Determining the age and evolution of the Universe: The Hubble Law and Friedmann equations

Edwin Hubble was perhaps the first astronomer to show that most galaxies (i.e. objects with distances of 10Mpc or more) have a recessional velocity v which is proportional to the distance d away from Earth*. This is called the *Hubble Law*

$$v = H_0 d$$

H_0 is the 'Hubble constant', which has a modern value of about $H_0 \approx 71.9\,\mathrm{kms^{-1}/Mpc}$
It is *not really a constant*, as (see below) it relates to the *scale* of Universe expansion, which is thought not to be linear. The zero suffix therefore means 'at the current epoch.'

Hubble's law implies that *the Universe is expanding*. If we consider just the radial motion due to expansion (imagine a sponge being continuously enlarged, and tracking the relative distances between pairs of holes) and assume this is at a *constant* rate throughout time t, we can therefore make an estimate of the age of the Universe.

$$v = \frac{d}{t} \qquad v = H_0 d$$

$$\therefore H_0 d = \frac{d}{t} \qquad \therefore t = \frac{1}{H_0}$$

$$t = \left(\frac{71.9 \times 10^3\,\mathrm{ms^{-1}}}{3.086 \times 10^{22}\,\mathrm{m}} \right)^{-1} = 13.6\,\text{billion years}$$

1Mega-parsec (Mpc) = 3.086 x10^{22} m

*The **Cosmological Principle** means *all parts of the Universe are expanding uniformly* relative to everywhere else, *at a given time* since the Big Bang. The Hubble law would therefore be the *same* from the perspective of a planet in another galaxy as it is on Earth.

As of 2017, the best estimate for the age of the Universe is **13.799 +/- 0.021 billion years** using the **Lambda-CDM model** and observations of the **Cosmic Microwave Background** (CMB) radiation via **Planck** and **Wilkinson Microwave Anisotropy (WMAP)** probe (and others). The Lamda CDM model assumes a non-constant Hubble parameter, and is based upon the solutions of the *Friedmann equations* – see below.

https://en.wikipedia.org/wiki/Age_of_the_universe

The expansion of the Universe implies a time, 13.8 billion years ago, when the entire cosmos occupied a singular point. Georges Lemaître proposed the idea of a **Big Bang**, essentially a 'moment of creation.'

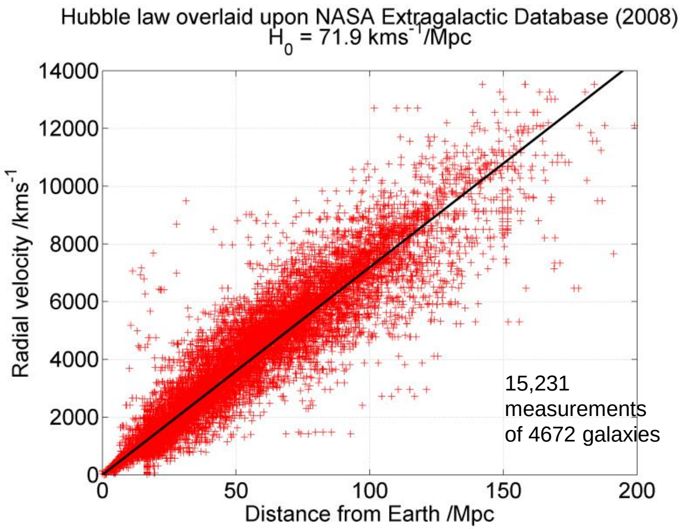
The expansion of the Universe also helps to resolve **Olbers' Paradox**. If the Universe is filled with a constant density n of stars of approximately uniform average luminosity (i.e. irradiative power) L, and the Universe is infinite, using an inverse-square law the received power / unit area on Earth should also be *infinite*, which is clearly not the case!

$$P = \int_0^\infty 4\pi r^2 dr \times n \times \frac{L}{4\pi r^2} = nL \int_0^\infty dr = \infty$$

Number of stars in shell of radius r and thickness dr

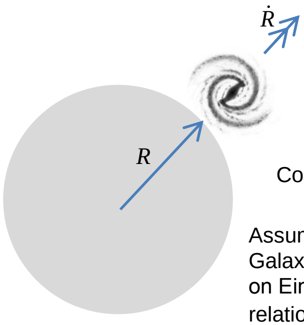
Inverse square law

https://ned.ipac.caltech.edu/level5/NED05D/ned05D_6.html



The deviation of galactic motion from the Hubble law is not surprising, since galaxies will gravitationally attract each other and therefore not be stationary relative to the (expanding) fabric of space. The Hubble law, and hence the expansion of the Universe, is therefore an overall trend.

