IGCSE Extended Mathematics at Winchester College

- **Numbers**
  - Cubes
  - Squares
    - 2 integer
  - Percentages
  - Errors and approximations
  - Standard form
    - 2.998×10^8
    - 6.63×10^{-34}
  - Units & conversions
  - Number facts
  - Fractions
  - Revision of long multiplication and division
  - π, $\sqrt{2}$, e …
  - How to use a calculator
    - Memory use
  - Surds
  - Sequences
  - Laws of indices
  - Prime factorization of integers
    - Highest common factor
    - Lowest common multiple
  - Ratio ⟷ fractions
  - Speed, distance & time
    - $v = \frac{x}{t}$
- **Statistics and probability**
  - Add $p(A)+p(B)$ …
  - Multiply $p(A)\,p(B)$ …
    - Tree diagrams
  - Independent vs conditional $p(a|b, c \dots)$
  - Averages (mean, median, mode)
  - Lower quartiles, upper quartiles
  - IQR
  - Box & whisker
  - Cumulative frequency
  - Bar charts & histograms
  - Display of data
- **Extra**
  - Standard deviation
  - Growth curves
  - Function transforms
  - Pascal's triangle and the Binomial Expansion of $(a+b)^n$
  - Graph ⊛ sketching
    - $\sqrt{x}$, x^3 …
  - Logarithms
  - Arithmetic and geometric series
  - Introductory calculus
    - Trapezium rule
- **Sets and logic**
  - $\cap, \cup, \subset$
  - $n(\,)$, $\mathcal{E}$, $\varnothing$
  - Venn diagrams ⟷ set notation
- **Matrices and transformations**
  - Inverse & determinant
  - Vectors
  - Solving equations?
  - Applications
  - Addition & multiplication
  - + translations
  - About origin → Geometrical transforms
    - Stretch / shear / rotation / reflection
- **Trigonometry**
  - Sin, cos, tan as multipliers
  - Sine and cosine rules
    - Any angle …
  - Pythagoras' theorem
  - Bearings
  - 3D problems
- **Algebra**
  - $x_{n+1} = \dots$
    - Iteration
  - Functions $f(x)$
  - Rearranging equations
  - Brackets | Grouping | Simplification
  - Fractions
  - Factorization
    - linear
    - quadratic
  - Variation / proportion
    - $y \propto x$
    - $y \propto \frac{1}{x}$
  - Laws of indices and surds
  - Solution of equations $f(x) = 0$
    - linear
    - Quadratics
      - Factorization
      - Complete the square
      - Formula
    - By graphical method
  - Inequalities
  - Simultaneous equations
    - linear +2, +3?
- **Geometry, volumes and areas**
  - Areas and volumes of basic shapes*
  - Loci
  - Constructions
  - Proportion / Similarity scale factors
  - Circle theorems
- **Graphs**
  - Linear programming
  - $y = mx + c$
  - Sine and cosine and tangent curves
  - Inverse proportion
  - Extra ⊛
  - Quadratics
    - Roots, vertex, y intercept
  - Kinematics
    - Velocity, time graph (t, v)
    - Displacement → Area of (t, v)
    - Acceleration → gradient of (t, v)

* Circles

Triangles	Spheres
Circles	Cones
Polygons	Cylinders
Rectangles	Cuboids
Trapezia	Trapezoids

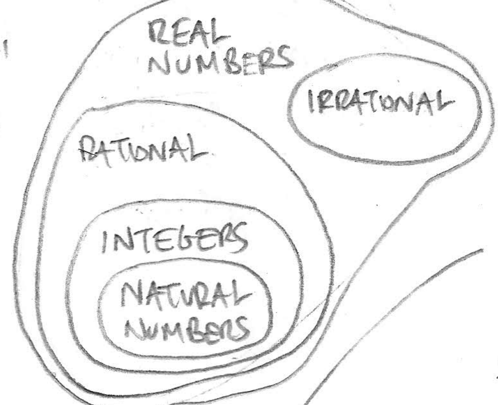
+ POSTERS
+ MENTAL ARITHMETIC
+ INVESTIGATIONS & PUZZLES
+ PRACTICAL APPLICATIONS
  + Programming

n	n^2	n^3	n^4	n^5	n^6	n^7	n^8	n^9	n^{10}	n^{11}	n^{12}	n^{13}
2	4	8	16	32	64	128	256	512	1024	2048	4096	8192
3	9	27	81	243	729							
4	16	64	256	1024								
5	25	125	625	3125								
6	36	216	1296									
7	49	343	2401									
8	64	512										
9	81	729										
10	100	1000										
11	121	1331										
12	144	1728										
13	169	2197										
14	196	2744										
15	225	3375										
16	256	4096										
17	289	4913										
18	324	5832										
19	361	6859										
20	400	8000										

