac{m_j \left(\mathbf{r}_j(t_n) - \mathbf{r}_i(t_n)\right)}{\left|\mathbf{r}_j(t_n) - \mathbf{r}_i(t_n)\right|^{P+1}} - \frac{\Lambda}{\chi} \sum_{j \neq i} R_j \frac{m_j \left(\mathbf{r}_j(t_n) - \mathbf{r}_i(t_n)\right)}{\left|\mathbf{r}_j(t_n) - \mathbf{r}_i(t_n)\right|^{Q+1}} \\
\mathbf{r}_i(t_{n+1}) &= \mathbf{r}_i(t_{n+1}) + \mathbf{v}_i(t_n)\Delta t + \tfrac{1}{2}\mathbf{a}_i(t_n)\left(\Delta t\right)^2 \\
\mathbf{a}_i(t_{n+1}) &= \Lambda \sum_{j \neq i} \frac{m_j \left(\mathbf{r}_j(t_{n+1}) - \mathbf{r}_i(t_{n+1})\right)}{\left|\mathbf{r}_j(t_{n+1}) - \mathbf{r}_i(t_{n+1})\right|^{P+1}} - \frac{\Lambda}{\chi} \sum_{j \neq i} R_j \frac{m_j \left(\mathbf{r}_j(t_{n+1}) - \mathbf{r}_i(t_{n+1})\right)}{\left|\mathbf{r}_j(t_{n+1}) - \mathbf{r}_i(t_{n+1})\right|^{Q+1}} \\
\mathbf{v}_i(t_{n+1}) &= \mathbf{v}_i(t_n) + \tfrac{1}{2}\left(\mathbf{a}_i(t_n) + \mathbf{a}_i(t_{n+1})\right)
\end{aligned} \tag{14}$$

1 A two body system

1.1 Dynamics of a two body system: a model

In summary, the following recipe describes the dynamics of a two body system interacting via Newtonian gravitation. This is in a form which could be implemented in a computer program which could be used to animate a two body system and generate graphs of various physical quantities. Derivations of the equations involved are given in subsequent sections.

Inputs

1. The masses m_1 and m_2
2. The gravitational force constant G
3. Mass separation s along semi-major axis of elliptical orbit
4. Orbit eccentricity ε such that $0 \leq \varepsilon \leq 1$. Note semi-major axis of ellipse $a = \frac{s}{1+\varepsilon}$
5. Initial position vector of centre of mass of system $\mathbf{h}_0$ relative to a fixed Cartesian coordinate system $\left\{\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}}\right\}$
6. Velocity of centre of mass $\dot{\mathbf{h}}$ (which is a constant of the motion)
7. Initial orbital phase angle θ_0 of elliptical orbit
8. Cartesian basis $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$, with origin $\mathbf{h}$. The orbit will confined to the $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$ plane. Note this basis set is time invariant, but the origin of the basis set could translate uniformly if $\dot{\mathbf{h}} \neq \mathbf{0}$.
9. The $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$ orbital plane will be defined in terms of a 3D rotation of the $\hat{\mathbf{x}}$ axis to align with a defined direction vector $\mathbf{d} = d_x\hat{\mathbf{X}} + d_y\hat{\mathbf{Y}} + d_{\mathbf{y}}\hat{\mathbf{Z}}$, followed by an anti-clockwise roll about this vector of angle α

Rotation of orbital plane

$$\hat{\mathbf{x}} = k\left(\hat{\mathbf{X}},\alpha, \mathbf{d}, \mathbf{0}, \hat{\mathbf{X}}\right) \quad (15)$$

$$\hat{\mathbf{y}} = k\left(\hat{\mathbf{Y}},\alpha, \mathbf{d}, \mathbf{0}, \hat{\mathbf{X}}\right) \quad (16)$$

$$\hat{\mathbf{z}} = k\left(\hat{\mathbf{Z}},\alpha, \mathbf{d}, \mathbf{0}, \hat{\mathbf{X}}\right) \quad (17)$$

Where $\mathbf{r}^{''} = k\left(\mathbf{r}, \theta, \mathbf{d}, \mathbf{h}, \hat{\mathbf{b}}\right)$ describes the transformation of vector $\mathbf{r}$ in the following steps:

1.

$$\begin{aligned} \mathbf{\Omega} &= \hat{\mathbf{b}} + \frac{\mathbf{d}}{|\mathbf{d}|} \\ \mathbf{r}' &= f\left(\mathbf{r}, \mathbf{h}, \mathbf{\Omega},\pi\right) \\ \mathbf{r}^{''} &= f\left(\mathbf{r}', \mathbf{h}, \mathbf{d},\pi\right) \\ \mathbf{r}^{'''} &= f\left(\mathbf{r}^{''}, \mathbf{h}, \mathbf{d},\alpha\right) \end{aligned}$$

Rodrigues' rotation formula is

$$f\left(\mathbf{r}, \mathbf{a}, \boldsymbol{\omega},\theta\right) = \mathbf{a}\left(1-\cos\theta\right) + \mathbf{r}\cos\theta + \frac{\boldsymbol{\omega}\times\left(\mathbf{r}-\mathbf{a}\right)}{|\boldsymbol{\omega}|}\sin\theta + \frac{\left(\mathbf{r}\cdot\boldsymbol{\omega} - \mathbf{a}\cdot\boldsymbol{\omega}\right)\left(1-\cos\theta\right)}{|\boldsymbol{\omega}|^2}\boldsymbol{\omega} \quad (18)$$

Initial conditions

$$\mathbf{w}_0 = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\theta_0}\left(\hat{\mathbf{x}}\cos\theta_0 + \hat{\mathbf{y}}\sin\theta_0\right) \quad (19)$$

$$\hat{\mathbf{r}}_0 = \cos\theta_0\hat{\mathbf{x}} + \sin\theta_0\hat{\mathbf{y}} \quad (20)$$

$$\hat{\boldsymbol{\theta}}_0 = -\sin\theta_0\hat{\mathbf{x}} + \cos\theta_0\hat{\mathbf{y}} \quad (21)$$

$$\dot{\mathbf{w}}_0 = \sqrt{\frac{G\left(m_1+m_2\right)}{a\left(1-\varepsilon^2\right)}}\left(1+\varepsilon\cos\theta_0\right)\left(\frac{\varepsilon\sin\theta_0}{1+\varepsilon\cos\theta_0}\hat{\mathbf{r}} + \hat{\boldsymbol{\theta}}\right) \quad (22)$$

Constant angular momentum and energy

$$\mathbf{j} = \frac{m_1 m_2}{m_1+m_2}\mathbf{w}_0 \times \dot{\mathbf{w}}_0 \tag{23}$$

$$\mathbf{J} = \mathbf{j} + (m_1+m_2)\,\mathbf{h}_0 \times \dot{\mathbf{h}} \tag{24}$$

$$E = \tfrac{1}{2}(m_1+m_2)\left|\dot{\mathbf{h}}\right|^2 - \frac{G m_1 m_2}{2a} \tag{25}$$

Orbital period

$$P = \sqrt{\frac{4\pi^2}{G(m_1+m_2)}a^3} \tag{26}$$

Dynamics at subsequent polar angles θ

$$t = \sqrt{\frac{a^3(1-\varepsilon^2)^3}{G(m_1+m_2)}}\int_{\theta_0}^{\theta}\frac{d\theta'}{\left(1+\varepsilon\cos\theta'\right)^2} \tag{27}$$

$$\tag{28}$$

$$\mathbf{h} = \mathbf{h}_0 + \dot{\mathbf{h}}t \tag{29}$$

$$\mathbf{w} = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\theta}\left(\hat{\mathbf{x}}\cos\theta + \hat{\mathbf{y}}\sin\theta\right) \tag{30}$$

$$\mathbf{r}_1 = \mathbf{h} - \frac{m_2\mathbf{w}}{m_1+m_2} \tag{31}$$

$$\mathbf{r}_2 = \mathbf{h} + \frac{m_1\mathbf{w}}{m_1+m_2} \tag{32}$$

$$\hat{\mathbf{r}} = \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}} \tag{33}$$

$$\hat{\boldsymbol{\theta}} = -\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}} \tag{34}$$

$$\dot{\mathbf{w}} = \sqrt{\frac{G(m_1+m_2)}{a(1-\varepsilon^2)}}(1+\varepsilon\cos\theta)\left(\frac{\varepsilon\sin\theta}{1+\varepsilon\cos\theta}\hat{\mathbf{r}} + \hat{\boldsymbol{\theta}}\right) \tag{35}$$

$$\dot{\theta} = (1+\varepsilon\cos\theta)^2\sqrt{\frac{G(m_1+m_2)}{a^3(1-\varepsilon^2)^3}} \tag{36}$$

$$\dot{\mathbf{r}}_1 = \dot{\mathbf{h}} - \frac{m_2\dot{\mathbf{w}}}{m_1+m_2} \tag{37}$$

$$\dot{\mathbf{r}}_2 = \dot{\mathbf{h}} + \frac{m_1\dot{\mathbf{w}}}{m_1+m_2} \tag{38}$$

Note $G = 6.67\times10^{-11}$ $\mathrm{m^3kg^{-1}s^{-2}}$ can be expressed in the natural units of a solar system simumation, i.e. in terms of solar masses $M_\odot = 1.99\times10^{30}$ kg, Earth years (1 Yr = 3.1536×10^7 s) and Astronomical units.

$$M_\odot = 1.99\times10^{30}\mathrm{kg} \tag{39}$$

$$1\mathrm{Yr} = 3.1536\times10^{7}\mathrm{s} \tag{40}$$

$$1\mathrm{AU} = 1.496\times10^{11}\mathrm{m} \tag{41}$$

Hence

$$G = 6.67\times10^{-11}\mathrm{m^3kg^{-1}s^{-2}} \tag{42}$$

$$= 6.67\times10^{-11}\left(\frac{\mathrm{AU}}{1.496\times10^{11}}\right)^3\left(\frac{M_\odot}{1.99\times10^{30}}\right)^{-1}\left(\frac{\mathrm{Yr}}{3.1536\times10^{7}}\right)^{-2} \tag{43}$$

$$= 39.43\ \mathrm{AU}^3 M_\odot^{-1}\mathrm{Yr}^{-2} \tag{44}$$

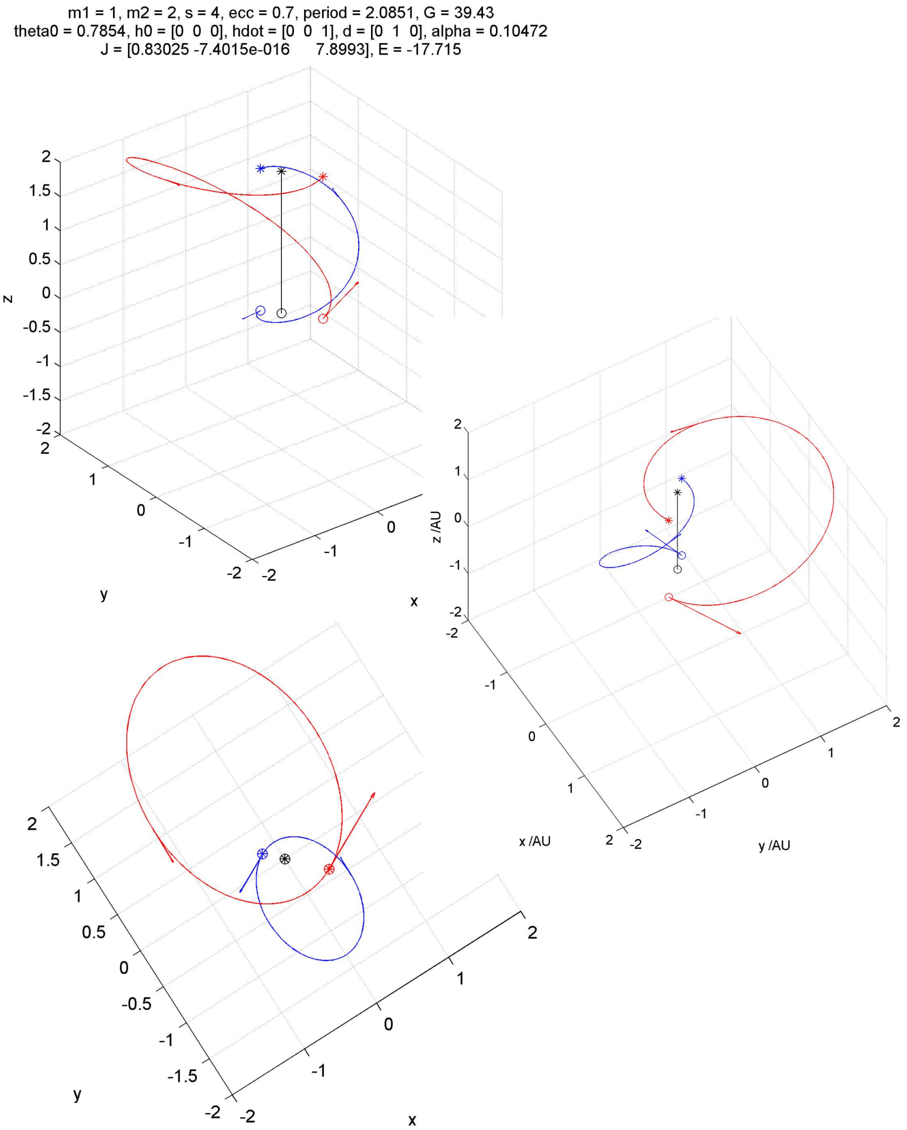


Figure 3: Model of an inclined elliptical orbit. Each graph shows a different viewpont of the same cirve. The centre of mass $\mathbf{h}$ is allowed to move with velocity $\dot{\mathbf{h}}$ i.e. $\mathbf{h} = \mathbf{h}_0 + t\dot{\mathbf{h}}$. Although the motion looks complicated, suitable rotation of the viewpoint reveals the elliptical trace of the separation vector between the two masses in an inverse-square gravitational embrace.

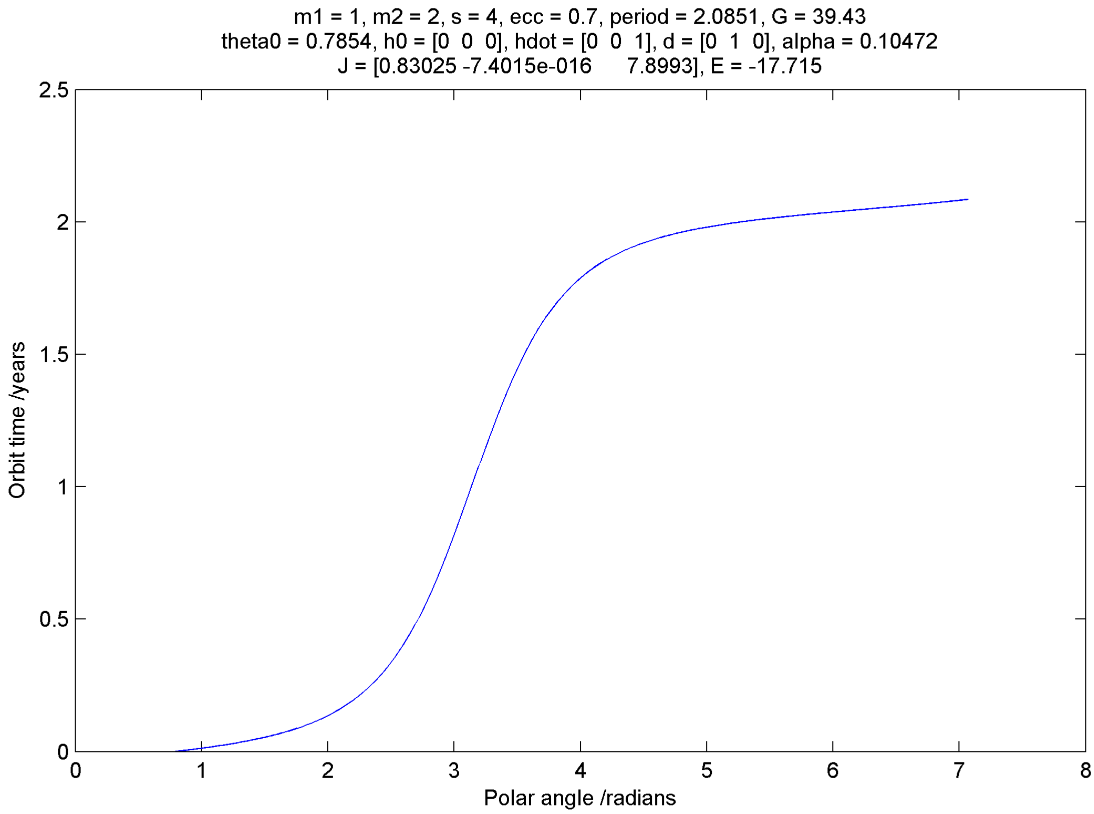


Figure 4: Orbital time is calculated by numerically evaluating the integral $t = \sqrt{\frac{a^3(1-\varepsilon^2)^3}{G(m_1+m_2)}} \int_{\theta_0}^{\theta} \frac{d\theta'}{(1+\varepsilon\cos\theta')^2}$

1.2 Equations of motion

Consider two masses m_1 and m_2 which attract each other with a gravitational force. At time $t = 0$ the masses have respective initial velocities $\mathbf{u}_1$ and $\mathbf{u}_2$. The positions of the masses are described by vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. Newton's second law states:

$$\begin{aligned} m_1 \frac{d\mathbf{u}_1}{dt} &= \frac{G m_1 m_2 (\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \\ m_2 \frac{d\mathbf{u}_2}{dt} &= -\frac{G m_1 m_2 (\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \end{aligned} \tag{45}$$

or, using $\frac{d\mathbf{u}_i}{dt} \equiv \frac{d\dot{\mathbf{r}}_i}{dt} \equiv \ddot{\mathbf{r}}_i$

$$\begin{aligned} \ddot{\mathbf{r}}_1 &= \frac{G m_2 (\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \\ \ddot{\mathbf{r}}_2 &= -\frac{G m_1 (\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \end{aligned} \tag{46}$$

1.3 Define position vectors relative to overall centre of mass

To analyse the subsequent motion of the masses for $t > 0$ let us redefine the position vectors in terms of the centre of mass of the system $\mathbf{h}$

$$\mathbf{h} = \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} \tag{47}$$

and the difference between the position vectors

$$\mathbf{w} = \mathbf{r}_2 - \mathbf{r}_1 \tag{48}$$

Hence

$$\begin{aligned} \mathbf{h} &= \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} \\ &= \frac{m_1 \mathbf{r}_1 + m_2 (\mathbf{w} + \mathbf{r}_1)}{m_1 + m_2} \\ &= \mathbf{r}_1 + \frac{m_2 \mathbf{w}}{m_1 + m_2} \end{aligned} \tag{49}$$

Therefore

$$\mathbf{r}_1 = \mathbf{h} - \frac{m_2 \mathbf{w}}{m_1 + m_2} \tag{50}$$

Similarly

$$\begin{aligned} \mathbf{h} &= \frac{m_1 \mathbf{r}_1 + m_2 \mathbf{r}_2}{m_1 + m_2} \\ &= \frac{m_1 (\mathbf{r}_2 - \mathbf{w}) + m_2 \mathbf{r}_2}{m_1 + m_2} \\ &= \mathbf{r}_2 - \frac{m_1 \mathbf{w}}{m_1 + m_2} \end{aligned} \tag{51}$$

Therefore

$$\mathbf{r}_2 = \mathbf{h} + \frac{m_1 \mathbf{w}}{m_1 + m_2} \tag{52}$$

1.4 Acceleration of the centre of mass position is zero

Double differentiating the centre of mass vector $\mathbf{h}$ with respect to time yields, using 46

$$\begin{aligned} \ddot{\mathbf{h}} &= \frac{m_1 \ddot{\mathbf{r}}_1 + m_2 \ddot{\mathbf{r}}_2}{m_1 + m_2} \\ &= \frac{m_1 \frac{G m_2 (\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} - m_2 \frac{G m_1 (\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3}}{m_1 + m_2} \\ &= \mathbf{0} \end{aligned}$$