The **Friedmann equations** provide cosmologists with a mathematical model of the large scale structure of the Universe. They can be derived by applying ideas of **General Relativity**, but it is also possible to determine their essential form using classical mechanics.



Consider a galaxy of mass m on the edge of an expanding spherical volume of the Universe of radius R. Let's assume the density of the mass within R is ρ.

Conservation of energy: $-\tfrac{1}{2}kmc^2 = \tfrac{1}{2}m\dot{R}^2 - \dfrac{Gm \times \tfrac{4}{3}\pi R^3 \rho}{R}$

KE GPE

Assume total energy of Galaxy has this form, based on Einstein's mass-energy relationship. k is a dimensionless constant.

If the galaxy (and hence the Universe) is gravitationally bound then we expect k to be positive and total energy to be *negative*.

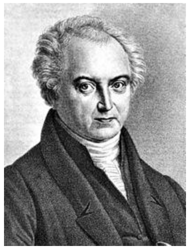
Hence: $$\left(\frac{\dot{R}}{R}\right)^2 = \tfrac{8}{3}\pi G \rho - \frac{kc^2}{R^2}$$

Speed of light $c = 2.998 \times 10^8\,\mathrm{ms^{-1}}$

Alexander Friedmann
1888-1925

Edwin Hubble
1889-1953

Georges Lemaître
1894–1966



Heinrich Olbers
1758-1840

Notes adapted from John Cullerne's Winchester College Pre-U Physics Course (2016) and my own notes from the Cambridge University *Gravitational Astrophysics & Cosmology course* (2001)

$$\left(\frac{\dot{R}}{R}\right)^2 = \tfrac{8}{3}\pi G\rho - \frac{kc^2}{R^2}$$ (From quasi-Classical argument)

This is very close to the form of the first Friedmann equation (F1) derivable from General Relativity. The full version incorporates the **Cosmological Constant** (which can be interpreted as a non-zero form of 'vacuum energy'), and parameter k is related to the geometric *curvature* of the spacetime which constitutes the large scale structure of the Universe.

$$\left(\frac{\dot{R}}{R}\right)^2 = \tfrac{8}{3}\pi G\rho - \frac{kc^2}{R^2} + \tfrac{1}{3}\Lambda c^2$$

Current observations* imply the Cosmological Constant is:

$$\Lambda \approx 1.19\times10^{-52}\,\mathrm{m}^{-2}$$

It is suggested that k = -1,0 or +1, and current observations imply that the Universe is 'geometrically flat' i.e. k = 0.

We can re-write F1 using the Hubble law: $\dot{R} = HR \quad \therefore \frac{\dot{R}}{R} = H$

$$H^2 = \tfrac{8}{3}\pi G\rho - \frac{kc^2}{R^2} + \tfrac{1}{3}\Lambda c^2$$

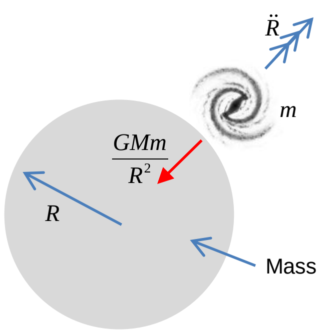
It is useful to find the time derivative of the first Friedmann equation:

$$2H\dot{H} = \tfrac{8}{3}\pi G\dot{\rho} + \frac{2kc^2}{R^3}\dot{R}$$

$$2H\dot{H} = \tfrac{8}{3}\pi G\dot{\rho} + \frac{2kc^2}{R^2}H$$

$$\therefore \dot{H} = \tfrac{4}{3}\pi G\frac{\dot{\rho}}{H} + \frac{kc^2}{R^2}$$

Now let us consider the **gravitational force** on the galaxy due to the spherical mass



Mass $M = \tfrac{4}{3}\pi R^3\rho$

Newton II:

$$m\ddot{R} = -\frac{Gm\tfrac{4}{3}\pi R^3\rho}{R^2}$$

$$\therefore \frac{\ddot{R}}{R} = -\tfrac{4}{3}\pi G\rho$$

Differentiating the Hubble law:

$$H = \frac{\dot{R}}{R}$$

$$\therefore \dot{H} = \frac{R\ddot{R} - \dot{R}^2}{R^2}$$

$$\therefore \dot{H} = \frac{\ddot{R}}{R} - \left(\frac{\dot{R}}{R}\right)^2$$

$$\therefore \frac{\ddot{R}}{R} = \dot{H} + H^2$$

Hence: $\dot{H} + H^2 = -\tfrac{4}{3}\pi G\rho$

To form the second Friedmann equation we must (i) incorporate the cosmological constant and (ii) take into account *radiation pressure.*

Although EM radiation has no mass, it does convey energy. The fundamental idea of General Relativity is that the *curvature of space-time is proportional to the local energy density*.

From thermodynamics we have the result that radiation pressure p is a *third* of the energy density of a 'photon gas'.

Hence if we assign a 'mass-equivalent energy density'

$$\rho c^2 = 3p$$

To account for radiation pressure (let's assume this is uniform on a cosmological scale)

$$\rho \to \rho + \frac{3p}{c^2}$$

We can therefore form the second Friedmann equation (F2)

$$\frac{\ddot{R}}{R} = \dot{H} + H^2 = -\tfrac{4}{3}\pi G\left(\rho + \frac{3p}{c^2}\right) + \tfrac{1}{3}\Lambda c^2$$

i.e. add the Cosmological constant in the same way as in F1

Interestingly, we *don't* incorporate radiation pressure into F1. I guess this is because radiation has no mass, but does have momentum and therefore has the potential to exert a force. I'm not sure this holds as a valid argument, but given GR is the 'proper' method, I guess we can gloss over these technicalities ....

Summary of Friedmann equations

$$\left(\frac{\dot{R}}{R}\right)^2 = H^2 = \tfrac{8}{3}\pi G\rho - \frac{kc^2}{R^2} + \tfrac{1}{3}\Lambda c^2 \quad \text{F1}$$

$$\Rightarrow \dot{H} = \tfrac{4}{3}\pi G\frac{\dot{\rho}}{H} + \frac{kc^2}{R^2}$$

$$\frac{\ddot{R}}{R} = \dot{H} + H^2 = -\tfrac{4}{3}\pi G\left(\rho + \frac{3p}{c^2}\right) + \tfrac{1}{3}\Lambda c^2 \quad \text{F2}$$

Hubble law

$$H = \frac{\dot{R}}{R}$$

$$\frac{\ddot{R}}{R} = \dot{H} + H^2$$

$$\Lambda \approx 1.19\times10^{-52}\,\mathrm{m}^{-2}$$

$$G = 6.67384\times10^{-11}\,\mathrm{m^3kg^{-1}s^{-2}}$$

Subtracting F1 from F2:

$$\dot{H} + H^2 - H^2 = -\tfrac{4}{3}\pi G\left(\rho + \frac{3p}{c^2}\right) + \tfrac{1}{3}\Lambda c^2 - \tfrac{8}{3}\pi G\rho + \frac{kc^2}{R^2} - \tfrac{1}{3}\Lambda c^2$$

$$\dot{H} = -\tfrac{4}{3}\pi G\left(3\rho + \frac{3p}{c^2}\right) + \frac{kc^2}{R^2}$$

Hence using $\dot{H} = \tfrac{4}{3}\pi G\frac{\dot{\rho}}{H} + \frac{kc^2}{R^2}$

$$\tfrac{4}{3}\pi G\frac{\dot{\rho}}{H} + \frac{kc^2}{R^2} = -\tfrac{4}{3}\pi G\rho\left(3 + \frac{3p}{c^2}\right) + \frac{kc^2}{R^2}$$

$$\dot{\rho} = -3\left(\rho + \frac{p}{c^2}\right)H$$

The ratio of 'mass-equivalent energy density' for photons to matter is perhaps more important on a cosmological scale than the absolute values of each quantity.