Therefore if the forces upon the masses are as described, *the acceleration of the centre of mass of the system is zero.* This means the centre of mass of the system can move at constant velocity $\dot{\mathbf{h}}$, which of course could be zero.

$$\dot{\mathbf{h}} = \frac{m_1\mathbf{u}_1 + m_2\mathbf{u}_2}{m_1 + m_2} \tag{53}$$

We can re-write 46 using our new definitions of position in terms of the centre of mass position $\mathbf{h}$ and the difference between the mass positions $\mathbf{w}$

$$\begin{aligned} \ddot{\mathbf{r}}_1 &= \ddot{\mathbf{h}} - \frac{m_2\ddot{\mathbf{w}}}{m_1 + m_2} = \frac{Gm_2\mathbf{w}}{|\mathbf{w}|^3} \\ \ddot{\mathbf{r}}_2 &= \ddot{\mathbf{h}} + \frac{m_1\ddot{\mathbf{w}}}{m_1 + m_2} = -\frac{Gm_1\mathbf{w}}{|\mathbf{w}|^3} \end{aligned} \tag{54}$$

Hence our coupled system of differential equations become just one, in $\mathbf{w}$

$$\ddot{\mathbf{w}} = -\frac{GM}{|\mathbf{w}|^3}\mathbf{w} \tag{55}$$

where

$$M = m_1 + m_2 \tag{56}$$

1.5 Angular momentum

The angular momentum of the two mass system is

$$\mathbf{J} = m_1\mathbf{r}_1 \times \dot{\mathbf{r}}_1 + m_2\mathbf{r}_2 \times \dot{\mathbf{r}}_2 \tag{57}$$

The rate of change of angular momentum is (assuming masses m_1 and m_2 are time invariant)

$$\begin{aligned} \dot{\mathbf{J}} &= m_1\frac{d}{dt}\left(\mathbf{r}_1 \times \dot{\mathbf{r}}_1\right) + m_2\frac{d}{dt}\left(\mathbf{r}_2 \times \dot{\mathbf{r}}_2\right) \\ &= m_1\left(\mathbf{r}_1 \times \ddot{\mathbf{r}}_1 + \dot{\mathbf{r}}_1 \times \dot{\mathbf{r}}_1\right) + m_2\left(\mathbf{r}_2 \times \ddot{\mathbf{r}}_2 + \dot{\mathbf{r}}_2 \times \dot{\mathbf{r}}_2\right) \\ &= m_1\mathbf{r}_1 \times \ddot{\mathbf{r}}_1 + m_2\mathbf{r}_2 \times \ddot{\mathbf{r}}_2 \end{aligned} \tag{58}$$

Since for any vector $\mathbf{k}$

$$\mathbf{k} \times \mathbf{k} = \mathbf{0} \tag{59}$$

Now recall

$$\begin{aligned} \ddot{\mathbf{r}}_1 &= \frac{Gm_2\left(\mathbf{r}_2 - \mathbf{r}_1\right)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \\ \ddot{\mathbf{r}}_2 &= -\frac{Gm_1\left(\mathbf{r}_2 - \mathbf{r}_1\right)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \end{aligned} \tag{60}$$

Hence

$$\begin{aligned} \dot{\mathbf{J}} &= m_1\mathbf{r}_1 \times \frac{Gm_2\left(\mathbf{r}_2 - \mathbf{r}_1\right)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} - m_2\mathbf{r}_2 \times \frac{Gm_1\left(\mathbf{r}_2 - \mathbf{r}_1\right)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \\ &= \left(\mathbf{r}_1 - \mathbf{r}_2\right) \times \frac{Gm_1m_2\left(\mathbf{r}_2 - \mathbf{r}_1\right)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \\ &= -\left(\mathbf{r}_2 - \mathbf{r}_1\right) \times \frac{Gm_1m_2\left(\mathbf{r}_2 - \mathbf{r}_1\right)}{|\mathbf{r}_2 - \mathbf{r}_1|^3} \\ &= \mathbf{0} \end{aligned} \tag{61}$$

So for a two body system interacting with Newton's inverse-square law of gravity, the total angular momentum $\mathbf{J}$ is *a constant of the motion.* i.e. there is no torque on the system.

$$\mathbf{J} = m_1\mathbf{r}_1 \times \dot{\mathbf{r}}_1 + m_2\mathbf{r}_2 \times \dot{\mathbf{r}}_2 \tag{62}$$

$$= m_1\mathbf{r}_{1,t=0} \times \mathbf{u}_1 + m_2\mathbf{r}_{2,t=0} \times \mathbf{u}_2 \tag{63}$$

1.6 Using the angular momentum to define the plane of motion for the system

Let us write $\mathbf{J}$ in terms of the centre of mass of the system $\mathbf{h} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1+m_2}$ and the mass separation vector $\mathbf{w} = \mathbf{r}_2 - \mathbf{r}_1$

$$\mathbf{r}_1 = \mathbf{h} - \frac{m_2\mathbf{w}}{m_1+m_2} \tag{64}$$

$$\mathbf{r}_2 = \mathbf{h} + \frac{m_1\mathbf{w}}{m_1+m_2} \tag{65}$$

Therefore

$$\begin{aligned} \mathbf{J} &= m_1\mathbf{r}_1 \times \dot{\mathbf{r}}_1 + m_2\mathbf{r}_2 \times \dot{\mathbf{r}}_2 && (66)\\ &= m_1\mathbf{r}_1 \times \left(\dot{\mathbf{h}} - \frac{m_2\dot{\mathbf{w}}}{m_1+m_2}\right) + m_2\mathbf{r}_2 \times \left(\dot{\mathbf{h}} + \frac{m_1\dot{\mathbf{w}}}{m_1+m_2}\right) && (67)\\ &= (m_1\mathbf{r}_1 + m_2\mathbf{r}_2) \times \dot{\mathbf{h}} + (\mathbf{r}_2 - \mathbf{r}_1) \times \frac{m_1 m_2 \dot{\mathbf{w}}}{m_1+m_2} \\ \therefore\ & \mathbf{J} = (m_1+m_2)\,\mathbf{h} \times \dot{\mathbf{h}} + \frac{m_1 m_2}{m_1+m_2}\mathbf{w} \times \dot{\mathbf{w}} \end{aligned}$$

Since $\dot{\mathbf{h}}$ is a constant we can write

$$\mathbf{h} = \mathbf{h}_0 + \dot{\mathbf{h}}t \tag{68}$$

Therefore

$$\mathbf{h} \times \dot{\mathbf{h}} = \left(\mathbf{h}_0 + \dot{\mathbf{h}}t\right) \times \dot{\mathbf{h}} = \mathbf{h}_0 \times \dot{\mathbf{h}} \tag{69}$$

Hence the $(m_1+m_2)\,\mathbf{h} \times \dot{\mathbf{h}}$ term is a constant, i.e. time invariant vector. Let us make use of this fact by defining an alternative (and still constant) angular momentum $\mathbf{j}$

$$\mathbf{j} = \mathbf{J} - (m_1+m_2)\,\mathbf{h}_0 \times \dot{\mathbf{h}} \tag{70}$$

Therefore

$$\mathbf{j} = \frac{m_1 m_2}{m_1+m_2}\mathbf{w} \times \dot{\mathbf{w}} \tag{71}$$

The fact that $\mathbf{j}$ is a constant means that vectors $\mathbf{w}$ and $\dot{\mathbf{w}}$ *must occupy the same plane throughout the motion.* Also, $\mathbf{j} = \frac{m_1 m_2}{m_1+m_2}\mathbf{w} \times \dot{\mathbf{w}}$ implies $\mathbf{j}$ is perpendicular to both $\mathbf{w}$ and $\dot{\mathbf{w}}$. We can therefore choose a plane where spherical polar angle $\phi = \frac{\pi}{2}$ at all times without loss of generality. Plane polar coordinates will be sufficient to describe the motion fully. The plane, characterized by Cartesian basis vectors $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ will be defined as

$$\begin{aligned} \hat{\mathbf{x}} &= \frac{\mathbf{w}_{\theta=0}}{|\mathbf{w}_{\theta=0}|} && (72)\\ \hat{\mathbf{y}} &= \hat{\mathbf{z}} \times \hat{\mathbf{x}} = \frac{\mathbf{j}}{|\mathbf{j}|} \times \frac{\mathbf{w}_{\theta=0}}{|\mathbf{w}_{\theta=0}|} \\ \hat{\mathbf{z}} &= \frac{\mathbf{j}}{|\mathbf{j}|} \end{aligned}$$

$\mathbf{w}_{\theta=0}$ is the mass separation vector when the plane polar angle is zero. We will see later that this will correspond to the semi-major axis of an elliptical orbit, if indeed the orbit is bound.

$$\mathbf{w}_0 = \mathbf{r}_{2,t=0} - \mathbf{r}_{1,t=0} \tag{73}$$

1.7 Describing orbits within the plane of motion

1.7.1 General considerations using spherical polars

$\mathbf{w}$ can now be described using plane polars, where θ is measured anti-clockwise from the $\hat{\mathbf{x}}$ axis. For completeness (and possible utility in future analyses where $\mathbf{j}$ may not be constant) we will leave in the ϕ of the more general spherical polar coordinates until we are forced to make use of $\phi = \frac{\pi}{2}$ to make further analytical progress. In this coordinate system (w, θ, ϕ), with orthonormal basis vectors $\left\{\hat{\mathbf{w}}, \hat{\boldsymbol{\theta}}, \hat{\boldsymbol{\phi}}\right\}$

$$\mathbf{w} = w\hat{\mathbf{w}} \tag{74}$$

$$\dot{\mathbf{w}} = \dot{w}\hat{\mathbf{w}} + w\dot{\theta}\sin\phi\hat{\boldsymbol{\theta}} + w\dot{\phi}\hat{\boldsymbol{\phi}} \tag{75}$$

$$
\begin{aligned}
\ddot{\mathbf{w}} &= \left(\ddot{w} - w\dot{\phi}^2 - w\dot{\theta}^2\sin^2\phi\right)\hat{\mathbf{w}} + ... \\
&\quad \left(2\dot{w}\dot{\theta}\sin\phi + 2w\dot{\theta}\dot{\phi}\cos\phi + w\ddot{\theta}\sin\phi\right)\hat{\boldsymbol{\theta}} + ... \\
&\quad \left(2\dot{w}\dot{\phi} + w\ddot{\phi} - w\dot{\theta}^2\sin\phi\cos\phi\right)\hat{\boldsymbol{\phi}}
\end{aligned} \tag{76}
$$

Hence $\ddot{\mathbf{w}} = -\frac{GM}{|\mathbf{w}|^3}\mathbf{w}$ becomes three coupled non-linear equations in w, θ, ϕ

$\parallel \hat{\mathbf{w}}$

$$
\ddot{w} - w\dot{\phi}^2 - w\dot{\theta}^2\sin^2\phi = -\frac{GM}{w^2} \tag{77}
$$

$\parallel \hat{\boldsymbol{\theta}}$

$$
2\dot{w}\dot{\theta}\sin\phi + 2w\dot{\theta}\dot{\phi}\cos\phi + w\ddot{\theta}\sin\phi = 0 \tag{78}
$$

$\parallel \hat{\boldsymbol{\phi}}$

$$
2\dot{w}\dot{\phi} + w\ddot{\phi} - w\dot{\theta}^2\sin\phi\cos\phi = 0 \tag{79}
$$

Therefore

$$
\begin{aligned}
\mathbf{w}\times\dot{\mathbf{w}} &= w\hat{\mathbf{w}}\times\left(\dot{w}\hat{\mathbf{w}} + w\dot{\theta}\sin\phi\hat{\boldsymbol{\theta}} + w\dot{\phi}\hat{\boldsymbol{\phi}}\right) \\
&= w^2\dot{\theta}\sin\phi\hat{\mathbf{w}}\times\hat{\boldsymbol{\theta}} + w^2\dot{\phi}\hat{\mathbf{w}}\times\hat{\boldsymbol{\phi}} \\
&= -w^2\dot{\theta}\sin\phi\hat{\boldsymbol{\phi}} + w^2\dot{\phi}\hat{\boldsymbol{\theta}}
\end{aligned} \tag{80}
$$

Since $\hat{\mathbf{w}}\times\hat{\boldsymbol{\theta}} = -\hat{\boldsymbol{\phi}}$ and $\hat{\mathbf{w}}\times\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\theta}}$.

Therefore

$$
\mathbf{j} = \frac{m_1 m_2}{m_1 + m_2} w^2\left(\dot{\phi}\hat{\boldsymbol{\theta}} - \dot{\theta}\sin\phi\hat{\boldsymbol{\phi}}\right) \tag{81}
$$

Hence

$$
\mathbf{j}\cdot\mathbf{j} \equiv |\mathbf{j}|^2 = \left(\frac{m_1 m_2}{m_1 + m_2}\right)^2\left(\dot{\phi}^2 + \dot{\theta}^2\sin^2\phi\right) \tag{82}
$$

which means

$$
\dot{\phi}^2 + \dot{\theta}^2\sin^2\phi = \frac{|\mathbf{j}|^2}{w^4}\left(\frac{m_1 + m_2}{m_1 m_2}\right)^2 \tag{83}
$$

Now the $\hat{\mathbf{w}}$ component of 55 is

$$
\ddot{w} - w\dot{\phi}^2 - w\dot{\theta}^2\sin^2\phi = -\frac{GM}{w^2} \tag{84}
$$

hence

$$
\ddot{w} - w\left(\dot{\phi}^2 + \dot{\theta}^2\sin^2\phi\right) = -\frac{GM}{w^2} \tag{85}
$$

Therefore

$$
\ddot{w} - w\left(\frac{|\mathbf{j}|^2}{w^4}\left(\frac{m_1 + m_2}{m_1 m_2}\right)^2\right) = -\frac{GM}{w^2} \tag{86}
$$

$$
\Rightarrow \ddot{w}w^3 - |\mathbf{j}|^2\left(\frac{m_1 + m_2}{m_1 m_2}\right)^2 + GMw = 0 \tag{87}
$$

1.7.2 Solving for the orbit using the fact that motion is in the plane

Now let us make use of the fact that $\phi = \frac{\pi}{2}$ (and therefore $\dot{\phi} = 0$) for this motion

$$
\begin{aligned}
\mathbf{j} &= -\frac{m_1 m_2}{m_1 + m_2} w^2\dot{\theta}\hat{\boldsymbol{\phi}} \\
|\mathbf{j}|^2 &= \left(\frac{m_1 m_2}{m_1 + m_2}\right)^2 w^4\dot{\theta}^2
\end{aligned} \tag{88}
$$

Since the motion is planar we can also express $w = w(\theta)$ and therefore

$$
\dot{w} = \frac{dw}{d\theta}\dot{\theta} \tag{89}
$$

Hence

$$\ddot{w} = \frac{d^2w}{d\theta^2}\dot{\theta}^2 + \frac{dw}{d\theta}\ddot{\theta} \tag{90}$$

Now the $\hat{\boldsymbol{\theta}}$ of 55 is

$$2\dot{w}\dot{\theta}\sin\phi + 2w\dot{\theta}\dot{\phi}\cos\phi + w\ddot{\theta}\sin\phi = 0 \tag{91}$$

Since $\phi = \frac{\pi}{2}$ and $\dot{\phi} = 0$ this reduces to

$$2\dot{w}\dot{\theta} + w\ddot{\theta} = 0 \tag{92}$$

Hence

$$\ddot{\theta} = -\frac{2\dot{w}\dot{\theta}}{w} \tag{93}$$

Now $\dot{w} = \frac{dw}{d\theta}\dot{\theta}$ which means

$$\ddot{\theta} = -\frac{2\dot{\theta}^2}{w}\frac{dw}{d\theta} \tag{94}$$

Therefore

$$\begin{aligned}
\ddot{w} &= \frac{d^2w}{d\theta^2}\dot{\theta}^2 + \frac{dw}{d\theta}\ddot{\theta} \\
&= \frac{d^2w}{d\theta^2}\dot{\theta}^2 + \frac{dw}{d\theta}\left(-\frac{2\dot{\theta}^2}{w}\frac{dw}{d\theta}\right) \\
&= \left(\frac{d^2w}{d\theta^2} - \frac{2}{w}\left(\frac{dw}{d\theta}\right)^2\right)\dot{\theta}^2
\end{aligned} \tag{95}$$

Now

$$|\mathbf{j}|^2 = \left(\frac{m_1m_2}{m_1+m_2}\right)^2 w^4\dot{\theta}^2 \tag{96}$$

Hence

$$\dot{\theta}^2 = \frac{1}{w^4}\left(\frac{m_1+m_2}{m_1m_2}\right)^2 |\mathbf{j}|^2 \tag{97}$$

and therefore

$$\ddot{w} = \left(\frac{1}{w^4}\frac{d^2w}{d\theta^2} - \frac{2}{w^5}\left(\frac{dw}{d\theta}\right)^2\right)\left(\frac{m_1+m_2}{m_1m_2}\right)^2 |\mathbf{j}|^2 \tag{98}$$

Putting this all together, we can re-write $\ddot{w}w^3 - |\mathbf{j}|^2\left(\frac{m_1+m_2}{m_1m_2}\right)^2 + GMw = 0$ derived in the previous section.

$$\begin{aligned}
\ddot{w}w^3 - |\mathbf{j}|^2\left(\frac{m_1+m_2}{m_1m_2}\right)^2 + GMw &= 0 \\
\left(\frac{1}{w}\frac{d^2w}{d\theta^2} - \frac{2}{w^2}\left(\frac{dw}{d\theta}\right)^2\right)\left(\frac{m_1+m_2}{m_1m_2}\right)^2|\mathbf{j}|^2 - |\mathbf{j}|^2\left(\frac{m_1+m_2}{m_1m_2}\right)^2 + GMw &= 0 \\
w\frac{d^2w}{d\theta^2} - 2\left(\frac{dw}{d\theta}\right)^2 - w^2 + \frac{GM}{|\mathbf{j}|^2}\left(\frac{m_1m_2}{m_1+m_2}\right)^2 w^3 &= 0 \\
w\frac{d^2w}{d\theta^2} - 2\left(\frac{dw}{d\theta}\right)^2 - w^2 + \frac{G}{|\mathbf{j}|^2}\frac{m_1^2m_2^2}{m_1+m_2}w^3 &= 0
\end{aligned} \tag{99}$$

To solve this consider the substitution

$$w = \frac{1}{u} \tag{100}$$

Therefore

$$\begin{aligned}
\frac{dw}{d\theta} &= -\frac{1}{u^2}\frac{du}{d\theta} \\
\frac{d^2w}{d\theta^2} &= -\frac{1}{u^2}\frac{d^2u}{d\theta^2} + \frac{2}{u^3}\left(\frac{du}{d\theta}\right)^2
\end{aligned} \tag{101}$$

Hence

$$\begin{aligned}
w\frac{d^2w}{d\theta^2} - 2\left(\frac{dw}{d\theta}\right)^2 - w^2 + \frac{G}{|\mathbf{j}|^2}\frac{m_1^2 m_2^2}{m_1+m_2}w^3 &= 0 \qquad (102)\\
\frac{1}{u}\left(-\frac{1}{u^2}\frac{d^2u}{d\theta^2} + \frac{2}{u^3}\left(\frac{du}{d\theta}\right)^2\right) - 2\left(-\frac{1}{u^2}\frac{du}{d\theta}\right)^2 - \frac{1}{u^2} + \frac{G}{|\mathbf{j}|^2}\frac{m_1^2 m_2^2}{m_1+m_2}\frac{1}{u^3} &= 0\\
-\frac{1}{u^3}\frac{d^2u}{d\theta^2} + \frac{2}{u^4}\left(\frac{du}{d\theta}\right)^2 - \frac{2}{u^4}\left(\frac{du}{d\theta}\right)^2 - \frac{1}{u^2} + \frac{G}{|\mathbf{j}|^2}\frac{m_1^2 m_2^2}{m_1+m_2}\frac{1}{u^3} &= 0\\
-\frac{1}{u^3}\frac{d^2u}{d\theta^2} - \frac{1}{u^2} + \frac{G}{|\mathbf{j}|^2}\frac{m_1^2 m_2^2}{m_1+m_2}\frac{1}{u^3} &= 0\\
\frac{d^2u}{d\theta^2} + u - \frac{G}{|\mathbf{j}|^2}\frac{m_1^2 m_2^2}{m_1+m_2} &= 0
\end{aligned}$$

To solve for $u(\theta)$ define, for brevity,

$$\alpha = \frac{G}{|\mathbf{j}|^2}\frac{m_1^2 m_2^2}{m_1+m_2} \qquad (103)$$

Hence

$$\frac{d^2u}{d\theta^2} + u = \alpha \qquad (104)$$

This is a linear second order ordinary differential equation in $u(\theta)$. Consider solutions of the form[2]

$$u = \pm\varepsilon\alpha\cos B\left(\theta - \theta'\right) + \alpha \qquad (105)$$

where ε, B, θ are constants and $\varepsilon \geq 0$. Substituting into $\frac{d^2u}{d\theta^2} + u = \alpha$ yields

$$\mp B\varepsilon\alpha\cos B\left(\theta - \theta'\right) \pm \varepsilon\alpha\cos B\left(\theta - \theta'\right) + \alpha = \alpha \qquad (106)$$

Hence

$$B = 1 \qquad (107)$$

Which means

$$u = \pm\varepsilon\alpha\cos\left(\theta - \theta'\right) + \alpha \qquad (108)$$

Therefore

$$w = \frac{|\mathbf{j}|^2\left(m_1+m_2\right)}{Gm_1^2m_2^2}\frac{1}{1 \pm \varepsilon\cos\left(\theta - \theta'\right)} \qquad (109)$$

If we set the sensible constraint that $w \geq 0$ then in the negative solution $\varepsilon \leq 1$.