Define a dimensionless parameter $\varepsilon = \frac{3p}{\rho c^2} \quad \therefore \frac{p}{c^2} = \tfrac{1}{3}\varepsilon\rho$

$$\varepsilon = \begin{cases} 1 & \text{radiation dominated} \\ \approx 0 & \text{matter dominated} \end{cases}$$

$$\therefore \dot{\rho} = -3\rho\left(1 + \tfrac{1}{3}\varepsilon\right)H \quad \therefore \frac{\dot{\rho}}{\rho} = -(3+\varepsilon)\frac{\dot{R}}{R}$$

$$\therefore \int\frac{d\rho}{\rho} = -(3+\varepsilon)\int\frac{dR}{R}$$

$$\ln\rho = -(3+\varepsilon)\ln R + \text{constant}$$

$$\rho = \frac{\rho_0}{(R/R_0)^{3+\varepsilon}}$$

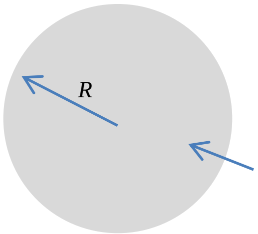
R_0 Radius at current epoch

$\rho \propto \frac{1}{R^3}$ Matter dominated

$\rho \propto \frac{1}{R^4}$ Radiation dominated

*https://en.wikipedia.org/wiki/Cosmological_constant

Alternative derivation of density equation $\dot{\rho} = -3\left(\rho + \frac{p}{c^2}\right)H$ using the First Law of Thermodynamics



If we only consider the expansion of the Universe and ignore motions relative to the 'fabric of space-time,' then we can consider the internal energy U of a large spherical subset of the Universe to be determined by Einstein's mass-energy relationship:

Mass $M = \frac{4}{3}\pi R^3 \rho$

$$U = Mc^2 = \tfrac{4}{3}\pi R^3 \rho c^2$$

We shall assume a uniform average density ρ within the sphere.

The expansion of the Universe ought to obey the **First Law of Thermodynamics**:

$$dU = dQ + dW$$

Change in internal energy — Heat added to sphere — Work done on sphere

Since the Universe is (by definition!) a closed system, we might confidently predict the *average* heat input to the 'average spherical subset' to be *zero*.
i.e. thermodynamic change on a large scale is *adiabatic*.

The change in internal energy is therefore a consequence of the work done by the Universe external to the sphere. Given the sparse matter distribution of the Universe, we might expect this work to be done via the application of *radiation pressure.* Let us assume this is of uniform pressure p in the vicinity of the sphere

$$dU = dQ + dW$$

$$dQ \approx 0 \qquad V = \tfrac{4}{3}\pi R^3 \qquad dW = -pdV$$

$$\therefore dU + pdV = 0$$

$$\therefore \frac{dU}{dt} + p\frac{dV}{dt} = 0$$

$$\frac{d}{dt}\left(\tfrac{4}{3}\pi R^3 \rho c^2\right) + p\frac{d}{dt}\left(\tfrac{4}{3}\pi R^3\right) = 0$$

$$\tfrac{4}{3}\pi c^2\left(R^3\dot{\rho} + 3R^2\dot{R}\rho\right) + \tfrac{4}{3}\pi p\left(3R^2\dot{R}\right) = 0$$

$$R^3\dot{\rho} + 3R^2\dot{R}\rho + \frac{p}{c^2}\left(3R^2\dot{R}\right) = 0$$

$$\dot{\rho} + 3\rho\frac{\dot{R}}{R} + 3\frac{p}{c^2}\frac{\dot{R}}{R} = 0$$

$$\dot{\rho} = -3\left(\rho + \frac{p}{c^2}\right)H = 0$$

Using the **Hubble relationship** $\dot{R} = HR \quad \therefore \frac{\dot{R}}{R} = H$

This argument appears to be routed in scientific orthodoxy, but it does pose an awkward problem when scaled up to the entire Universe. If the Universe represents all that there is, then what pressure could be pushing against it?

The Friedmann analysis appears to be concerned, literally, about universal trends, rather than specific local phenomenon like stars and galaxies. Therefore the subset idea can probably be justified as long as the subset is large enough to fit with the adiabatic assumption (i.e. no heat change), but no so large that the sphere is not expanding into an essentially infinite reservoir of radiation.

To solve the Friedmann equations, we can use the above results to express the density term in terms of, suitably scaled matter and radiation parts, based upon observed 'densities' measured in the present epoch.