The polar equation of an *ellipse*, taking w to be the radius from the focus of the ellipse is

$$w = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\theta} \qquad (110)$$

Eccentricity ε is defined in terms of the semi-major axis of the ellipse a and semi-minor axis b

$$\begin{aligned}
\varepsilon &= \sqrt{1-\frac{b^2}{a^2}} \qquad (111)\\
b &= a\sqrt{1-\varepsilon^2} \qquad (112)
\end{aligned}$$

Note this must also be $0 \leq \varepsilon \leq 1$ to make b real.

So indeed the range of possible values of ε for both positive and negative solutions of $w(\theta)$ is

$$0 \leq \varepsilon \leq 1 \qquad (113)$$

For the elliptical solution we can equate the constants to write the semi-major axis a in terms of the other orbital parameters

[2] Note this has two constants of integration, ε and θ' appropriate to the second order nature of the differential equation.

$$\begin{aligned} a\left(1-\varepsilon^2\right) &= \frac{|\mathbf{j}|^2\,(m_1+m_2)}{Gm_1^2m_2^2} \\ a &= \frac{|\mathbf{j}|^2\,(m_1+m_2)}{Gm_1^2m_2^2\,(1-\varepsilon^2)} \end{aligned} \tag{114}$$

Therefore if we consider the positive solution of our orbit equation, and also define a coordinate system $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ such that $\theta' = 0$ [3]

$$\begin{aligned} w &= \frac{l}{1+\varepsilon\cos\theta} \\ l &= a\left(1-\varepsilon^2\right) = \frac{|\mathbf{j}|^2\,(m_1+m_2)}{Gm_1^2m_2^2} \\ a &= \frac{|\mathbf{j}|^2\,(m_1+m_2)}{Gm_1^2m_2^2\,(1-\varepsilon^2)} \\ b &= a\sqrt{1-\varepsilon^2} \end{aligned} \tag{115}$$

The time dependency of w can be found by numerically solving

$$\ddot{w}w^3 - |\mathbf{j}|^2\left(\frac{m_1+m_2}{m_1m_2}\right)^2 + GMw = 0 \tag{116}$$

and then θ found from

$$\begin{aligned} w &= \frac{|\mathbf{j}|^2\,(m_1+m_2)}{Gm_1^2m_2^2}\frac{1}{1+\varepsilon\cos\theta} \\ 1+\varepsilon\cos\theta &= \frac{|\mathbf{j}|^2\,(m_1+m_2)}{wGm_1^2m_2^2} \\ \theta &= \cos^{-1}\left(\frac{|\mathbf{j}|^2\,(m_1+m_2)}{\varepsilon wGm_1^2m_2^2} - \frac{1}{\varepsilon}\right) \end{aligned} \tag{117}$$

Alternatively one can find θ as a function of time t using

$$\begin{aligned} \dot{\theta} &= \frac{1}{w^2}\left(\frac{m_1+m_2}{m_1m_2}\right)|\mathbf{j}| \\ &= \frac{G^2m_1^4m_2^4\,(1+\varepsilon\cos\theta)^2}{|\mathbf{j}|^4\,(m_1+m_2)^2}\left(\frac{m_1+m_2}{m_1m_2}\right)|\mathbf{j}| \\ &= \frac{G^2m_1^3m_2^3\,(1+\varepsilon\cos\theta)^2}{|\mathbf{j}|^3\,(m_1+m_2)} \end{aligned} \tag{118}$$

Therefore

$$t = \frac{|\mathbf{j}|^3\,(m_1+m_2)}{G^2m_1^3m_2^3}\int_{\theta_0}^{\theta}\frac{d\theta'}{\left(1+\varepsilon\cos\theta'\right)^2} \tag{119}$$

So, via numerical integration, one can find t as a function of θ. If this is indeed a one-to-one function, $\theta(t)$ can be obtained.

1.8 Position and velocity of masses in a two body system

Let us summarize the results derived so far:

Define centre of mass

$$\mathbf{h} = \frac{m_1\mathbf{r}_1 + m_2\mathbf{r}_2}{m_1+m_2} \tag{120}$$

[3] This lack of generality will be accounted for by setting a three-dimensional orientation of the ellipse as an input parameter of an orbital model. The key point is at $t = 0$, θ may not in general $= 0$. This will be accounted for in the calculation of time from polar angle.

and the mass separation vector

$$\mathbf{w} = \mathbf{r}_2 - \mathbf{r}_1 \tag{121}$$

Let the position of the centre of mass at $t = 0$ be $\mathbf{h}_0$. Hence $\mathbf{w}_0$ is the initial mass separation vector at simulation time $t = 0$.

$$\mathbf{w}_0 = \mathbf{r}_{2,t=0} - \mathbf{r}_{1,t=0} \tag{122}$$

The position vectors of masses are

$$\mathbf{r}_1 = \mathbf{h} - \frac{m_2 \mathbf{w}}{m_1 + m_2} \tag{123}$$

$$\mathbf{r}_2 = \mathbf{h} + \frac{m_1 \mathbf{w}}{m_1 + m_2} \tag{124}$$

The centre of mass velocity $\dot{\mathbf{h}}$ is a constant of the motion is the masses don't change, hence to find the mass velocities $\dot{\mathbf{r}}_1$ and $\dot{\mathbf{r}}_2$ all we have to find is $\dot{\mathbf{w}}$

$$\dot{\mathbf{r}}_1 = \dot{\mathbf{h}} - \frac{m_2 \dot{\mathbf{w}}}{m_1 + m_2} \tag{125}$$

$$\dot{\mathbf{r}}_2 = \dot{\mathbf{h}} + \frac{m_1 \dot{\mathbf{w}}}{m_1 + m_2} \tag{126}$$

Define total angular momentum $\mathbf{J}$, which is a *constant* of the system

$$\mathbf{J} = m_1 \mathbf{r}_1 \times \dot{\mathbf{r}}_1 + m_2 \mathbf{r}_2 \times \dot{\mathbf{r}}_2 \tag{127}$$

Define the difference in angular momentum bewen the entire system and a single mass equal to the sum of the two masses, whose dynamics are described by the centre of mass of the two-mass system.

$$\mathbf{j} = \mathbf{J} - (m_1 + m_2)\, \mathbf{h}_0 \times \dot{\mathbf{h}} \tag{128}$$

It can be shown

$$\mathbf{j} = \frac{m_1 m_2}{m_1 + m_2} \mathbf{w} \times \dot{\mathbf{w}} \tag{129}$$

The fact that $\mathbf{j}$ is a constant means that vectors $\mathbf{w}$ and $\dot{\mathbf{w}}$ *must occupy the same plane throughout the motion.* Also, $\mathbf{j} = \frac{m_1 m_2}{m_1 + m_2} \mathbf{w} \times \dot{\mathbf{w}}$ implies $\mathbf{j}$ is perpendicular to both $\mathbf{w}$ and $\dot{\mathbf{w}}$. We can therefore choose a plane where spherical polar angle $\phi = \frac{\pi}{2}$ at all times without loss of generality.

Define polar coordinates to describe the elliptical orbit of $\mathbf{w}$ about the centre of mass $\mathbf{h}$

$$\mathbf{w} = w(\theta) \hat{\mathbf{r}} \tag{130}$$

$$\hat{\mathbf{r}} = \cos\theta \hat{\mathbf{x}} + \sin\theta \hat{\mathbf{y}} \tag{131}$$

$$\hat{\boldsymbol{\theta}} = -\sin\theta \hat{\mathbf{x}} + \cos\theta \hat{\mathbf{y}} \tag{132}$$

For closed elliptical orbits with eccentricity $0 \leq \varepsilon \leq 1$:

$$w = \frac{l}{1 + \varepsilon \cos\theta} \tag{133}$$

The semi-latus rectum l of the elliptical orbit is

$$l = \frac{|\mathbf{j}|^2 (m_1 + m_2)}{G m_1^2 m_2^2} \tag{134}$$

The rate of change of $\mathbf{w}$ is

$$\dot{\mathbf{w}} = \frac{a\left(1 - \varepsilon^2\right)\dot{\theta}}{1 + \varepsilon \cos\theta} \left(\frac{\varepsilon \sin\theta}{1 + \varepsilon \cos\theta} \hat{\mathbf{r}} + \hat{\boldsymbol{\theta}} \right) \tag{135}$$

Hence

$$w = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\theta} \tag{136}$$

$$\dot{\mathbf{w}} = \frac{a\left(1-\varepsilon^2\right)\dot{\theta}}{1+\varepsilon\cos\theta}\left(\frac{\varepsilon\sin\theta}{1+\varepsilon\cos\theta}\hat{\mathbf{r}}+\hat{\boldsymbol{\theta}}\right) \tag{137}$$

We can define $\mathbf{w}_0$ using the polar equation of the ellipse

$$\mathbf{w}_0 = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\theta_0}\left(\hat{\mathbf{x}}\cos\theta_0+\hat{\mathbf{y}}\sin\theta_0\right)$$

The elliptical height radii b is

$$b = a\sqrt{1-\varepsilon^2}$$

The elliptical width a relates to the eccentricity ε and angular momentum $\mathbf{j}$ via

$$a = \frac{|\mathbf{j}|^2\left(m_1+m_2\right)}{Gm_1^2m_2^2\left(1-\varepsilon^2\right)}$$

Hence

$$|\mathbf{j}| = \sqrt{\frac{aGm_1^2m_2^2\left(1-\varepsilon^2\right)a}{m_1+m_2}}$$

Now for a two body gravitational system

$$|\mathbf{j}| = \frac{m_1m_2}{m_1+m_2}w^2\dot{\theta} \tag{138}$$

Hence

$$\frac{m_1m_2}{m_1+m_2}w^2\dot{\theta} = \sqrt{\frac{Gm_1^2m_2^2\left(1-\varepsilon^2\right)a}{m_1+m_2}} \tag{139}$$

$$\dot{\theta}w^2\sqrt{\frac{m_1^2m_2^2}{\left(m_1+m_2\right)^2}} = \sqrt{\frac{Gm_1^2m_2^2\left(1-\varepsilon^2\right)a}{m_1+m_2}} \tag{140}$$

And therefore

$$\dot{\theta} = \frac{\sqrt{G\left(m_1+m_2\right)\left(1-\varepsilon^2\right)a}}{w^2} \tag{141}$$

Using $w = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\theta}$

$$\dot{\theta} = \frac{\left(1+\varepsilon\cos\theta\right)^2\sqrt{G\left(m_1+m_2\right)\left(1-\varepsilon^2\right)a}}{a^2\left(1-\varepsilon^2\right)^2} \tag{142}$$

$$= \frac{\left(1+\varepsilon\cos\theta\right)^2\sqrt{G\left(m_1+m_2\right)\left(1-\varepsilon^2\right)a}}{\sqrt{a^4\left(1-\varepsilon^2\right)^4}} \tag{143}$$

Hence

$$\dot{\theta} = \left(1+\varepsilon\cos\theta\right)^2\sqrt{\frac{G\left(m_1+m_2\right)}{a^3\left(1-\varepsilon^2\right)^3}} \tag{144}$$

This result gives us an alternative method for finding t without having to previously compute the magnitude of the angular momentum $|\mathbf{j}|$.

$$t = \sqrt{\frac{a^3\left(1-\varepsilon^2\right)^3}{G\left(m_1+m_2\right)}}\int_{\theta_0}^{\theta}\frac{d\theta'}{\left(1+\varepsilon\cos\theta'\right)^2} \tag{145}$$

Now $\dot{\mathbf{w}} = \frac{a\left(1-\varepsilon^2\right)\dot{\theta}}{1+\varepsilon\cos\theta}\left(\frac{\varepsilon\sin\theta}{1+\varepsilon\cos\theta}\hat{\mathbf{r}}+\hat{\boldsymbol{\theta}}\right)$, hence

$$\dot{\mathbf{w}} = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\theta}\left(1+\varepsilon\cos\theta\right)^2\sqrt{\frac{G\left(m_1+m_2\right)}{a^3\left(1-\varepsilon^2\right)^3}}\left(\frac{\varepsilon\sin\theta}{1+\varepsilon\cos\theta}\hat{\mathbf{r}}+\hat{\boldsymbol{\theta}}\right)$$

$$= \sqrt{a^2\left(1-\varepsilon^2\right)^2}\left(1+\varepsilon\cos\theta\right)\sqrt{\frac{G\left(m_1+m_2\right)}{a^3\left(1-\varepsilon^2\right)^3}}\left(\frac{\varepsilon\sin\theta}{1+\varepsilon\cos\theta}\hat{\mathbf{r}}+\hat{\boldsymbol{\theta}}\right)$$

Which simplifies to

$$\dot{\mathbf{w}} = \sqrt{\frac{G(m_1+m_2)}{a(1-\varepsilon^2)}}(1+\varepsilon\cos\theta)\left(\frac{\varepsilon\sin\theta}{1+\varepsilon\cos\theta}\hat{\mathbf{r}}+\hat{\boldsymbol{\theta}}\right) \tag{146}$$

Hence

$$\dot{\mathbf{w}}_0 = \sqrt{\frac{G(m_1+m_2)}{a(1-\varepsilon^2)}}(1+\varepsilon\cos\theta_0)\left(\frac{\varepsilon\sin\theta_0}{1+\varepsilon\cos\theta_0}\hat{\mathbf{r}}+\hat{\boldsymbol{\theta}}\right) \tag{147}$$

1.9 Total energy of the two body system

The total energy of a two-mass gravitational system is

$$E = \tfrac{1}{2}m_1|\dot{\mathbf{r}}_1|^2 + \tfrac{1}{2}m_2|\dot{\mathbf{r}}_2|^2 - \frac{Gm_1m_1}{|\mathbf{r}_2-\mathbf{r}_1|} \tag{148}$$

Now

$$\mathbf{w} = \mathbf{r}_2 - \mathbf{r}_1 \tag{149}$$

And from the results derived in the previous sections

$$\begin{aligned}
\mathbf{w} &= \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\theta}(\hat{\mathbf{x}}\cos\theta+\hat{\mathbf{y}}\sin\theta) && (150)\\
\hat{\mathbf{r}} &= \cos\theta\hat{\mathbf{x}}+\sin\theta\hat{\mathbf{y}} && (151)\\
\hat{\boldsymbol{\theta}} &= -\sin\theta\hat{\mathbf{x}}+\cos\theta\hat{\mathbf{y}} && (152)\\
\dot{\mathbf{w}} &= \sqrt{\frac{G(m_1+m_2)}{a(1-\varepsilon^2)}}(1+\varepsilon\cos\theta)\left(\frac{\varepsilon\sin\theta}{1+\varepsilon\cos\theta}\hat{\mathbf{r}}+\hat{\boldsymbol{\theta}}\right) && (153)\\
\dot{\mathbf{r}}_1 &= \dot{\mathbf{h}}-\frac{m_2\dot{\mathbf{w}}}{m_1+m_2} && (154)\\
\dot{\mathbf{r}}_2 &= \dot{\mathbf{h}}+\frac{m_1\dot{\mathbf{w}}}{m_1+m_2} && (155)
\end{aligned}$$

Hence

$$\begin{aligned}
E &= \tfrac{1}{2}m_1\left|\dot{\mathbf{h}}-\frac{m_2\dot{\mathbf{w}}}{m_1+m_2}\right|^2 + \tfrac{1}{2}m_2\left|\dot{\mathbf{h}}+\frac{m_1\dot{\mathbf{w}}}{m_1+m_2}\right|^2 - \frac{Gm_1m_1}{|\mathbf{w}|} && (156)\\
&= \tfrac{1}{2}m_1\left\{\left|\dot{\mathbf{h}}\right|^2 - \frac{2m_2\dot{\mathbf{w}}\cdot\dot{\mathbf{h}}}{m_1+m_2} + \left(\frac{m_2}{m_1+m_2}\right)^2|\dot{\mathbf{w}}|^2\right\} + \ ... && (157)\\
&\quad ... + \tfrac{1}{2}m_2\left\{\left|\dot{\mathbf{h}}\right|^2 + \frac{2m_1\dot{\mathbf{w}}\cdot\dot{\mathbf{h}}}{m_1+m_2} + \left(\frac{m_1}{m_1+m_2}\right)^2|\dot{\mathbf{w}}|^2\right\} - \frac{Gm_1m_1}{|\mathbf{w}|} && (158)\\
&= \tfrac{1}{2}(m_1+m_2)\left|\dot{\mathbf{h}}\right|^2 + \tfrac{1}{2}\frac{m_2^2m_1+m_1^2m_2}{(m_1+m_2)^2}|\dot{\mathbf{w}}|^2 - \frac{Gm_1m_1}{|\mathbf{w}|} && (159)\\
&= \tfrac{1}{2}(m_1+m_2)\left|\dot{\mathbf{h}}\right|^2 + \tfrac{1}{2}\frac{m_1m_2}{m_1+m_2}|\dot{\mathbf{w}}|^2 - \frac{Gm_1m_1}{|\mathbf{w}|} && (160)
\end{aligned}$$

Now

$$|\mathbf{w}| = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\theta} \tag{161}$$

and

$$
\begin{aligned}
|\dot{\mathbf{w}}|^2 &= \frac{G(m_1+m_2)}{a(1-\varepsilon^2)}(1+\varepsilon\cos\theta)^2\left|\frac{\varepsilon\sin\theta}{1+\varepsilon\cos\theta}\hat{\mathbf{r}}+\hat{\boldsymbol{\theta}}\right|^2 &(162)\\
&= \frac{G(m_1+m_2)}{a(1-\varepsilon^2)}(1+\varepsilon\cos\theta)^2\left\{\left(\frac{\varepsilon\sin\theta}{1+\varepsilon\cos\theta}\right)^2+1\right\} &(163)\\
&= \frac{G(m_1+m_2)}{a(1-\varepsilon^2)}(1+\varepsilon\cos\theta)^2\left\{\frac{\varepsilon^2\sin^2\theta+(1+\varepsilon\cos\theta)^2}{(1+\varepsilon\cos\theta)^2}\right\} &(164)\\
&= \frac{G(m_1+m_2)}{a(1-\varepsilon^2)}(1+\varepsilon\cos\theta)^2\left\{\frac{\varepsilon^2\left(1-\cos^2\theta\right)+(1+\varepsilon\cos\theta)^2}{(1+\varepsilon\cos\theta)^2}\right\} &(165)\\
&= \frac{G(m_1+m_2)}{a(1-\varepsilon^2)}(1+\varepsilon\cos\theta)^2\left\{\frac{\varepsilon^2-\varepsilon^2\cos^2\theta+1+2\varepsilon\cos\theta+\varepsilon^2\cos^2\theta}{(1+\varepsilon\cos\theta)^2}\right\} &(166)
\end{aligned}
$$

Which simplifies to

$$|\dot{\mathbf{w}}|^2 = \frac{G(m_1+m_2)}{a(1-\varepsilon^2)}\left(\varepsilon^2+1+2\varepsilon\cos\theta\right) \tag{167}$$

Hence

$$
\begin{aligned}
E &= \tfrac{1}{2}(m_1+m_2)\left|\dot{\mathbf{h}}\right|^2+\tfrac{1}{2}\frac{m_1m_2}{m_1+m_2}|\dot{\mathbf{w}}|^2-\frac{Gm_1m_1}{|\mathbf{w}|} &(168)\\
&= \tfrac{1}{2}(m_1+m_2)\left|\dot{\mathbf{h}}\right|^2+\tfrac{1}{2}\frac{m_1m_2}{m_1+m_2}\frac{G(m_1+m_2)}{a(1-\varepsilon^2)}\left(\varepsilon^2+1+2\varepsilon\cos\theta\right)-\frac{Gm_1m_1(1+\varepsilon\cos\theta)}{a(1-\varepsilon^2)} &(169)\\
&= \tfrac{1}{2}(m_1+m_2)\left|\dot{\mathbf{h}}\right|^2+\frac{Gm_1m_2}{2a(1-\varepsilon^2)}\left(\varepsilon^2+1+2\varepsilon\cos\theta\right)-\frac{Gm_1m_1(1+\varepsilon\cos\theta)}{a(1-\varepsilon^2)} &(170)\\
&= \tfrac{1}{2}(m_1+m_2)\left|\dot{\mathbf{h}}\right|^2+\frac{Gm_1m_2}{2a(1-\varepsilon^2)}\left\{\varepsilon^2+1+2\varepsilon\cos\theta-2-2\varepsilon\cos\theta\right\} &(171)\\
&= \tfrac{1}{2}(m_1+m_2)\left|\dot{\mathbf{h}}\right|^2+\frac{Gm_1m_2}{2a(1-\varepsilon^2)}\left\{\varepsilon^2-1\right\} &(172)
\end{aligned}
$$

Therefore

$$E = \tfrac{1}{2}(m_1+m_2)\left|\dot{\mathbf{h}}\right|^2-\frac{Gm_1m_2}{2a} \tag{173}$$

Now recall for an elliptical orbit

$$a = \frac{|\mathbf{j}|^2(m_1+m_2)}{Gm_1^2m_2^2(1-\varepsilon^2)} \tag{174}$$

Hence

$$
\begin{aligned}
E &= \tfrac{1}{2}(m_1+m_2)\left|\dot{\mathbf{h}}\right|^2-\frac{Gm_1m_2}{2}\frac{Gm_1^2m_2^2\left(1-\varepsilon^2\right)}{|\mathbf{j}|^2(m_1+m_2)} &(175)\\
&= \tfrac{1}{2}(m_1+m_2)\left|\dot{\mathbf{h}}\right|^2-\frac{G^2m_1^3m_2^3\left(1-\varepsilon^2\right)}{2|\mathbf{j}|^2(m_1+m_2)} &(176)
\end{aligned}
$$

Hence we can now describe the orbit in terms of a given magnitude of angular momentum $|\mathbf{j}|$ and eccentricity ε. For a given $|\mathbf{j}|$, this means E varies quadratically with ε. If we set $\left|\dot{\mathbf{h}}\right| = 0$ then $\varepsilon = 0$ will be the minimum of a parabola. This is the lowest energy solution and corresponds to a circular orbit. For an elliptical orbit, $0 < \varepsilon \leq 1$. When $\varepsilon = 1$ then $E = 0$, i.e. the orbit is unbound - there is not ne gravitational potential energy.

1.10 Kepler's laws

Kepler's three laws are:

1. *The orbit of every planet in the solar system is an ellipse with the Sun at one of the two foci.*
2. *A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.*
3. *The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.*

   The wording of Kepler's laws implies a specific application to the solar system. However, the laws are more generally applicable to any system of two masses whose mutual attraction is an inverse-square law.

1.10.1 Kepler's first law

Bound orbits of a two-mass system undergoing attraction via Newton's law of gravitation are ellipses.

We have shown above for two-mass system interacting via an inverse-square law of attraction, the centre of mass $\mathbf{h}$ is at one focus of an ellipse. The displacement vector $\mathbf{w}$ of the masses traces out the elliptical orbit. The magnitude of the displacement is given by

$$|\mathbf{w}| = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\theta} \tag{177}$$

where ε is the eccentricity of the elliptical orbit, a is the semi-major axis of the ellipse and θ is the polar angle of the orbit measured anticlockwise from the semi-major axis of the ellipse. The position vectors of the masses are given by

$$\mathbf{r}_1 = \mathbf{h} - \frac{m_2\mathbf{w}}{m_1+m_2} \tag{178}$$

$$\mathbf{r}_2 = \mathbf{h} + \frac{m_1\mathbf{w}}{m_1+m_2} \tag{179}$$

1.10.2 Kepler's second law

The the rate of area $\frac{dA}{dt}$ swept out by the elliptical orbit from the focus of the ellipse is a constant.

A sector of arc of angle $d\theta$ is given by

$$dA = \tfrac{1}{2}\left|\mathbf{w}\right|^2 d\theta \tag{180}$$

Hence

$$\frac{dA}{dt} = \tfrac{1}{2}\left|\mathbf{w}\right|^2 \dot{\theta} \tag{181}$$

Now

$$\dot{\theta} = \left(1+\varepsilon\cos\theta\right)^2 \sqrt{\frac{G\left(m_1+m_2\right)}{a^3\left(1-\varepsilon^2\right)^3}} \tag{182}$$

and

$$\left|\mathbf{w}\right|^2 = \frac{a^2\left(1-\varepsilon^2\right)^2}{\left(1+\varepsilon\cos\theta\right)^2} \tag{183}$$

Hence

$$\frac{dA}{dt} = \tfrac{1}{2}\frac{a^2\left(1-\varepsilon^2\right)^2}{\left(1+\varepsilon\cos\theta\right)^2}\left(1+\varepsilon\cos\theta\right)^2 \sqrt{\frac{G\left(m_1+m_2\right)}{a^3\left(1-\varepsilon^2\right)^3}} \tag{184}$$

$$= \tfrac{1}{2}a^2\left(1-\varepsilon^2\right)^2 \sqrt{\frac{G\left(m_1+m_2\right)}{a^3\left(1-\varepsilon^2\right)^3}} \tag{185}$$

$$= \tfrac{1}{2}\sqrt{G\left(m_1+m_2\right)\left(1-\varepsilon^2\right)a} \tag{186}$$

which is a constant of the motion.

1.10.3 Kepler's third law

The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.

Since $\frac{dA}{dt}$ is a constant and the area of an ellipse is πab, the orbital period P must equal

$$P = \frac{\pi ab}{\frac{dA}{dt}} \tag{187}$$

Now

$$b = a\sqrt{1-\varepsilon^2} \tag{188}$$

Hence

$$P = \frac{\pi a^2\sqrt{1-\varepsilon^2}}{\frac{1}{2}\sqrt{G\left(m_1+m_2\right)\left(1-\varepsilon^2\right)a}} \tag{189}$$

$$= \frac{2\pi\sqrt{a^4\left(1-\varepsilon^2\right)}}{\sqrt{G\left(m_1+m_2\right)a\left(1-\varepsilon^2\right)}} \tag{190}$$

$$= 2\pi\sqrt{\frac{a^3}{G\left(m_1+m_2\right)}} \tag{191}$$

Hence

$$P^2 = \frac{4\pi^2}{G\left(m_1+m_2\right)}a^3 \tag{192}$$

as required. The constant of proportionality of P^2 vs a^3 is $\frac{4\pi^2}{G(m_1+m_2)}$

2 Mass rings initial conditions

2.1 Scope

A set of mass rings will be a collection of planets making concentric circular orbits about a central star of mass M. The gravitational interaction between the planets shall be ignored, only the gravitational field of the star is important. In this section we will derive the initial positions and velocities of the masses. A more general numerical method (i.e. involving other bodies, which may perturb the position of the star as well as the ring masses) shall be employed to compute the subsequent dynamics.

2.2 Define quantities

Inputs

Quantity	Symbol	Typical value
Number of rings	N_{rings}	30
$\hat{\mathbf{x}}$ coordinate of ring centre /AU	x_c	0
$\hat{\mathbf{y}}$ coordinate of ring centre /AU	y_c	0
$\hat{\mathbf{z}}$ coordinate of ring centre /AU	z_c	0
$\hat{\mathbf{x}}$ velocity of ring centre / AU/Yr	$\dot{x}_c$	0
$\hat{\mathbf{y}}$ velocity of ring centre / AU/Yr	$\dot{y}_c$	0
$\hat{\mathbf{z}}$ velocity of ring centre / AU/Yr	$\dot{z}_c$	0
Ring rotation vector	$\mathbf{d} = d_x\hat{\mathbf{x}} + d_y\hat{\mathbf{y}} + d_{\mathbf{y}}\hat{\mathbf{z}}$	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
Radius of first ring /AU	R_0	1
Mass of star /solar masses	M	1
Inter-ring radial distance /AU	ΔR	0.1
Arc separation of ring masses /AU	δ	$\frac{1}{30}\pi$

Derived quantities for masses m_{ij} corresponding to the j^{th} mass in ring i

Quantity	Symbol
Set of ring mass $\hat{\mathbf{x}}$ coordinates	$\{x_{ij}\}$
Set of ring mass $\hat{\mathbf{y}}$ coordinates	$\{y_{ij}\}$
Set of ring mass $\hat{\mathbf{z}}$ coordinates	$\{z_{ij}\}$
Set of ring mass $\hat{\mathbf{x}}$ velocities	$\{\dot{x}_{ij}\}$
Set of ring mass $\hat{\mathbf{y}}$ velocities	$\{\dot{y}_{ij}\}$
Set of ring mass $\hat{\mathbf{z}}$ velocities	$\{\dot{z}_{ij}\}$

2.3 Derived quantities

For a two-mass system, the rate of change of separation vector between masses is

$$\dot{\mathbf{w}} = \sqrt{\frac{G\left(m_1+m_2\right)}{a\left(1-\varepsilon^2\right)}}\left(1+\varepsilon\cos\theta\right)\left(\frac{\varepsilon\sin\theta}{1+\varepsilon\cos\theta}\hat{\mathbf{r}}+\hat{\boldsymbol{\theta}}\right) \tag{193}$$

The conversion between plane polar coordinates and Cartesians is

$$\begin{aligned} \hat{\mathbf{r}} &= \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}} \\ \hat{\boldsymbol{\theta}} &= -\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}} \end{aligned} \tag{194}$$

For circular orbits $\varepsilon = 0$. Since we can ignore the mass of the planets, $m_1 + m_2 = M$, the mass of the star. Since the planets don't move the star, the semi-major axis of the orbit ellipse is the radius of the ring R. Hence $a = R$.The rate of change of separation vector is now the actual velocity $\mathbf{u}$ of the planet.

Hence

$$\mathbf{u} = \sqrt{\frac{GM}{R}}\hat{\boldsymbol{\theta}} \tag{195}$$

or in Cartesians

$$\mathbf{u} = \sqrt{\frac{GM}{R}}\left(-\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}\right) \tag{196}$$

We can use the above result to define the velocities of of mass m_{ij} corresponding to the j^{th} mass in ring i. The radius of the i^{th} ring is

$$R_i = R_0 + (i-1)\Delta R \tag{197}$$

The number of planets in this ring will be

$$n_i = \text{floor}\left(\frac{2\pi R_i}{\delta}\right) \tag{198}$$

The Cartesian coordinates of the ring masses m_{ij} are therefore

$$x_{ij} = x_c + R_i \cos\left(\frac{2\pi(j-1)}{n_i}\right) \tag{199}$$

$$y_{ij} = y_c + R_i \sin\left(\frac{2\pi(j-1)}{n_i}\right) \tag{200}$$

$$z_{ij} = z_c \tag{201}$$

The corresponding initial velocities are

$$\dot{x}_{ij} = \dot{x}_c - \sqrt{\frac{GM}{R_i}} \sin\left(\frac{2\pi(j-1)}{n_i}\right) \tag{202}$$

$$\dot{y}_{ij} = \dot{y}_c + \sqrt{\frac{GM}{R_i}} \cos\left(\frac{2\pi(j-1)}{n_i}\right) \tag{203}$$

$$\dot{z}_{ij} = \dot{z}_c \tag{204}$$

The desired plane of rotation can be achieved via the following transformation of the mass position and velocity vectors

$$\mathbf{r}_{ij} = x_{ij}\hat{\mathbf{x}} + y_{ij}\hat{\mathbf{y}} + z_{ij}\hat{\mathbf{z}} \tag{205}$$

$$\mathbf{u}_{ij} = \dot{x}_{ij}\hat{\mathbf{x}} + \dot{y}_{ij}\hat{\mathbf{y}} + \dot{z}_{ij}\hat{\mathbf{z}} \tag{206}$$

$$\mathbf{r}'_{ij} = f\left(\mathbf{r}_{ij}, 0, \mathbf{d}, \mathbf{0}, \hat{\mathbf{z}}\right) \tag{207}$$

$$\mathbf{u}'_{ij} = f\left(\mathbf{u}_{ij}, 0, \mathbf{d}, \mathbf{0}, \hat{\mathbf{z}}\right) \tag{208}$$

Where $\mathbf{r}'' = f(\mathbf{r}, \theta, \boldsymbol{\omega}, \mathbf{h}, \mathbf{b})$ describes the transformation of vector $\mathbf{r}$ in the following steps:

1. Rotate $\mathbf{r}$ such that vector $\mathbf{b}$ aligns with the rotation vector $\boldsymbol{\omega}$. Let this be $\mathbf{r}'$.
2. Rotate $\mathbf{r}'$ anticlockwise around vector $\boldsymbol{\omega}$ by θ, about the point $\mathbf{h}$

$$\begin{aligned}
\mathbf{r}'' &= f(\mathbf{r}, \theta, \boldsymbol{\omega}, \mathbf{h}, \mathbf{b}) \\
\boldsymbol{\Omega} &= \mathbf{b} + \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|} \\
\mathbf{r}' &= 2\mathbf{h} - \mathbf{r} - \frac{2\left(\mathbf{h}\cdot\boldsymbol{\Omega} - \mathbf{r}\cdot\boldsymbol{\Omega}\right)}{|\boldsymbol{\Omega}|^2}\boldsymbol{\Omega} \\
\mathbf{r}'' &= \mathbf{a}(1+\cos\theta) - \mathbf{r}'\cos\theta - \frac{\boldsymbol{\omega}\times(\mathbf{r}'-\mathbf{a})}{|\boldsymbol{\omega}|}\sin\theta + \frac{\left(\mathbf{r}'\cdot\boldsymbol{\omega} - \mathbf{a}\cdot\boldsymbol{\omega}\right)(1+\cos\theta)}{|\boldsymbol{\omega}|^2}\boldsymbol{\omega}
\end{aligned}$$

3 Mass cluster initial conditions

In a mass cluster, a set of essentially massless planets orbit a central star in a circular fashion. Inter-planet interactions are ignored. We will define a cluster in the following way

- Let the number of planets per AU^2 in each spherical shell be ρ
- Let the number of shells be N
- Let the radius of the first shell be R_0
- Let the shell separation be ΔR

The radius of the i^{th} shell is

$$R_i = R_0 + (i-1)\Delta R \tag{209}$$

The number of masses in shell i shall be n_i^2 where

$$n_i = \mathrm{floor}\left(\sqrt{4\pi R_i^2 \rho}\right) \tag{210}$$

The position and velocity of mass m_{ijk} , where $1 \leq i \leq N$ is the shell number and $1 \leq j \leq n_i$ and $1 \leq k \leq n_i$ index the masses in shell i are

$$x_{ij} = x_c + R_i \cos\left(-\frac{\pi}{2} + \frac{\pi\,(j-1)}{n_i}\right) \sin\left(\frac{2\pi\,(k-1)}{n_i}\right) \tag{211}$$

$$y_{ij} = y_c + R_i \cos\left(-\frac{\pi}{2} + \frac{\pi\,(j-1)}{n_i}\right) \cos\left(\frac{2\pi\,(k-1)}{n_i}\right) \tag{212}$$

$$z_{ij} = z_c + R_i \sin\left(-\frac{\pi}{2} + \frac{\pi\,(j-1)}{n_i}\right) \tag{213}$$

$$\dot{x}_{ij} = \dot{x}_c - \sqrt{\frac{GM}{R_i}} \sin\left(-\frac{\pi}{2} + \frac{\pi\,(j-1)}{n_i}\right) \sin\left(\frac{2\pi\,(k-1)}{n_i}\right) \tag{214}$$

$$\dot{y}_{ij} = \dot{y}_c - \sqrt{\frac{GM}{R_i}} \sin\left(-\frac{\pi}{2} + \frac{\pi\,(j-1)}{n_i}\right) \cos\left(\frac{2\pi\,(k-1)}{n_i}\right) \tag{215}$$

$$\dot{z}_{ij} = \dot{z}_c + \sqrt{\frac{GM}{R_i}} \cos\left(-\frac{\pi}{2} + \frac{\pi\,(j-1)}{n_i}\right) \tag{216}$$

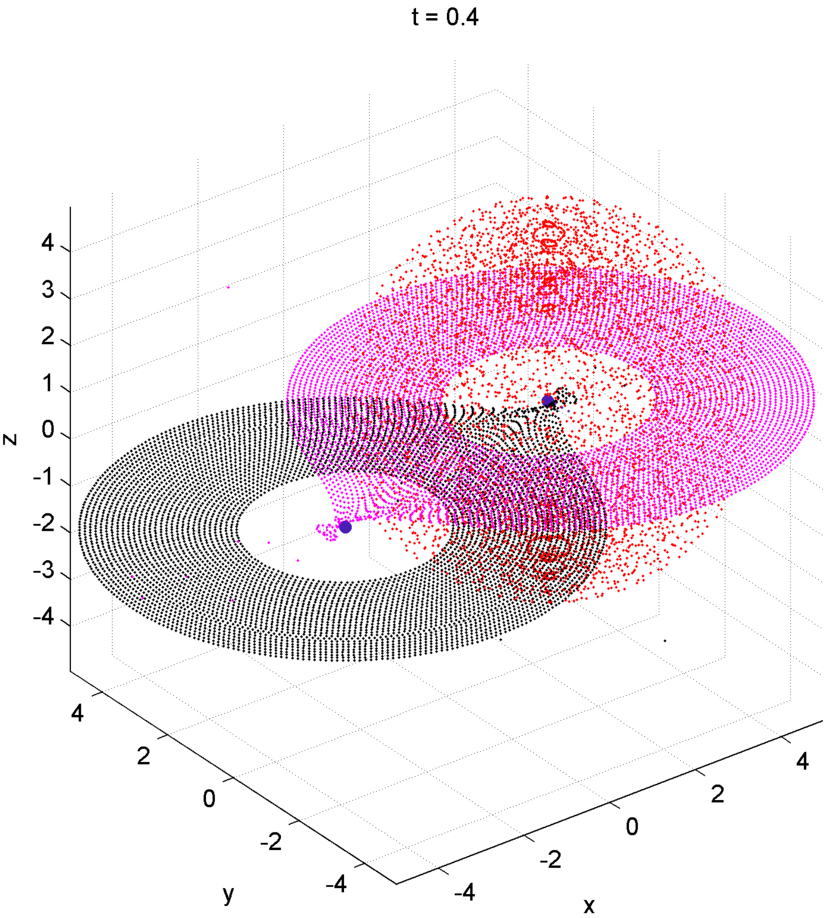


Figure 5: A spherical mass cluster of circular orbits is added to a binary star system with rings initial condition.