It is also instructive to define a dimensionless density parameter and a 'critical density'

$$H^2 = \tfrac{8}{3}\pi G\rho - \frac{kc^2}{R^2} + \tfrac{1}{3}\Lambda c^2 \qquad \text{F1}$$

$$k = 0, \Lambda = 0, \rho = \rho_c, H = H_0$$

$$\therefore \boxed{\rho_c = \frac{3H_0^2}{8\pi G}}$$

$$\boxed{\Omega = \frac{\rho}{\rho_c} \qquad \therefore \rho = \frac{3H_0^2}{8\pi G}\Omega}$$

Using the mass of a proton as $m_p = 1.6726 \times 10^{-27}\,\text{kg}$

$$\Rightarrow \rho_c = \frac{3 \times \left(\dfrac{71.9 \times 10^3 \text{ms}^{-1}}{3.086 \times 10^{22}\text{m}} \right)^2}{8\pi \times 6.67384 \times 10^{-11}\text{m}^3\text{kg}^{-1}\text{s}^{-2}}$$

$$\Rightarrow \rho_c = 9.7089 \times 10^{-27}\,\text{kgm}^{-3}$$

$$\Rightarrow \boxed{\rho_c \approx 5.8 \text{ protons}/\text{m}^3}$$

Recent observations imply that the average matter density is only about 0.2 atoms per cubic metre. Assuming matter is mostly Hydrogen molecules, this means 0.4 protons per cubic metre, only about 7% of the 'critical' density above.

Current theories (as of 2017) account for this difference by the presence of **Dark Matter** and **Dark Energy**. Unfortunately the physical existence of either of these mysterious quantities has yet to be confirmed! Perhaps they exist, or perhaps our model of gravity on a large scale is flawed ....

From the Planck Cosmology probe: Universe energy density: **Dark Energy 69.1% Dark Matter 25.9% Matter 4.9%**

$$\boxed{H^2 = \tfrac{8}{3}\pi G\rho - \frac{kc^2}{R^2} + \tfrac{1}{3}\Lambda c^2} \qquad \text{F1}$$

$$\therefore \frac{H^2}{H_0^2} = \frac{8\pi G}{3H_0^2}\rho - \frac{kc^2}{H_0^2 R^2} + \tfrac{1}{3}\frac{\Lambda c^2}{H_0^2}$$

$$\therefore \frac{H^2}{H_0^2} = \frac{\rho}{\rho_c} - \frac{kc^2}{H_0^2 R^2} + \tfrac{1}{3}\frac{\Lambda c^2}{H_0^2}$$

$\rho \propto \dfrac{1}{R^3}$ Matter dominated

$\rho \propto \dfrac{1}{R^4}$ Radiation dominated

(from previous page)

We can hence define the overall density as:

$$\rho = \frac{\rho_{matter}}{(R/R_0)^3} + \frac{\rho_{rad}}{(R/R_0)^4}$$

$$\therefore \frac{\rho}{\rho_c} = \frac{\Omega_{matter}}{(R/R_0)^3} + \frac{\Omega_{rad}}{(R/R_0)^4}$$

From the Planck probe results, dark plus 'normal' matter**

$$\Omega_{matter} \approx 0.14$$

Define (dimensionless) **scaling parameter:**

$$a(t) = R/R_0 \qquad \therefore H = \frac{\dot a}{a}; \quad \frac{\rho}{\rho_c} = \frac{\Omega_{matter}}{a^3} + \frac{\Omega_{rad}}{a^4}$$

If time is defined relative to current epoch:

$$a(t_0) = 1 \qquad t_0 = 13.8 \text{ billion years}$$

$$\therefore \frac{H^2}{H_0^2} = \frac{8\pi G}{3H_0^2}\rho - \frac{kc^2}{H_0^2 R^2} + \tfrac{1}{3}\frac{\Lambda c^2}{H_0^2} \qquad \text{F1}$$

$$\therefore \frac{H^2}{H_0^2} = \left(\frac{\Omega_{matter}}{a^3} + \frac{\Omega_{rad}}{a^4} \right) - \frac{kc^2}{H_0^2 R_0^2 a^2} + \tfrac{1}{3}\frac{\Lambda c^2}{H_0^2}$$

$$\boxed{\therefore \left(\frac{\dot a}{a}\right)^2 = H_0^2\left(\frac{\Omega_{matter}}{a^3} + \frac{\Omega_{rad}}{a^4} \right) - \frac{kc^2}{R_0^2 a^2} + \tfrac{1}{3}\Lambda c^2} \qquad \text{F3}$$

Special case #1:

$$k = 0, \ \Lambda = 0, \ \Omega_{matter} \approx 1, \ \Omega_{rad} \approx 0$$

Flat, matter dominated Universe

$$\left(\frac{\dot a}{a}\right)^2 = \frac{H_0^2}{a^3}$$

Consider scale factor time variations of the form: $a(t) = \left(\dfrac{t}{t_0}\right)^n$

$$\left(\frac{\dot a}{a}\right)^2 = \frac{H_0^2}{a^3} \Rightarrow \left(\frac{nt^{n-1}}{t^n}\right)^2 = \frac{H_0^2 t_0^{3n}}{t^{3n}}$$

$$n^2 t^{-2} = H_0^2 t_0^{3n} t^{-3n} \qquad \therefore n = \tfrac{2}{3}$$

$$\therefore \tfrac{4}{9} = H_0^2 t_0^2 \qquad \therefore t_0 = \tfrac{2}{3}H_0^{-1}$$

$$\therefore \boxed{a(t) = \left(\tfrac{3}{2}H_0 t\right)^{\frac{2}{3}}}$$

Hence present age of Universe is:

$$1 = \left(\tfrac{3}{2}H_0 t_0\right)^{\frac{2}{3}}$$

$$\therefore \boxed{t_0 = \tfrac{2}{3}H_0^{-1}}$$

Special case #2:

$$k = 0, \ \Lambda = 0, \ \Omega_{matter} \approx 0, \ \Omega_{rad} \approx 1$$

Flat, radiation dominated Universe

$$\left(\frac{\dot a}{a}\right)^2 = \frac{H_0^2}{a^4} \qquad a(t) = \left(\frac{t}{t_0}\right)^n$$

$$\left(\frac{\dot a}{a}\right)^2 = \frac{H_0^2}{a^4} \Rightarrow \left(\frac{nt^{n-1}}{t^n}\right)^2 = \frac{H_0^2 t_0^{4n}}{t^{4n}}$$

$$n^2 t^{-2} = H_0^2 t_0^{4n} t^{-4n} \qquad \therefore n = \tfrac{1}{2}$$

$$\therefore \tfrac{1}{4} = H_0^2 t_0^2 \qquad \therefore t_0 = 2H_0^{-1}$$

$$\therefore \boxed{a(t) = \left(2H_0 t\right)^{\frac{1}{2}}}$$

Hence present age of Universe is:

$$1 = \left(2H_0 t_0\right)^{\frac{1}{2}}$$

$$\therefore \boxed{t_0 = \tfrac{1}{2}H_0^{-1}}$$

Special case #3:

$$k = 0, \ \Lambda \neq 0, \ \Omega_{matter} \approx 0, \ \Omega_{rad} \approx 0$$

Exponential expansion

$$\left(\frac{\dot a}{a}\right)^2 = \tfrac{1}{3}\Lambda c^2$$

$$\therefore \frac{1}{a}\frac{da}{dt} = c\sqrt{\tfrac{1}{3}\Lambda}$$

$$\int_1^a \frac{da}{a} = c\sqrt{\tfrac{1}{3}\Lambda}\int_{t_0}^t dt$$

$$\ln a = c\sqrt{\tfrac{1}{3}\Lambda}\,(t - t_0)$$

$$\therefore \boxed{a(t) = e^{c\sqrt{\frac{1}{3}\Lambda}(t-t_0)}}$$

Special case #4:

Linear expansion i.e. simple Hubble model

$$\boxed{a(t) = \frac{t}{t_0}}$$

Define $t_0 = H_0^{-1}$

Note this isn't really compatible with F3 since the time derivative of a is a constant and therefore F3 would imply a polynomial equation in a (which will have a finite number of solutions).

NOTE: In various literature where k = 1,-1 is considered, the dependence on a fixed R_0 is ignored. I'm not sure how to resolve this!

https://en.wikipedia.org/wiki/Planck_(spacecraft)

** I think this what is meant.

Summary

Best estimate (2017) for the age of the Universe is **13.799 +/- 0.021 billion years**

$$\frac{1}{H_0}=\left(\frac{71.9\times10^{3}\mathrm{ms}^{-1}}{3.086\times10^{22}\mathrm{m}}\right)^{-1}=13.6\,\text{billion years}$$

with 2016 calculation of **Hubble constant** $H_0 \approx 71.9\,\mathrm{kms}^{-1}/\mathrm{Mpc}$

1 billion = 10^9

1 year = 365 x 24 x 3600 s

The radiation and matter special cases are therefore much less close than the linear expansion model.