4 Example simulations

Typically simulations will be configured in terms of

- Pairs of stars in initially elliptical orbits
- Rings of planetlets initially centered on the stars
- Mass clusters initially centred on the stars

4.1 Two stars with initially circular rings

Stars 1 and 2

Quantity	Symbol	Value
Mass #1 $/M_\odot$	m_1	1
Mass #2 $/M_\odot$	m_2	1
Mass separation along semi-major axis of elliptical orbit	$2a$	4
Orbit eccentricity	ε	0
Initial position vector of centre of mass of system relative to simulation coordinate system $\left\{\hat{\mathbf{X}}, \hat{\mathbf{Y}}, \hat{\mathbf{Z}}\right\}$	$\mathbf{h}_0$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
Velocity of centre of mass / Au/Yr	$\dot{\mathbf{h}}$	$\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$
Initial orbital phase angle of elliptical orbit /radians	θ_0	0
The $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}\}$ orbital plane will be defined in terms of a 3D rotation of the $\hat{\mathbf{x}}$ axis to align with a defined direction vector	$\mathbf{d} = d_x\hat{\mathbf{X}} + d_y\hat{\mathbf{Y}} + d_{\mathbf{y}}\hat{\mathbf{Z}}$	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
Anticloclockwise rotation of ellipse aligned with $\mathbf{d}$, about $\mathbf{d}$ /radians	α	0

Each star shall have associated rings of planetlets. In addition to extra tangential velocity to result in mutually circular orbits, the entire ring shall also initally move with a bulk velocity equal to the initial velocity of stars 1 and 2 respectively.

Ring 1

Quantity	Symbol	Value
Number of rings	N_{rings}	30
Ring rotation vector	$\mathbf{d} = d_x\hat{\mathbf{x}} + d_y\hat{\mathbf{y}} + d_{\mathbf{y}}\hat{\mathbf{z}}$	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
Anticloclockwise rotation of ring aligned with $\mathbf{d}$, about $\mathbf{d}$ /radians	α	0
Radius of first ring /AU	R_0	1
Mass of star /solar masses	M_1	m_1
Inter-ring radial distance /AU	ΔR	0.1
Arc separation of ring masses /AU	δ	$\frac{1}{30}\pi$

Ring 2

Quantity	Symbol	Value
Number of rings	N_{rings}	30
Ring rotation vector	$\mathbf{d} = d_x\hat{\mathbf{x}} + d_y\hat{\mathbf{y}} + d_{\mathbf{y}}\hat{\mathbf{z}}$	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
Anticloclockwise rotation of ring aligned with $\mathbf{d}$, about $\mathbf{d}$ /radians	α	$\frac{1}{2}\pi$
Radius of first ring /AU	R_0	1
Mass of star /solar masses	M_2	m_2
Inter-ring radial distance /AU	ΔR	0.1
Arc separation of ring masses /AU	δ	$\frac{1}{30}\pi$

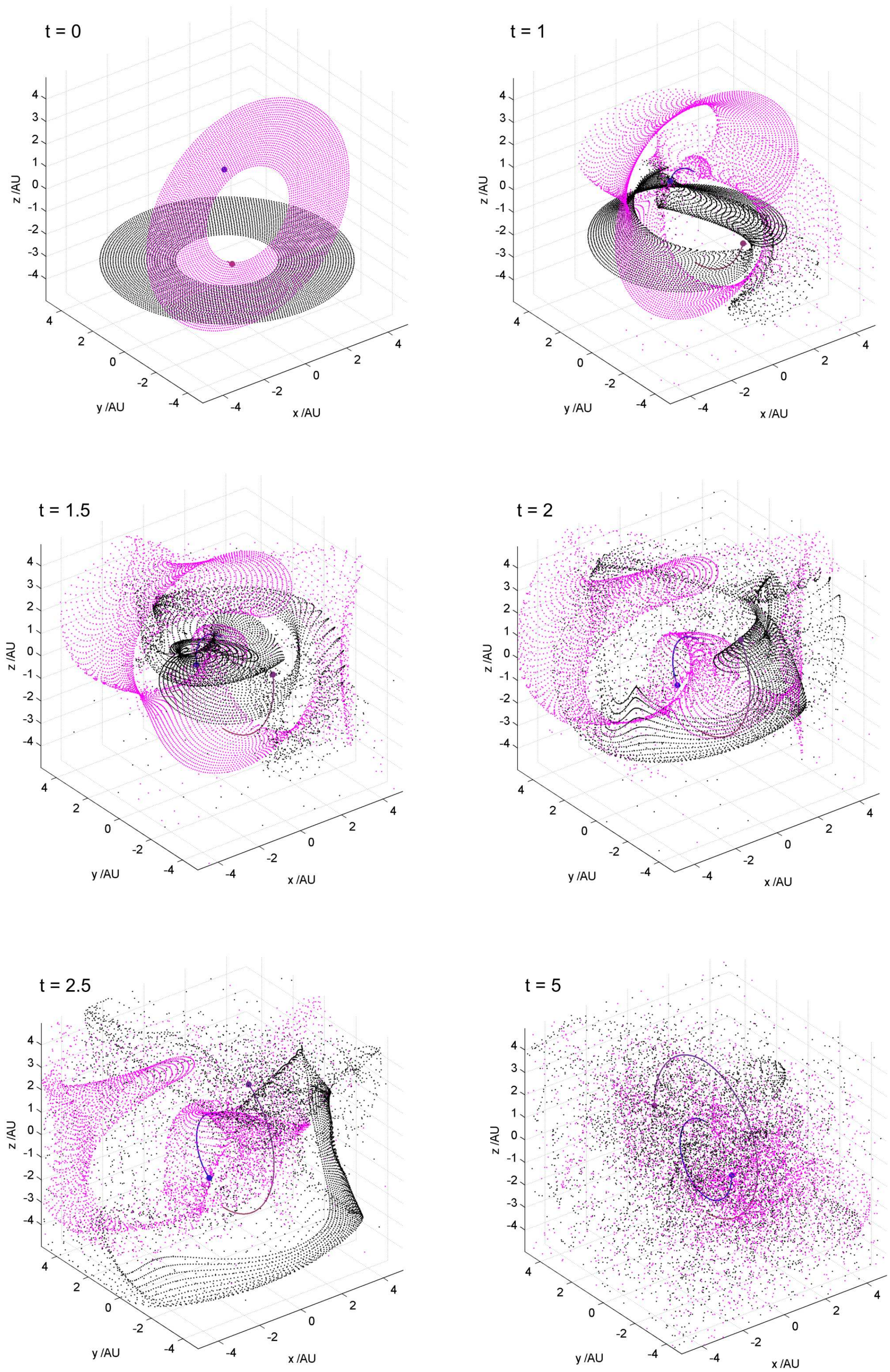


Figure 6: An initial condition of circular rings about two stars in the ratio 1:2 solar masses is distorted into complex cloud-like forms following gravitational attraction. In addition, the system is modelled to be in a box with partially elastic walls. The coefficient of restitution is $k = 0.4$.

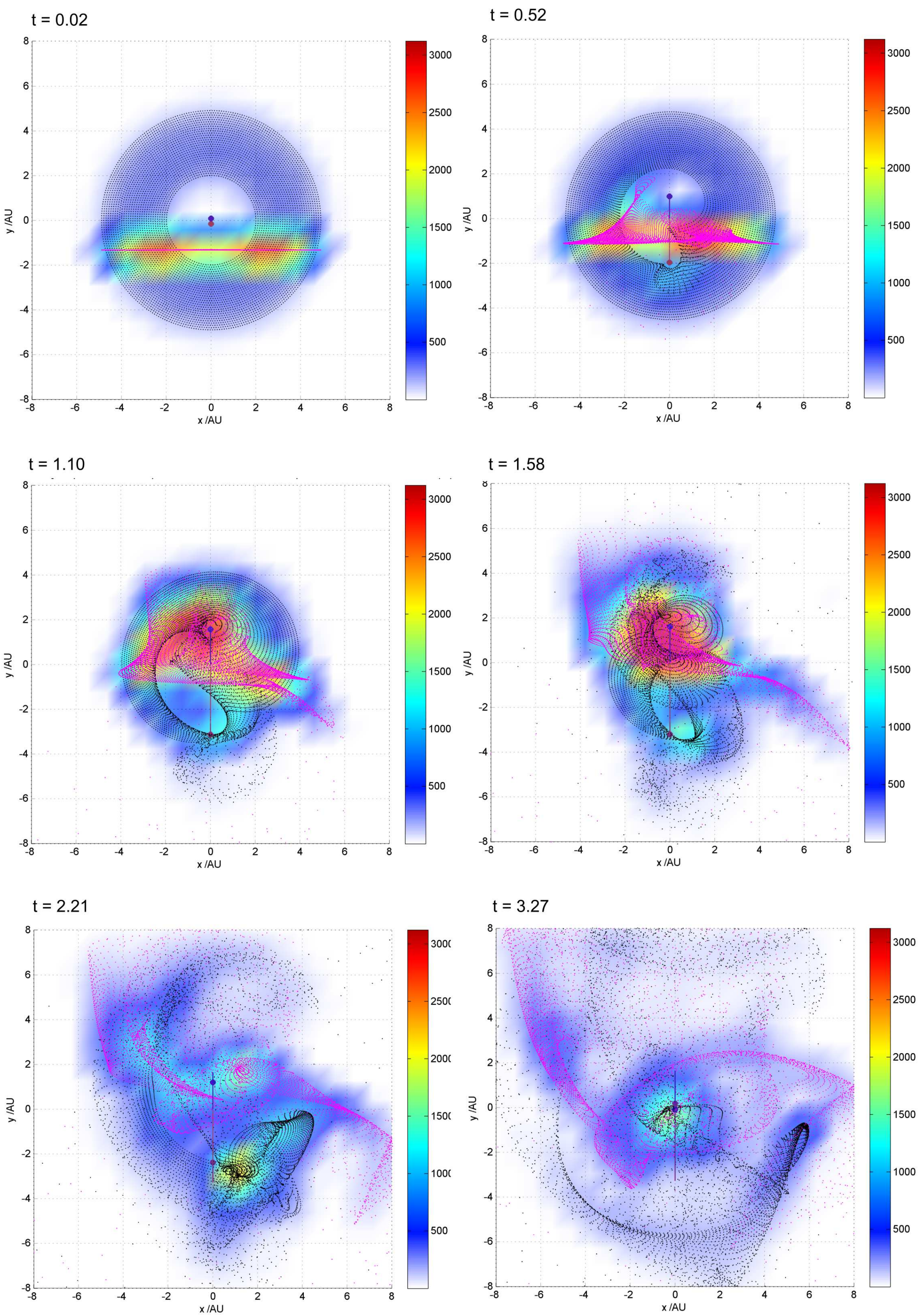


Figure 7: Spatially averaged speed map in the x, y plane corresponding the the binary stars and rings system. The orbital ellipse of the masses is viewed end on in this plane so the masses will apear to oscillate in the y direction.

4.2 Two stars with rings with rings in mutually circular orbits. Star orbit perpendicular to ring orbit

4.3 The solar system

The Solar System has the following parameters. (Woan, 2000 pp176). All orbits are assumed to be elliptical about the sun. Note

$$\frac{M_\odot}{M_\oplus} \approx 332{,}948 \tag{217}$$

and

$$R_\oplus \approx \frac{\text{AU}}{23,455} \tag{218}$$

In SI units:

$$M_\odot = 1.9891 \times 10^{30} \text{ kg} \tag{219}$$
$$R_\odot = 6.960 \times 10^{8} \text{ m} \tag{220}$$
$$M_\oplus = 5.9742 \times 10^{24} \text{ kg} \tag{221}$$
$$R_\oplus = 6.37814 \times 10^{6} \text{ m} \tag{222}$$
$$1\text{AU} = 1.495979 \times 10^{11} \text{ m} \tag{223}$$

Object	$M/M_\oplus$	a /AU	ε[4]	θ_0	β	α	$R/R_\oplus$	T_{rot} / days	P /Yr
Sun	332,837	-	-	-	-	-	109.123	-	-
Mercury	0.055	0.387	0.21	*	7.00	0	0.383	58.646	0.241
Venus†	0.815	0.723	0.01	*	3.39	0	0.949	243.018	0.615
Earth	1.000	1.000	0.02	*	0.00	0	1.000	0.997	1.000
Mars	0.107	1.523	0.09	*	1.85	0	0.533	1.026	1.881
Jupiter	317.85	5.202	0.05	*	1.31	0	11.209	0.413	11.861
Saturn	95.159	9.576	0.06	*	2.49	0	9.449	0.444	29.628
Uranus†	14.500	19.293	0.05	*	0.77	0	4.007	0.718	84.747
Neptune	17.204	30.246	0.01	*	1.77	0	3.883	0.671	166.344
Pluto†	0.003	39.509	0.25	*	17.5	0	0.187	6.387	248.348

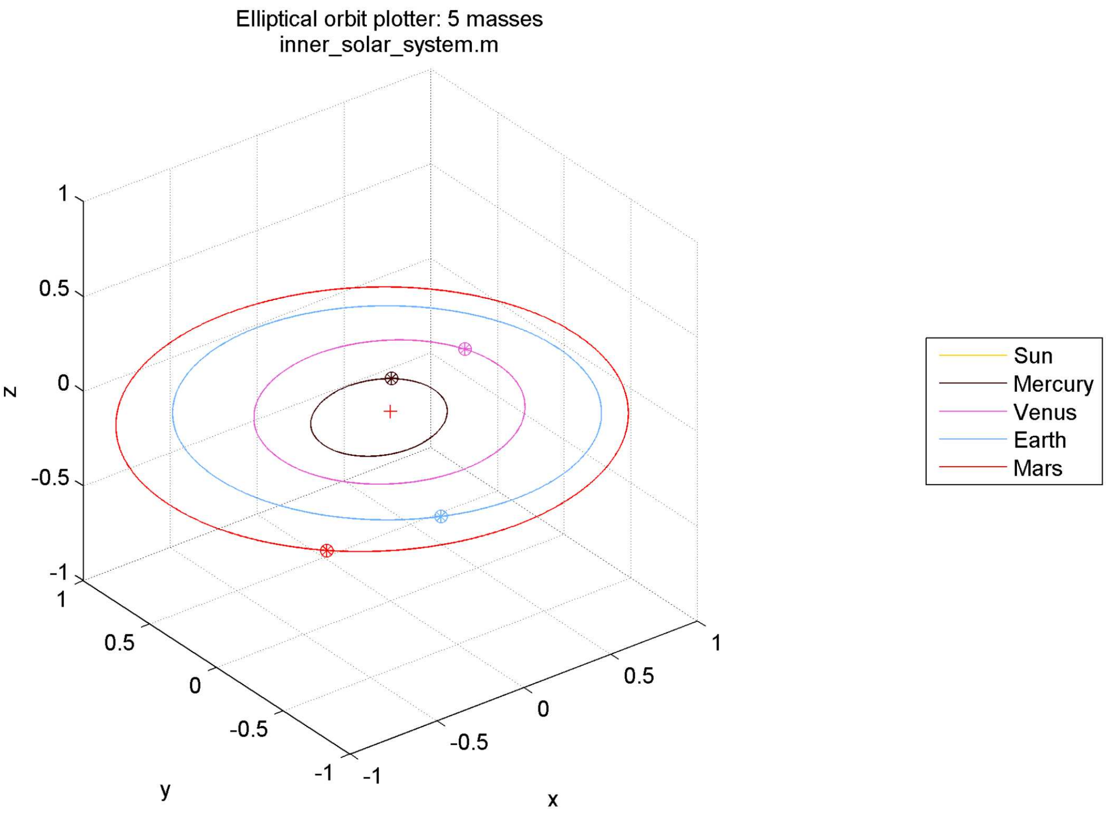
where β is the orbital inclination /degrees. In all cases the semi-major axis pointing direction is

$$\mathbf{d} = d_x\hat{\mathbf{x}} + d_y\hat{\mathbf{y}} + d_{\mathbf{y}}\hat{\mathbf{z}} = \cos\beta\hat{\mathbf{x}} + \sin\beta\hat{\mathbf{z}} \tag{224}$$

* For the current orbital polar angle θ_0 (and indeed more accurate values for solar system parameters) see the website of the Jet Propulsion Laboratory (JPL) `http://ssd.jpl.nasa.gov/`

†These planets rotate clockwise about their own internal polar axis. ("Retrograde"). All the other planets rotate anti-clockwise about their own internal axis. All the planets orbit the sun in an anticlockwise direction.

[4] http://nineplanets.org/data.html



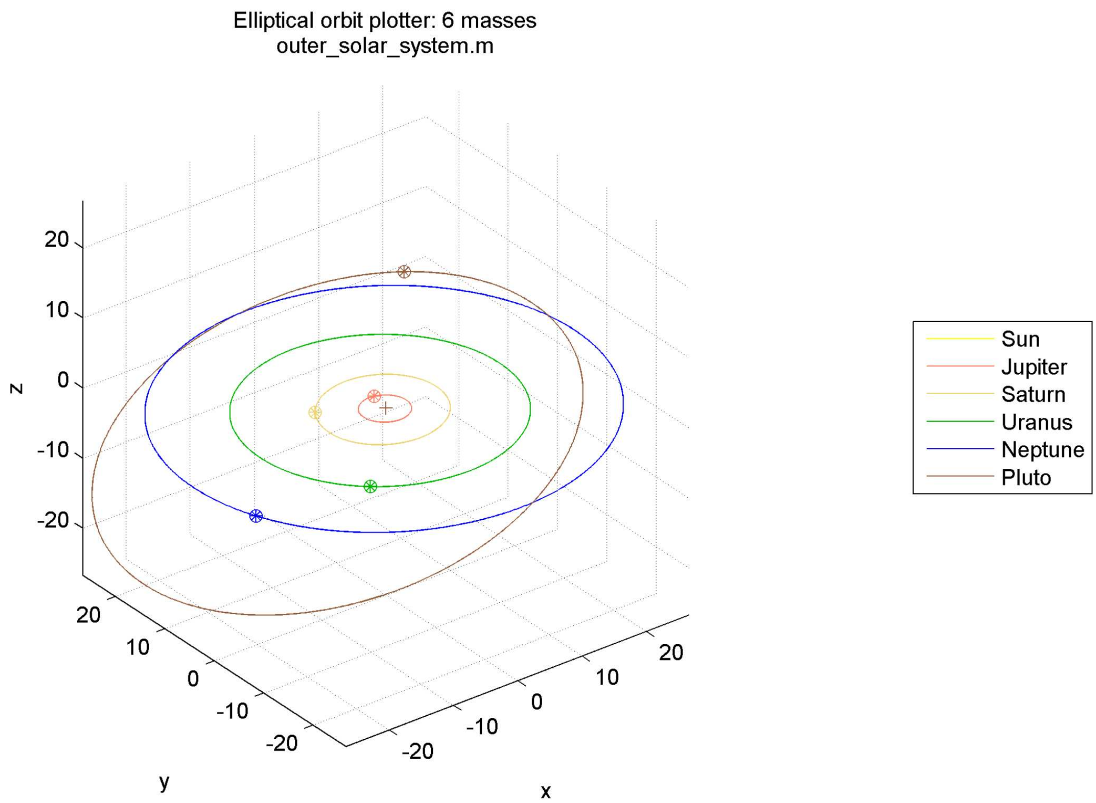


Figure 8: Model of the solar system assuming an elliptical orbit about the sun for each of the planets. Pluto is included for interest! In these plots the orbits are worked out exactly assuming no inter-plant interaction. The initial polar angle of the orbital ellipses have been chosen at random for each planet.

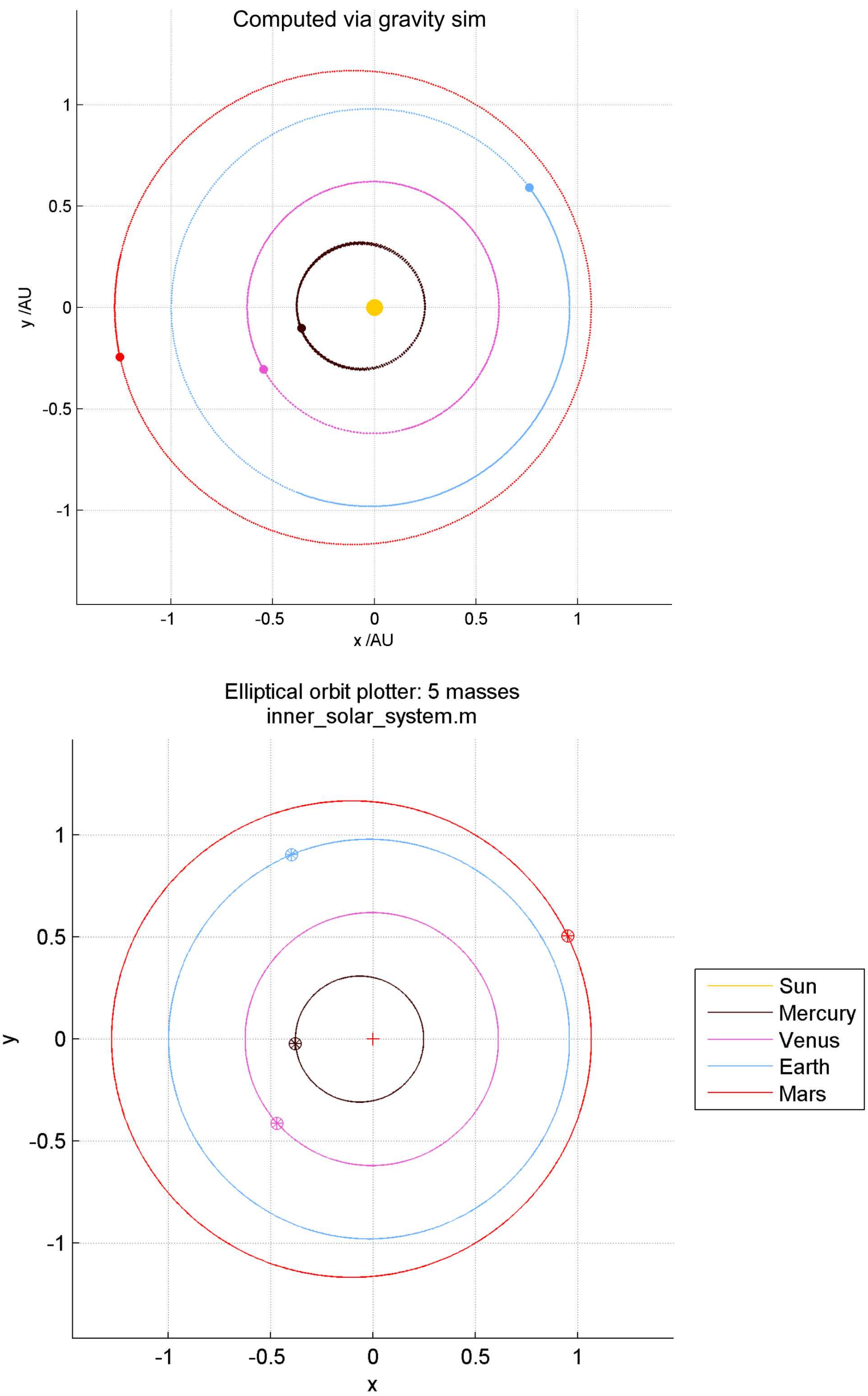


Figure 9: The exact elliptical orbits of the inner four planets (computed assuming no inter-planet interaction) is compared with the corresponding `gravity_sim` model using a time step of $\Delta t = 0.01$ years. In the latter case, inter-planet attraction is modelled. On this timescale, deviations caused by the finite timestep of the Verlet algorithm are likely to be more significant (especially for Mercury where the acceleration is most signifcant) than real gravitational perturbations.

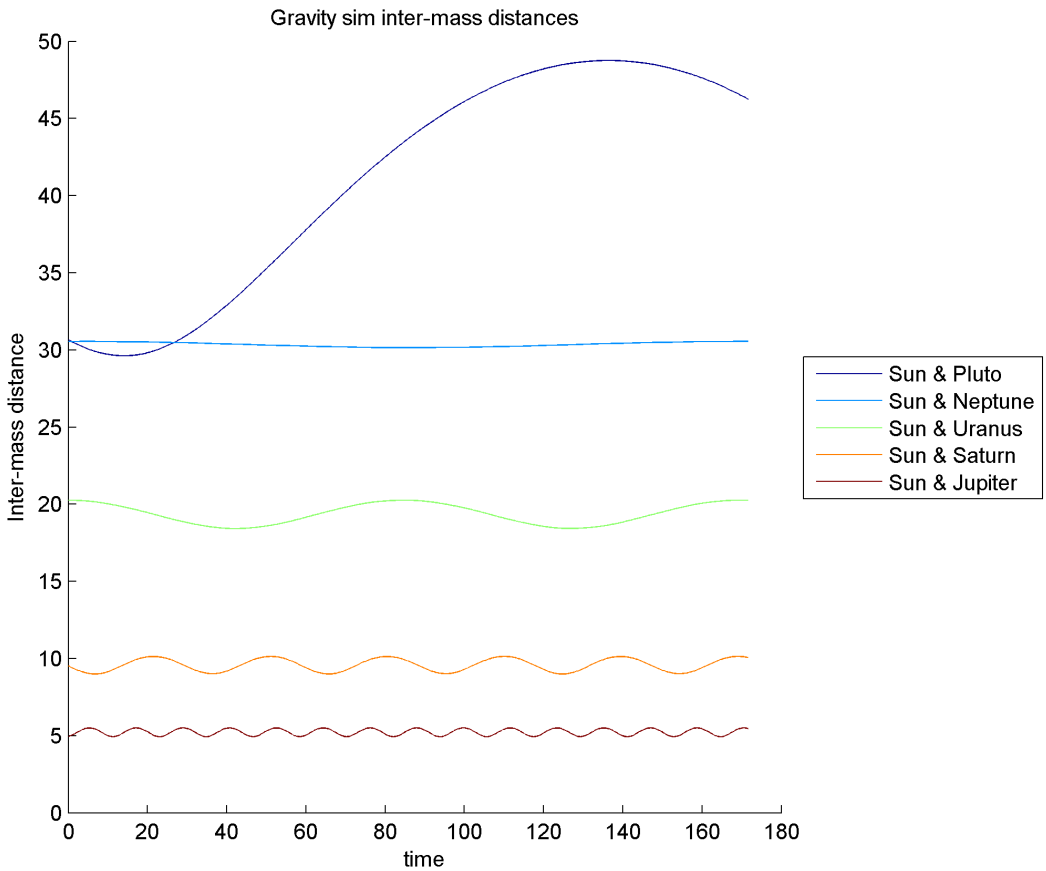


Figure 10: Inter-mass distances /AU for the outer planets of the Solar System vs time /years.

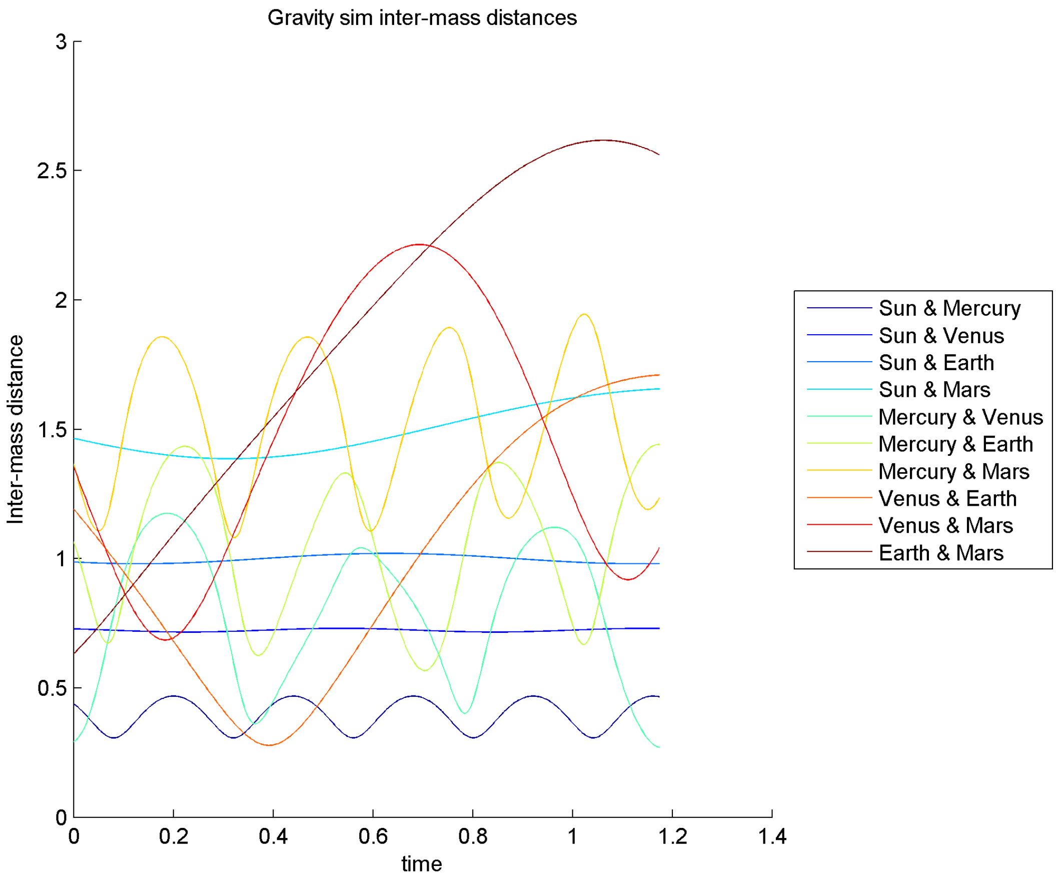


Figure 11: Inter-mass distances /AU for the inner planets of the solar system vs time /years.

5 References

1. Woan, G., *The Cambridge Handbook of Physics Formulas.* Cambridge University Press. 2001.
2. Goldstein,H., Poole, C., Safko, J., *Classical Mechanics.* Pearson Education Inc /Addison Wesley. Third Edition. 2002.
3. Morin, D., *Introduction to Classical Mechanics.* Cambridge University Press. 2007
4. NASA Jet Propulsion Laboratory (JPL) Solar System Simulator. `http://space.jpl.nasa.gov/`

6 Appendix 1: Derivatives of position vectors in Cartesian and spherical polar coordinates

6.1 Correspondance between Cartesian and spherical polar coordinates

Define a position vector $\mathbf{r}$ in both Cartesian (x, y, z) and spherical polar coordinates (r, θ, ϕ). θ is measured anti-clockwise from the x Cartesian axis and ϕ is measured from the z axis towards the x, y plane

$$\begin{aligned} \mathbf{r} &= x\hat{\mathbf{x}}+y\hat{\mathbf{y}}+z\hat{\mathbf{z}} \\ \mathbf{r} &= r\hat{\mathbf{r}} \end{aligned} \tag{225}$$

Cartesian coordinates use the right handed set of unit vectors $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ that are defined to have the following properties

$$\begin{aligned} \hat{\mathbf{x}} \cdot \hat{\mathbf{x}} &= \hat{\mathbf{y}} \cdot \hat{\mathbf{y}} = \hat{\mathbf{z}} \cdot \hat{\mathbf{z}} = 1 \\ \hat{\mathbf{x}} \cdot \hat{\mathbf{y}} &= \hat{\mathbf{x}} \cdot \hat{\mathbf{z}} = \hat{\mathbf{y}} \cdot \hat{\mathbf{z}} = 0 \end{aligned} \tag{226}$$

$$\begin{aligned} \hat{\mathbf{x}} \times \hat{\mathbf{y}} &= \hat{\mathbf{z}} \\ \hat{\mathbf{z}} \times \hat{\mathbf{x}} &= \hat{\mathbf{y}} \\ \hat{\mathbf{y}} \times \hat{\mathbf{z}} &= \hat{\mathbf{x}} \end{aligned} \tag{227}$$

$$\begin{aligned} \frac{d\hat{\mathbf{x}}}{dt} &= 0 \\ \frac{d\hat{\mathbf{y}}}{dt} &= 0 \\ \frac{d\hat{\mathbf{z}}}{dt} &= 0 \end{aligned} \tag{228}$$

The basis vectors of the spherical polars coordinate system are also orthonormal, but are not fixed in position. i.e. they vary with coordinates r, θ, ϕ,

$$\hat{\mathbf{r}} \cdot \hat{\mathbf{r}} = \hat{\boldsymbol{\theta}} \cdot \hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\phi}} = 1 \tag{229}$$

$$\hat{\mathbf{r}} \cdot \hat{\boldsymbol{\theta}} = \hat{\mathbf{r}} \cdot \boldsymbol{\phi} = \hat{\boldsymbol{\phi}} \cdot \hat{\boldsymbol{\theta}} = 0 \tag{230}$$

The conversion between Cartesian and spherical polars is as follows

$$\begin{aligned} x &= r\cos\theta\sin\phi \\ y &= r\sin\theta\sin\phi \\ z &= r\cos\phi \end{aligned} \tag{231}$$

$$\begin{aligned} r &= \sqrt{x^2+y^2+z^2} \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right) \\ \phi &= \cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right) \end{aligned} \tag{232}$$

Hence

$$\mathbf{r} = r\cos\theta\sin\phi\hat{\mathbf{x}}+r\sin\theta\sin\phi\hat{\mathbf{y}}+r\cos\phi\hat{\mathbf{z}} \tag{233}$$

6.2 Unit vectors of the spherical polars coordinate system

The spherical polars unit vectors are defined in the following general way[5]. The following derivations make regular use of the invariant property of the Cartesian basis vectors defined above i.e. $\hat{\mathbf{x}}\cdot\hat{\mathbf{y}}=\hat{\mathbf{x}}\cdot\hat{\mathbf{z}}=\hat{\mathbf{y}}\cdot\hat{\mathbf{z}}=0$ and $\frac{d\hat{\mathbf{x}}}{dt}=0$, $\frac{d\hat{\mathbf{y}}}{dt}=0$, $\frac{d\hat{\mathbf{z}}}{dt}=0$.

$$\hat{\mathbf{r}}=\frac{\frac{\partial\mathbf{r}}{\partial r}}{\left|\frac{\partial\mathbf{r}}{\partial r}\right|},\quad \hat{\boldsymbol{\theta}}=\frac{\frac{\partial\mathbf{r}}{\partial\theta}}{\left|\frac{\partial\mathbf{r}}{\partial\theta}\right|},\quad \hat{\boldsymbol{\phi}}=\frac{\frac{\partial\mathbf{r}}{\partial\phi}}{\left|\frac{\partial\mathbf{r}}{\partial\phi}\right|}$$

$$\begin{aligned}
\mathbf{r} &= r\cos\theta\sin\phi\hat{\mathbf{x}}+r\sin\theta\sin\phi\hat{\mathbf{y}}+r\cos\phi\hat{\mathbf{z}} \\
\frac{\partial\mathbf{r}}{\partial r} &= \cos\theta\sin\phi\hat{\mathbf{x}}+\sin\theta\sin\phi\hat{\mathbf{y}}+\cos\phi\hat{\mathbf{z}} \\
\left|\frac{\partial\mathbf{r}}{\partial r}\right|^2 &= \frac{\partial\mathbf{r}}{\partial r}\cdot\frac{\partial\mathbf{r}}{\partial r}=(\cos\theta\sin\phi\hat{\mathbf{x}}+\sin\theta\sin\phi\hat{\mathbf{y}}+\cos\phi\hat{\mathbf{z}})\cdot(\cos\theta\sin\phi\hat{\mathbf{x}}+\sin\theta\sin\phi\hat{\mathbf{y}}+\cos\phi\hat{\mathbf{z}}) \\
&= \cos^2\theta\sin^2\phi+\sin^2\theta\sin^2\phi+\cos^2\phi \\
&= \left(\sin^2\theta+\cos^2\theta\right)\sin^2\phi+\cos^2\phi \\
&= 1 \\
&\Rightarrow \hat{\mathbf{r}}=\cos\theta\sin\phi\hat{\mathbf{x}}+\sin\theta\sin\phi\hat{\mathbf{y}}+\cos\phi\hat{\mathbf{z}}
\end{aligned} \tag{234}$$

$$\begin{aligned}
\mathbf{r} &= r\cos\theta\sin\phi\hat{\mathbf{x}}+r\sin\theta\sin\phi\hat{\mathbf{y}}+r\cos\phi\hat{\mathbf{z}} \\
\frac{\partial\mathbf{r}}{\partial\theta} &= -r\sin\theta\sin\phi\hat{\mathbf{x}}+r\cos\theta\sin\phi\hat{\mathbf{y}} \\
\left|\frac{\partial\mathbf{r}}{\partial\theta}\right|^2 &= \frac{\partial\mathbf{r}}{\partial\theta}\cdot\frac{\partial\mathbf{r}}{\partial\theta}=(-r\sin\theta\sin\phi\hat{\mathbf{x}}+r\cos\theta\sin\phi\hat{\mathbf{y}})\cdot(-r\sin\theta\sin\phi\hat{\mathbf{x}}+r\cos\theta\sin\phi\hat{\mathbf{y}}) \\
&= r^2\sin^2\theta\sin^2\phi+r^2\cos^2\theta\sin^2\phi \\
&= r^2\sin^2\phi\left(\sin^2\theta+\cos^2\theta\right) \\
&= r^2\sin^2\phi \\
&\therefore\quad \hat{\boldsymbol{\theta}}=\frac{-r\sin\theta\sin\phi\hat{\mathbf{x}}+r\cos\theta\sin\phi\hat{\mathbf{y}}}{r\sin\phi} \\
&\Rightarrow\quad \hat{\boldsymbol{\theta}}=-\sin\theta\hat{\mathbf{x}}+\cos\theta\hat{\mathbf{y}}
\end{aligned} \tag{235}$$

$$\begin{aligned}
\mathbf{r} &= r\cos\theta\sin\phi\hat{\mathbf{x}}+r\sin\theta\sin\phi\hat{\mathbf{y}}+r\cos\phi\hat{\mathbf{z}} \\
\frac{\partial\mathbf{r}}{\partial\phi} &= r\cos\theta\cos\phi\hat{\mathbf{x}}+r\sin\theta\cos\phi\hat{\mathbf{y}}-r\sin\phi\hat{\mathbf{z}} \\
\left|\frac{\partial\mathbf{r}}{\partial\phi}\right|^2 &= \frac{\partial\mathbf{r}}{\partial\phi}\cdot\frac{\partial\mathbf{r}}{\partial\phi}=(r\cos\theta\cos\phi\hat{\mathbf{x}}+r\sin\theta\cos\phi\hat{\mathbf{y}}-r\sin\phi\hat{\mathbf{z}})\cdot(r\cos\theta\cos\phi\hat{\mathbf{x}}+r\sin\theta\cos\phi\hat{\mathbf{y}}-r\sin\phi\hat{\mathbf{z}}) \\
&= r^2\cos^2\theta\cos^2\phi+r^2\sin^2\theta\cos^2\phi+r^2\sin^2\phi \\
&= r^2\left(\sin^2\theta+\cos^2\theta\right)\cos^2\phi+r^2\sin^2\phi \\
&= r^2 \\
&\Rightarrow \hat{\boldsymbol{\phi}}=\cos\theta\cos\phi\hat{\mathbf{x}}+\sin\theta\cos\phi\hat{\mathbf{y}}-\sin\phi\hat{\mathbf{z}}
\end{aligned} \tag{236}$$

In summary

$$\begin{aligned}
\hat{\mathbf{r}} &= \cos\theta\sin\phi\hat{\mathbf{x}}+\sin\theta\sin\phi\hat{\mathbf{y}}+\cos\phi\hat{\mathbf{z}} \\
\hat{\boldsymbol{\theta}} &= -\sin\theta\hat{\mathbf{x}}+\cos\theta\hat{\mathbf{y}} \\
\hat{\boldsymbol{\phi}} &= \cos\theta\cos\phi\hat{\mathbf{x}}+\sin\theta\cos\phi\hat{\mathbf{y}}-\sin\phi\hat{\mathbf{z}}
\end{aligned} \tag{237}$$

[5] i.e. this recipe works for any coordinate system. Start with cartesians, then express x, y, z in terms of the new coordinates, then find the new unit vectors by working out the partial derivative or the position vector $\mathbf{r}$ in terms of the new coordinates.