Universe expansion scale factor: $a(t)=R/R_0$

$$a(t)=\left(\frac{t}{t_0}\right)^{n}$$

t_0 is age of Universe at current Epoch. $a(t_0)=1$

$$a(t)=\left(2H_0t\right)^{\frac{1}{2}}, \quad t_0=\tfrac{1}{2}H_0^{-1}$$

Radiation dominated universe

$$a(t)=\left(\tfrac{3}{2}H_0t\right)^{\frac{2}{3}}, \quad t_0=\tfrac{2}{3}H_0^{-1}$$

Matter dominated universe

$$a(t)=\frac{t}{t_0}, \quad t_0=H_0^{-1}$$

Linear expansion

$$a(t)=e^{c\sqrt{\frac{1}{3}\Lambda}(t-t_0)}$$

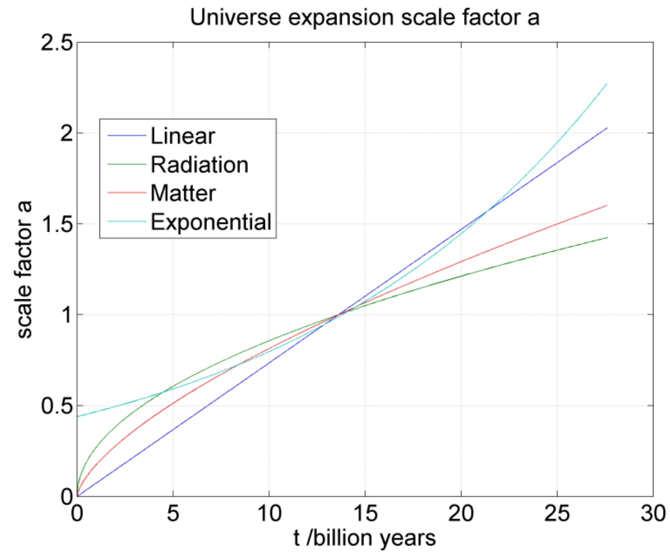
Exponential expansion, driven by **Cosmological Constant.**

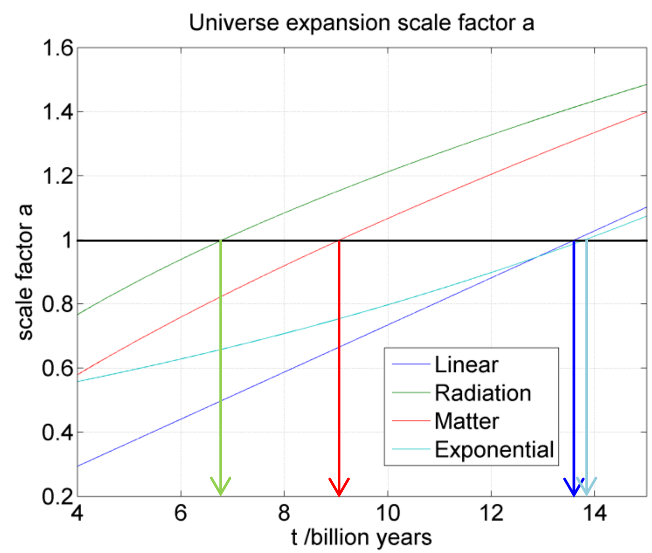
$t_0=13.8\times10^{9}$ (years)

Note this means the Universe has a *finite size* at the time of the Big Bang!

A ‘best’ model is probably a *hybrid* of all of the above solutions. The Universe certainly contains both matter and radiation, and ‘vacuum fluctuations’ may explain *why* the Cosmological Constant is non-zero.

(Although strictly speaking the linear solution is incompatible with the Friedmann equations. However, it might be an asymptotic trend?)





Zoom in to show different age of Universe predictions using different models for the scale factor variation with time since the Big Bang.

The arrows show the predicted age of the Universe at the current Epoch, given the modern value of Hubble’s constant.

Speed of light $c=2.998\times10^{8}\mathrm{ms}^{-1}$