6.3 Time derivatives of the spherical polars unit vectors

$$\begin{aligned}\hat{\mathbf{r}} &= \cos\theta\sin\phi\hat{\mathbf{x}}+\sin\theta\sin\phi\hat{\mathbf{y}}+\cos\phi\hat{\mathbf{z}} \\ \hat{\boldsymbol{\theta}} &= -\sin\theta\hat{\mathbf{x}}+\cos\theta\hat{\mathbf{y}} \\ \hat{\boldsymbol{\phi}} &= \cos\theta\cos\phi\hat{\mathbf{x}}+\sin\theta\cos\phi\hat{\mathbf{y}}-\sin\phi\hat{\mathbf{z}}\end{aligned} \tag{238}$$

Therefore

$$\begin{aligned}\frac{d\hat{\mathbf{r}}}{dt} &= \left(-\dot{\theta}\sin\theta\sin\phi+\dot{\phi}\cos\theta\cos\phi\right)\hat{\mathbf{x}}+\left(\dot{\theta}\cos\theta\sin\phi+\dot{\phi}\sin\theta\cos\phi\right)\hat{\mathbf{y}}-\dot{\phi}\sin\phi\hat{\mathbf{z}} \\ &= \dot{\theta}\sin\phi\left(-\sin\theta\hat{\mathbf{x}}+\cos\theta\hat{\mathbf{y}}\right)+\dot{\phi}\left(\cos\theta\cos\phi\hat{\mathbf{x}}+\sin\theta\cos\phi\hat{\mathbf{y}}-\sin\phi\hat{\mathbf{z}}\right) \\ &= \dot{\theta}\sin\phi\hat{\boldsymbol{\theta}}+\dot{\phi}\hat{\boldsymbol{\phi}}\end{aligned} \tag{239}$$

$$\begin{aligned}\frac{d\hat{\boldsymbol{\theta}}}{dt} &= -\dot{\theta}\cos\theta\hat{\mathbf{x}}-\dot{\theta}\sin\theta\hat{\mathbf{y}} \\ &= -\dot{\theta}\left(\cos\theta\hat{\mathbf{x}}+\sin\theta\hat{\mathbf{y}}\right) \\ &= -\dot{\theta}\left(\sin\phi\cos\theta\sin\phi\hat{\mathbf{x}}+\sin\phi\sin\theta\sin\phi\hat{\mathbf{y}}+\sin\phi\cos\phi\hat{\mathbf{z}}+\cos\phi\cos\theta\cos\phi\hat{\mathbf{x}}+\cos\phi\sin\theta\cos\phi\hat{\mathbf{y}}-\cos\phi\sin\phi\hat{\mathbf{z}}\right) \\ &= -\dot{\theta}\left(\sin\phi\hat{\mathbf{r}}+\cos\phi\hat{\boldsymbol{\phi}}\right)\end{aligned} \tag{240}$$

$$\begin{aligned}\frac{d\hat{\boldsymbol{\phi}}}{dt} &= \left(-\dot{\theta}\sin\theta\cos\phi-\dot{\phi}\cos\theta\sin\phi\right)\hat{\mathbf{x}}+\left(\dot{\theta}\cos\theta\cos\phi-\dot{\phi}\sin\theta\sin\phi\right)\hat{\mathbf{y}}-\dot{\phi}\cos\phi\hat{\mathbf{z}} \\ &= -\dot{\phi}\left(\cos\theta\sin\phi\hat{\mathbf{x}}+\sin\theta\sin\phi\hat{\mathbf{y}}+\cos\phi\hat{\mathbf{z}}\right)+\dot{\theta}\cos\phi\left(-\sin\theta\hat{\mathbf{x}}+\cos\theta\hat{\mathbf{y}}\right) \\ &= -\dot{\phi}\hat{\mathbf{r}}+\dot{\theta}\cos\phi\hat{\boldsymbol{\theta}}\end{aligned} \tag{241}$$

In summary, the first time derivatives of the spherical polars unit vectors are (in terms of spherical polars unit vectors, not Cartesian unit vectors)

$$\begin{aligned}\frac{d\hat{\mathbf{r}}}{dt} &= \dot{\theta}\sin\phi\hat{\boldsymbol{\theta}}+\dot{\phi}\hat{\boldsymbol{\phi}} \\ \frac{d\hat{\boldsymbol{\theta}}}{dt} &= -\dot{\theta}\left(\sin\phi\hat{\mathbf{r}}+\cos\phi\hat{\boldsymbol{\phi}}\right) \\ \frac{d\hat{\boldsymbol{\phi}}}{dt} &= -\dot{\phi}\hat{\mathbf{r}}+\dot{\theta}\cos\phi\hat{\boldsymbol{\theta}}\end{aligned} \tag{242}$$

6.4 Velocity and acceleration in spherical polars

We can use the results above to write expressions for velocity $\mathbf{v} \equiv \dot{\mathbf{r}} \equiv \frac{d\mathbf{r}}{dt}$ and acceleration $\mathbf{a} \equiv \ddot{\mathbf{r}} \equiv \frac{d\mathbf{v}}{dt}$ in terms of the spherical polars coordinates and basis vectors.

$$\begin{aligned}
\mathbf{r} &= r\cos\theta\sin\phi\hat{\mathbf{x}} + r\sin\theta\sin\phi\hat{\mathbf{y}} + r\cos\phi\hat{\mathbf{z}} \\
\dot{\mathbf{r}} &= \frac{d\mathbf{r}}{dt} \\
&= \left(\dot{r}\cos\theta\sin\phi - r\dot{\theta}\sin\theta\sin\phi + r\dot{\phi}\cos\theta\cos\phi\right)\hat{\mathbf{x}} + ... \\
&\quad \left(\dot{r}\sin\theta\sin\phi + r\dot{\theta}\cos\theta\sin\phi + r\dot{\phi}\sin\theta\cos\phi\right)\hat{\mathbf{y}} + ... \\
&\quad \left(\dot{r}\cos\phi - r\dot{\phi}\sin\phi\right)\hat{\mathbf{z}} \\
&= \dot{r}\left(\cos\theta\sin\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\phi\hat{\mathbf{z}}\right) + ... \\
&\quad r\dot{\theta}\sin\phi\left(-\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}\right) + ... \\
&\quad r\dot{\phi}\left(\cos\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\cos\phi\hat{\mathbf{y}} - \sin\phi\hat{\mathbf{z}}\right) \\
&= \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\sin\phi\hat{\boldsymbol{\theta}} + r\dot{\phi}\hat{\boldsymbol{\phi}}
\end{aligned} \tag{243}$$

Since

$$\begin{aligned}
\hat{\mathbf{r}} &= \cos\theta\sin\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\phi\hat{\mathbf{z}} \\
\hat{\boldsymbol{\theta}} &= -\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}} \\
\hat{\boldsymbol{\phi}} &= \cos\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\cos\phi\hat{\mathbf{y}} - \sin\phi\hat{\mathbf{z}}
\end{aligned} \tag{244}$$

and

$$\begin{aligned}
\frac{d\hat{\mathbf{r}}}{dt} &= \dot{\theta}\sin\phi\hat{\boldsymbol{\theta}} + \dot{\phi}\hat{\boldsymbol{\phi}} \\
\frac{d\hat{\boldsymbol{\theta}}}{dt} &= -\dot{\theta}\left(\sin\phi\hat{\mathbf{r}} + \cos\phi\hat{\boldsymbol{\phi}}\right) \\
\frac{d\hat{\boldsymbol{\phi}}}{dt} &= -\dot{\phi}\hat{\mathbf{r}} + \dot{\theta}\cos\phi\hat{\boldsymbol{\theta}}
\end{aligned} \tag{245}$$

Therefore

$$\begin{aligned}
\ddot{\mathbf{r}} &= \frac{d\dot{\mathbf{r}}}{dt} \\
&= \frac{d}{dt}\left(\dot{r}\hat{\mathbf{r}} + r\dot{\theta}\sin\phi\hat{\boldsymbol{\theta}} + r\dot{\phi}\hat{\boldsymbol{\phi}}\right) \\
&= \ddot{r}\hat{\mathbf{r}} + \dot{\mathbf{r}}\frac{d\hat{\mathbf{r}}}{dt} + \frac{d}{dt}\left(r\dot{\theta}\sin\phi\hat{\boldsymbol{\theta}}\right) + \frac{d}{dt}\left(r\dot{\phi}\hat{\boldsymbol{\phi}}\right) \\
&\quad \text{Algebra needed!}
\end{aligned} \tag{246}$$

$$\begin{aligned}
\ddot{\mathbf{r}} &= \left(\ddot{r} - r\dot{\phi}^2 - r\dot{\theta}^2\sin^2\phi\right)\hat{\mathbf{r}} + ... \\
&\quad \left(2\dot{r}\dot{\theta}\sin\phi + 2r\dot{\theta}\dot{\phi}\cos\phi + r\ddot{\theta}\sin\phi\right)\hat{\boldsymbol{\theta}} + ... \\
&\quad \left(2\dot{r}\dot{\phi} + r\ddot{\phi} - r\dot{\theta}^2\sin\phi\cos\phi\right)\hat{\boldsymbol{\phi}}
\end{aligned}$$

http://mathworld.wolfram.com/SphericalCoordinates.html

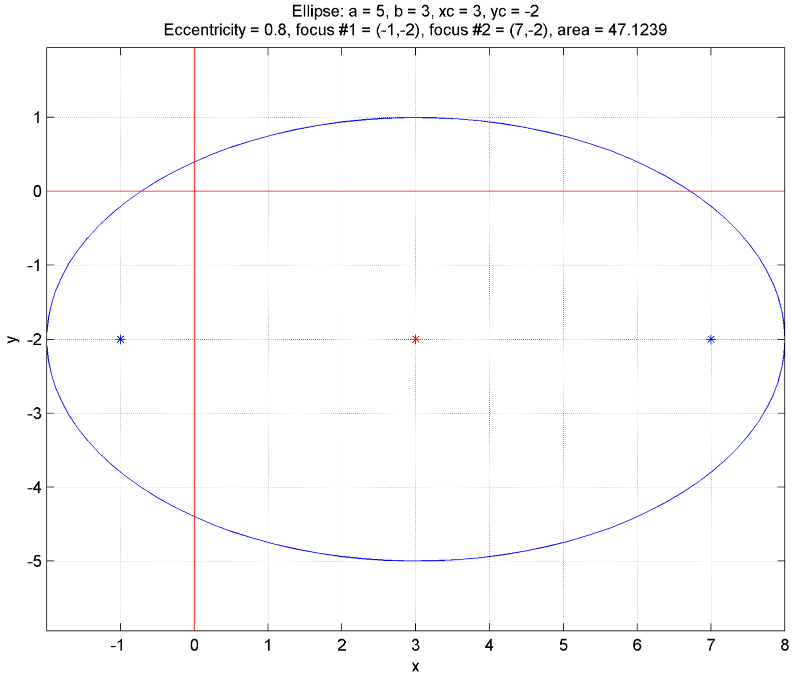


Figure 12: The ellipse.

7 Appendix 2: The Ellipse

7.1 Cartesian equation of the ellipse

An ellipse is defined in Caretsian x, y coordinates by the following equation

$$\frac{(x - x_c)^2}{a^2} + \frac{(y - y_c)^2}{b^2} = 1 \tag{247}$$

(x_c, y_c) is the centre of the ellipse, which cuts the x axis at $(x_c \pm a, 0)$ and the y axis at $(0, y_c \pm b)$

7.2 Parametric equation of the ellipse

The ellipse can be defined in terms of a single parameter ϕ via the following equations, which yield the Caretsian coordinates x, y

$$\begin{aligned} x &= x_c + a\cos\phi \\ y &= y_c + b\sin\phi \end{aligned} \tag{248}$$

Hence

$$\frac{(x - x_c)^2}{a^2} + \frac{(y - y_c)^2}{b^2} = \frac{(a\cos\phi)^2}{a^2} + \frac{(b\sin\phi)^2}{b^2} = \cos^2\phi + \sin^2\phi = 1 \tag{249}$$

so the parametric form is indeed consistent with the Cartesian form.

7.3 Polar equation of the ellipse

Define the *eccentricity* ε

$$\varepsilon = \sqrt{1 - \frac{b^2}{a^2}} \tag{250}$$

Hence

$$\begin{aligned} \varepsilon^2 a^2 &= a^2 - b^2 \\ \therefore\ & b^2 = a^2 - a^2\varepsilon^2 \end{aligned} \tag{251}$$

The ellipse can also be described in (shifted) polar coordinates (r, θ) where θ is measured anticlockwise from the x axis. The following conversions between Cartesian and polar coordinates will prove to be sensible.

$$\begin{aligned} x &= x_c + a\varepsilon + r\cos\theta \\ y &= y_c + r\sin\theta \end{aligned} \tag{252}$$

$$\begin{aligned} r &= \sqrt{(x - x_c - a\varepsilon)^2 + (y - y_c)^2} \\ \theta &= \tan^{-1}\left(\frac{y - y_c}{x - x_c - a\varepsilon}\right) \end{aligned} \tag{253}$$

$$\begin{aligned} \frac{(x - x_c)^2}{a^2} + \frac{(y - y_c)^2}{b^2} &= 1 \\ \frac{(a\varepsilon + r\cos\theta)^2}{a^2} + \frac{r^2\sin^2\theta}{b^2} &= 1 \\ \frac{a^2\varepsilon^2 + 2ra\varepsilon\cos\theta + r^2\cos^2\theta}{a^2} + \frac{r^2\left(1 - \cos^2\theta\right)}{b^2} &= 1 \\ \frac{b^2a^2\varepsilon^2 + 2b^2ra\varepsilon\cos\theta + b^2r^2\cos^2\theta}{b^2a^2} + \frac{a^2r^2 - a^2r^2\cos^2\theta}{a^2b^2} &= 1 \\ b^2a^2\varepsilon^2 + 2b^2ra\varepsilon\cos\theta + b^2r^2\cos^2\theta + a^2r^2 - a^2r^2\cos^2\theta &= b^2a^2 \\ b^2\left(a^2\varepsilon^2 + 2ra\varepsilon\cos\theta + r^2\cos^2\theta - a^2\right) + a^2r^2 - a^2r^2\cos^2\theta &= 0 \end{aligned} \tag{254}$$

Now substituting for $b^2 = a^2 - a^2\varepsilon^2$

$$\begin{aligned} \left(a^2 - a^2\varepsilon^2\right)\left(a^2\varepsilon^2 + 2ra\varepsilon\cos\theta + r^2\cos^2\theta - a^2\right) + a^2r^2 - a^2r^2\cos^2\theta &= 0 \\ \left(\varepsilon^2 - 1\right)\left(a^2\varepsilon^2 + 2ra\varepsilon\cos\theta + r^2\cos^2\theta - a^2\right) - r^2 + r^2\cos^2\theta &= 0 \\ r^2\left(\cos^2\theta - 1 + \left(\varepsilon^2 - 1\right)\cos^2\theta\right) + 2ra\varepsilon\cos\theta\left(\varepsilon^2 - 1\right) + \left(\varepsilon^2 - 1\right)\left(a^2\varepsilon^2 - a^2\right) &= 0 \\ r^2\left(\varepsilon^2\cos^2\theta - 1\right) + 2ra\varepsilon\cos\theta\left(\varepsilon^2 - 1\right) + a^2\left(\varepsilon^2 - 1\right)^2 &= 0 \end{aligned} \tag{255}$$

Hence using the quadratic formula

$$\begin{aligned} r &= \frac{-2a\varepsilon\cos\theta\left(\varepsilon^2 - 1\right) \pm \sqrt{4a^2\varepsilon^2\cos^2\theta\left(\varepsilon^2 - 1\right)^2 - 4\left(\varepsilon^2\cos^2\theta - 1\right)a^2\left(\varepsilon^2 - 1\right)^2}}{2\left(\varepsilon^2\cos^2\theta - 1\right)} \\ r &= \frac{-2a\varepsilon\cos\theta\left(\varepsilon^2 - 1\right) \pm \sqrt{4a^2\varepsilon^2\cos^2\theta\left(\varepsilon^2 - 1\right)^2 - 4a^2\varepsilon^2\cos^2\theta\left(\varepsilon^2 - 1\right)^2 + 4a^2\left(\varepsilon^2 - 1\right)^2}}{2\left(\varepsilon^2\cos^2\theta - 1\right)} \\ r &= \frac{-2a\varepsilon\cos\theta\left(\varepsilon^2 - 1\right) \pm \sqrt{4a^2\left(\varepsilon^2 - 1\right)^2}}{2\left(\varepsilon^2\cos^2\theta - 1\right)} \\ r &= \frac{-2a\varepsilon\cos\theta\left(\varepsilon^2 - 1\right) \pm 2a\left(\varepsilon^2 - 1\right)}{2\left(\varepsilon^2\cos^2\theta - 1\right)} \\ r &= \frac{a\left(\varepsilon^2 - 1\right)\left(-\varepsilon\cos\theta \pm 1\right)}{\left(\varepsilon^2\cos^2\theta - 1\right)} \\ r &= \frac{a\left(\varepsilon^2 - 1\right)\left(-\varepsilon\cos\theta \pm 1\right)}{\left(\varepsilon\cos\theta - 1\right)\left(\varepsilon\cos\theta + 1\right)} \\ r &= \frac{a\left(1 - \varepsilon^2\right)\left(\varepsilon\cos\theta \mp 1\right)}{\left(\varepsilon\cos\theta - 1\right)\left(\varepsilon\cos\theta + 1\right)} \end{aligned} \tag{256}$$

Therefore

$$\begin{aligned} r_+ &= \frac{a\left(1 - \varepsilon^2\right)\left(\varepsilon\cos\theta + 1\right)}{\left(\varepsilon\cos\theta - 1\right)\left(\varepsilon\cos\theta + 1\right)} = \frac{a\left(1 - \varepsilon^2\right)}{\left(\varepsilon\cos\theta - 1\right)} \\ r_- &= \frac{a\left(1 - \varepsilon^2\right)\left(\varepsilon\cos\theta - 1\right)}{\left(\varepsilon\cos\theta - 1\right)\left(\varepsilon\cos\theta + 1\right)} = \frac{a\left(1 - \varepsilon^2\right)}{\left(\varepsilon\cos\theta + 1\right)} \end{aligned} \tag{257}$$

So

$$r = \frac{a\left(1-\varepsilon^2\right)}{\varepsilon\cos\theta \pm 1} \tag{258}$$

Now r must be > 0 for all θ and eccentricities ε. For a circle $b = a$ and therefore $\varepsilon = 0$. Obviously $r = a$. Therefore the -ve solution has no physical meaning. For a slightly more expansive argument, note that from the definition $\varepsilon = \sqrt{1 - \frac{b^2}{a^2}}$

$$0 \leq \varepsilon \leq 1 \tag{259}$$

Therefore since the maximum value of $\varepsilon\cos\theta \leq 1$, this means the -ve solution will always yield -ve values of r which is unphysical.

Hence the polar equation of the ellipse is

$$r = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\theta} \tag{260}$$

7.4 Velocities for elliptical orbits

Consider a body undergoing an elliptical orbit. At time t the orbit makes angle θ with the horizontal axis of the ellipse. The ellipse has eccentricity ε and horizontal width a. The polar equation of the orbit is therefore

$$r = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\theta} \tag{261}$$

Now, the rate of change of position vector $\mathbf{r} = r\hat{\mathbf{r}}$ is given by

$$\dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} \tag{262}$$

where $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are, respectively, the unit vectors in the radial and polar angle directions. Note θ is measured anti-clockwise from the horizontal.

Now since $r = r(\theta)$

$$\dot{r} = \frac{dr}{d\theta}\dot{\theta} \tag{263}$$

Hence

$$\dot{\mathbf{r}} = \dot{\theta}\left(\frac{dr}{d\theta}\hat{\mathbf{r}} + r\hat{\boldsymbol{\theta}}\right) \tag{264}$$

Using the polar equation of the ellipse $r = \frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\theta}$

$$\frac{dr}{d\theta} = \frac{d}{d\theta}\left\{\frac{a\left(1-\varepsilon^2\right)}{1+\varepsilon\cos\theta}\right\} \tag{265}$$

$$= \frac{-a\left(1-\varepsilon^2\right)}{\left(1+\varepsilon\cos\theta\right)^2}\left(-\varepsilon\sin\theta\right) \tag{266}$$

$$\therefore \frac{dr}{d\theta} = \frac{a\varepsilon\left(1-\varepsilon^2\right)\sin\theta}{\left(1+\varepsilon\cos\theta\right)^2} = \frac{\varepsilon\sin\theta}{1+\varepsilon\cos\theta}r\dot{\theta} \tag{267}$$

Hence

$$\dot{\mathbf{r}} = \frac{a\left(1-\varepsilon^2\right)\dot{\theta}}{1+\varepsilon\cos\theta}\left(\frac{\varepsilon\sin\theta}{1+\varepsilon\cos\theta}\hat{\mathbf{r}} + \hat{\boldsymbol{\theta}}\right) \tag{268}$$

The conversion between the polar unit vectors and the Cartesian equivalents are

$$\begin{aligned}\hat{\mathbf{r}} &= \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}} \\ \hat{\boldsymbol{\theta}} &= -\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}\end{aligned} \tag{269}$$

and

$$\mathbf{r} = \left(x_c + a\varepsilon + r\cos\theta\right)\hat{\mathbf{x}} + \left(y_c + r\sin\theta\right)\hat{\mathbf{y}} \tag{270}$$

where (x_c, y_c) is the centre of the ellipse in Cartesians.

7.5 Link between parametric and polar equations of the ellipse

The following equations relate the Cartesian x coordinate of the ellipse, the parameter ϕ and the polar coorniates r, θ

$$\begin{aligned} x &= x_c + a\cos\phi \\ y &= y_c + b\sin\phi \\ \varepsilon &= \sqrt{1 - \frac{b^2}{a^2}} \\ \theta &= \tan^{-1}\left(\frac{y - y_c}{x - x_c - a\varepsilon}\right) \\ r &= \sqrt{(x - x_c - a\varepsilon)^2 + (y - y_c)^2} \end{aligned} \tag{271}$$

Hence

$$\theta = \tan^{-1}\left(\frac{b\sin\phi}{a\cos\phi - a\varepsilon}\right) \tag{272}$$

8 Appendix 3: Three dimensional rotation

8.1 Derivation of generalized Rodrigues' rotation formula

Let us define three-dimensional rotation[6] in the following way:

- Position vectors (i.e. displacements from a defined origin) in 3-dimensional space $\mathbf{r}$ shall be rotated to $\mathbf{r}'$ anti-clockwise by angle θ about a vector $\boldsymbol{\omega}$.
- The rotation shall occur about position vector $\mathbf{a}$. This means the tip of a vector $\mathbf{b} = \mathbf{r} - \mathbf{a}$ will trace a circular path around $\boldsymbol{\omega}$ by angle θ.

The circular path of $\mathbf{b}$ can allow us to express its new value $\mathbf{b}'$ following the rotation of θ about a vector $\boldsymbol{\omega}$. This achieved via firstly defining a set of Cartesian basis vectors $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ where $\hat{\mathbf{z}}$ is parallel to the rotation vector $\boldsymbol{\omega}$ and $\hat{\mathbf{y}}$ is perpendicular to both $\mathbf{b}$ and $\boldsymbol{\omega}$. Note $\{\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\}$ are a right handed set and are orthonormal.

$$\begin{aligned} \hat{\mathbf{z}} &= \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|} \\ \hat{\mathbf{y}} &= \frac{\boldsymbol{\omega} \times \mathbf{b}}{|\boldsymbol{\omega} \times \mathbf{b}|} \\ \hat{\mathbf{x}} &= \hat{\mathbf{y}} \times \hat{\mathbf{z}} \end{aligned} \tag{273}$$

Basis vector $\hat{\mathbf{x}}$ is a vector triple product, which can be simplified into dot-products[7]

$$\begin{aligned} \hat{\mathbf{x}} &= \hat{\mathbf{y}} \times \hat{\mathbf{z}} \\ &= \frac{\boldsymbol{\omega} \times \mathbf{b}}{|\boldsymbol{\omega} \times \mathbf{b}|} \times \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|} \\ &= \frac{\boldsymbol{\omega} \times \mathbf{b} \times \boldsymbol{\omega}}{|\boldsymbol{\omega} \times \mathbf{b}|\,|\boldsymbol{\omega}|} \end{aligned} \tag{274}$$

$$\therefore \hat{\mathbf{x}} = \frac{(\boldsymbol{\omega} \cdot \boldsymbol{\omega})\,\mathbf{b} - (\boldsymbol{\omega} \cdot \mathbf{b})\,\boldsymbol{\omega}}{|\boldsymbol{\omega} \times \mathbf{b}|\,|\boldsymbol{\omega}|} \tag{275}$$

Now α is the angle between $\mathbf{b}$ and $\boldsymbol{\omega}$ (indeed between $\mathbf{b}'$ and $\boldsymbol{\omega}$).

$$|\mathbf{b}|\,|\boldsymbol{\omega}| \cos\alpha = \mathbf{b} \cdot \boldsymbol{\omega} \tag{276}$$

Also note the general feature of cross-products

$$|\boldsymbol{\omega} \times \mathbf{b}| = |\boldsymbol{\omega}|\,|\mathbf{b}| \sin\alpha \tag{277}$$

Hence

$$\mathbf{b}' = |\mathbf{b}| \cos\alpha \hat{\mathbf{z}} + |\mathbf{b}| \sin\alpha \left(\cos\theta \hat{\mathbf{x}} + \sin\theta \hat{\mathbf{y}}\right) \tag{278}$$

Substituting for the results above

$$\begin{aligned} \mathbf{b}' &= |\mathbf{b}| \cos\alpha \hat{\mathbf{z}} + |\mathbf{b}| \sin\alpha \left(\cos\theta \hat{\mathbf{x}} + \sin\theta \hat{\mathbf{y}}\right) \\ \mathbf{b}' &= \frac{\mathbf{b} \cdot \boldsymbol{\omega}}{|\boldsymbol{\omega}|} \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|} + \frac{|\boldsymbol{\omega} \times \mathbf{b}|}{|\boldsymbol{\omega}|} \left(\cos\theta \left(\frac{(\boldsymbol{\omega} \cdot \boldsymbol{\omega})\,\mathbf{b} - (\boldsymbol{\omega} \cdot \mathbf{b})\,\boldsymbol{\omega}}{|\boldsymbol{\omega} \times \mathbf{b}|\,|\boldsymbol{\omega}|}\right) + \sin\theta \frac{\boldsymbol{\omega} \times \mathbf{b}}{|\boldsymbol{\omega} \times \mathbf{b}|}\right) \\ &= \frac{(\mathbf{b} \cdot \boldsymbol{\omega})}{|\boldsymbol{\omega}|^2} \boldsymbol{\omega} + \left(\frac{(\boldsymbol{\omega} \cdot \boldsymbol{\omega})\,\mathbf{b} - (\boldsymbol{\omega} \cdot \mathbf{b})\,\boldsymbol{\omega}}{|\boldsymbol{\omega}|^2}\right) \cos\theta + \frac{(\boldsymbol{\omega} \times \mathbf{b})}{|\boldsymbol{\omega}|} \sin\theta \\ &= \mathbf{b} \cos\theta + \frac{(\boldsymbol{\omega} \times \mathbf{b})}{|\boldsymbol{\omega}|} \sin\theta + \frac{(\mathbf{b} \cdot \boldsymbol{\omega})\,(1 - \cos\theta)}{|\boldsymbol{\omega}|^2} \boldsymbol{\omega} \end{aligned} \tag{279}$$

Since $\mathbf{b} = \mathbf{r} - \mathbf{a}$ and therefore $\mathbf{r}' = \mathbf{a} + \mathbf{b}'$

$$\mathbf{r}' = \mathbf{a} + (\mathbf{r} - \mathbf{a}) \cos\theta + \frac{\boldsymbol{\omega} \times (\mathbf{r} - \mathbf{a})}{|\boldsymbol{\omega}|} \sin\theta + \frac{((\mathbf{r} - \mathbf{a}) \cdot \boldsymbol{\omega})\,(1 - \cos\theta)}{|\boldsymbol{\omega}|^2} \boldsymbol{\omega} \tag{280}$$

Which gives the general result $\mathbf{r}' = f(\mathbf{r}, \mathbf{a}, \boldsymbol{\omega}, \theta)$

$$\mathbf{r}' = \mathbf{a}\,(1 - \cos\theta) + \mathbf{r} \cos\theta + \frac{\boldsymbol{\omega} \times (\mathbf{r} - \mathbf{a})}{|\boldsymbol{\omega}|} \sin\theta + \frac{(\mathbf{r} \cdot \boldsymbol{\omega} - \mathbf{a} \cdot \boldsymbol{\omega})\,(1 - \cos\theta)}{|\boldsymbol{\omega}|^2} \boldsymbol{\omega} \tag{281}$$

[6] This section derives *Rodrigues' rotation formula* http://en.wikipedia.org/wiki/Rodrigues%27_rotation_formula

[7] $\mathbf{a} \times \mathbf{b} \times \mathbf{c} = (a \cdot c)\,\mathbf{b} - (a \cdot b)\,\mathbf{c}$

8.2 Two dimensional rotation in a single plane

Lets consider the special case of a two dimensional rotation, i.e. an anti-clockwise rotation by θ about $\mathbf{a}$ where $\mathbf{r}, \mathbf{a}$ are in the Cartesian x, y plane and $\boldsymbol{\omega} = \hat{\mathbf{z}}$

$$\begin{aligned} \mathbf{a} &= a_x \hat{\mathbf{x}} + a_y \hat{\mathbf{y}} \\ \mathbf{r} &= x \hat{\mathbf{x}} + y \hat{\mathbf{y}} \\ \mathbf{r}' &= x' \hat{\mathbf{x}} + y' \hat{\mathbf{y}} \end{aligned} \tag{282}$$

$$\begin{aligned} \mathbf{r} \cdot \boldsymbol{\omega} = \mathbf{0} \\ \mathbf{a} \cdot \boldsymbol{\omega} = \mathbf{0} \end{aligned} \tag{283}$$

$$\begin{aligned} \boldsymbol{\omega} \times (\mathbf{r} - \mathbf{a}) &= (x - a_x) \hat{\mathbf{z}} \times \hat{\mathbf{x}} + (y - a_y) \hat{\mathbf{z}} \times \hat{\mathbf{y}} \\ &= (x - a_x) \hat{\mathbf{y}} - (y - a_y) \hat{\mathbf{x}} \end{aligned} \tag{284}$$

Hence

$$\begin{aligned} \mathbf{r}' &= \mathbf{a} (1 - \cos\theta) + \mathbf{r} \cos\theta + \frac{\boldsymbol{\omega} \times (\mathbf{r} - \mathbf{a})}{|\boldsymbol{\omega}|} \sin\theta + \frac{(\mathbf{r} \cdot \boldsymbol{\omega} - \mathbf{a} \cdot \boldsymbol{\omega}) (1 - \cos\theta)}{|\boldsymbol{\omega}|^2} \boldsymbol{\omega} \\ &= (a_x \hat{\mathbf{x}} + a_y \hat{\mathbf{y}}) (1 - \cos\theta) + (x \hat{\mathbf{x}} + y \hat{\mathbf{y}}) \cos\theta + ((x - a_x) \hat{\mathbf{y}} - (y - a_y) \hat{\mathbf{x}}) \sin\theta \\ &= (a_x (1 - \cos\theta) + x \cos\theta - (y - a_y) \sin\theta) \hat{\mathbf{x}} + (a_y (1 - \cos\theta) + y \cos\theta + (x - a_x) \sin\theta) \hat{\mathbf{y}} \\ &= (a_x + (x - a_x) \cos\theta - (y - a_y) \sin\theta) \hat{\mathbf{x}} + (a_y + (y - a_y) \cos\theta + (x - a_x) \sin\theta) \hat{\mathbf{y}} \end{aligned} \tag{285}$$

This can be written in matrix form as

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \left(\begin{pmatrix} x \\ y \end{pmatrix} - \begin{pmatrix} a_x \\ a_y \end{pmatrix} \right) + \begin{pmatrix} a_x \\ a_y \end{pmatrix} \tag{286}$$

which is consistent with a 2D rotation matrix implementation of rotation about a point (a_x, a_y)

8.3 Roll, pitch or yaw of a 3D object

A practical scenario is a rotation of an object defined by a set of position vectors $\{\mathbf{r}_i\}$ by α about a rotation axis $\boldsymbol{\omega}$ position at vector $\mathbf{a}$ within the scene. Unlike the generic problem discussed above, one typically desires to rotate the body about one of its *principle axis*. In the unrotated state[8] let's define these as the Cartesian basis vectors $\{\hat{\mathbf{x}}_B, \hat{\mathbf{y}}_B, \hat{\mathbf{z}}_B\}$. For brevity, define $\hat{\mathbf{b}}$ as one of these basis vectors.

- A *roll* is an anti-clockwise rotation about the $\hat{\mathbf{x}}_B$ axis
- A *pitch* is an anti-clockwise rotation about the $\hat{\mathbf{y}}_B$ axis
- A *yaw* is an anti-clockwise rotation about the $\hat{\mathbf{z}}_B$ axis

To achieve the desired rotation we must firstly rotate all position vectors such that the principle axis $\hat{\mathbf{b}}$ aligns with $\boldsymbol{\omega}$. This can be achieved by a rotation by π radians about a rotation axis

$$\boldsymbol{\Omega} = \hat{\mathbf{b}} + \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|} \tag{287}$$

and about the centre of mass of the object $\mathbf{h}$. Applying Rodrigues' rotation formula, and noting $\cos\pi = -1$, $\sin\pi = 0$

$$\mathbf{r}' = 2\mathbf{h} - \mathbf{r} + \frac{2 (\mathbf{r} \cdot \boldsymbol{\Omega} - \mathbf{h} \cdot \boldsymbol{\Omega})}{|\boldsymbol{\Omega}|^2} \boldsymbol{\Omega} \tag{288}$$

[8] A data set describing the vertices of a three dimensional shape will typically be oriented along the natural lines of symmetry, which are often likley to be the principle axis. e.g. the $\hat{\mathbf{x}}_B$ direction will typically be the longitudinal (tail to nose) axis of a aircraft. Note principle axis are actually a property of the body. They are the Cartesian basis vectors which diagonalizes the inertia matrix for the body.

Note if $\mathbf{h} = \mathbf{0}$

$$\mathbf{r}' = \frac{2\left(\mathbf{r}\cdot\boldsymbol{\Omega}\right)}{\left|\boldsymbol{\Omega}\right|^2}\boldsymbol{\Omega} - \mathbf{r} \tag{289}$$

Check: Let $\mathbf{b} = \hat{\mathbf{x}}_B = \boldsymbol{\omega}$ i.e. a pitch with the centre of mass stationary.

$$\mathbf{r} = x\hat{\mathbf{x}}_B, + y\hat{\mathbf{y}}_B + z\hat{\mathbf{z}}_B$$

$$\boldsymbol{\Omega} = \hat{\mathbf{b}} + \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|} = 2\hat{\mathbf{x}}_B$$

$$\begin{aligned} \mathbf{r}' &= \frac{2\left(2x\right)}{4}2\hat{\mathbf{x}}_B - x\hat{\mathbf{x}}_B - y\hat{\mathbf{y}}_B - z\hat{\mathbf{z}}_B \\ &= \hat{\mathbf{x}}_B - y\hat{\mathbf{y}}_B - z\hat{\mathbf{z}}_B \end{aligned}$$

So an extra rotation of π is needed to restore the original orientation. Taking this into account, the desired rotation of α about a rotation axis $\boldsymbol{\omega}$ position at vector $\mathbf{a}$ within the scene can be achieved via multiple applications of Rodrigues' formula using the following overall transformation $\mathbf{r} \longrightarrow \mathbf{r}^{'''}$.

$$\begin{aligned} \boldsymbol{\Omega} &= \hat{\mathbf{b}} + \frac{\boldsymbol{\omega}}{|\boldsymbol{\omega}|} \\ \mathbf{r}' &= f\left(\mathbf{r}, \mathbf{a}, \boldsymbol{\Omega}, \pi\right) \\ \mathbf{r}^{''} &= f\left(\mathbf{r}', \mathbf{a}, \boldsymbol{\omega}, \pi\right) \\ \mathbf{r}^{'''} &= f\left(\mathbf{r}^{''}, \mathbf{a}, \boldsymbol{\omega}, \alpha\right) \end{aligned}$$

where $\hat{\mathbf{b}} \in \{\hat{\mathbf{x}}_B, \hat{\mathbf{y}}_B, \hat{\mathbf{z}}_B\}$
and Rodrigues' formula is

$$f\left(\mathbf{r}, \mathbf{a}, \boldsymbol{\omega}, \theta\right) = \mathbf{a}\left(1 - \cos\theta\right) + \mathbf{r}\cos\theta + \frac{\boldsymbol{\omega}\times\left(\mathbf{r} - \mathbf{a}\right)}{|\boldsymbol{\omega}|}\sin\theta + \frac{\left(\mathbf{r}\cdot\boldsymbol{\omega} - \mathbf{a}\cdot\boldsymbol{\omega}\right)\left(1 - \cos\theta\right)}{|\boldsymbol{\omega}|^2}\boldsymbol{\omega} \tag{290}$$