Numbers

Integers

"whole numbers"



can't express irrationals as a fraction of integers

$\pi \approx 3.14$, $e \approx 2.72$ } two decimal places

$\sqrt{2} \approx 1.41$

$\sqrt{3} \approx 1.73$

$\sqrt{5} \approx 2.24$

$\sqrt{7} \approx 2.65$

$\ln 2 \approx 0.693$

$\log_{10} 2 \approx 0.301$

Long multiplication

137×59

=	100	30	7
50	5000	1500	350
9	900	270	63

5000
1500
350
+ 900
270
63
8083
21

Standard form

6.63×10^{-34}

2.998×10^8

4 significant figures

$0.00001 = 10^{-5}$

$1{,}000{,}000 = 10^6$

Percentages

39% of 73 is $0.39 \times 73 = 28.47$

3% loss on £1000 $= 0.03 \times 1000 = £30$ loss

Fractions

$\frac{3}{2} \times \frac{7}{5} = \frac{3 \times 7}{2 \times 5} = \frac{21}{10}$

$\frac{4}{3} \div \frac{11}{5} = \frac{4}{3} \times \frac{5}{11}$

$\frac{2}{3} + \frac{4}{9} = \left(\frac{2}{3}\right)\left(\frac{9}{9}\right) + \left(\frac{4}{9}\right)\left(\frac{3}{3}\right)$

$= \frac{18 + 12}{27}$

$= \frac{30}{27}$

$= \boxed{\frac{10}{9}}$

$= \boxed{1\frac{1}{9}}$

$1 + \frac{1}{9}$

Ratio

$4:7:2 \Rightarrow$ fractions

$4+7+2 = 13 \longrightarrow \frac{4}{13} : \frac{7}{13} : \frac{2}{13}$

Highest common factor (HCF) and Lowest common Multiple

LCM of 8, 3: $= \boxed{24}$

8, 16, 24, 32 (8)

3, 6, 9, 12, 15, 18, 21, 24, 27 (3)

HCF of 12, 8: $12 = 3 \times 2^2$

$8 = 2^3$

so HCF $= 2^2 = \boxed{4}$

laws of indices

"power Add"

$2^3 \times 2^7 = 2^{10}$

$2^4 \times 2^{-3} = 2^1$

$\frac{1}{2^5} = 2^{-5}$

$(2^3)^6 = 2^{18}$

$(2^{-3})^{-5} = 2^{-3 \times -5} = 2^{15}$

Surds

$\sqrt[n]{2} = 2^{1/n}$

$\sqrt[3]{2} = 2^{1/3}$

$\frac{1}{\sqrt{3}} = 3^{-\frac{1}{2}}$

$\frac{2}{\sqrt{3}} = \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2\sqrt{3}}{3}$

Algebra

Collecting terms via brackets

$$2x^2 + y - 3x + 7x^2 - 4y + 3 - y^2 + 7$$
$$= x^2\{2+7\} + x\{-3\} + y^2\{-1\} + y\{1-4\} + 10$$
$$= \boxed{9x^2 - 3x - y^2 - 3y + 10}$$

Inequalities

* Inequalities work like equations, except sign of inequality reverses if ÷ or × by a -ve.

$x \geq 3$ $\quad$ $y < -7$

Multiplying out brackets

$$(a+b)(c+d)$$
$$= ac + bc + da + bd$$
(First inside outside last)

use a matrix for higher numbers of elements

$$(x+y-3)(2x+y^2)$$

=

	$2x$	y^2
x	$2x^2$	xy^2
y	$2xy$	y^3
-3	$-6x$	$-3y^2$

$= y^3 + xy^2 - 3y^2 + 2xy + 2x^2 - 6x$

Fractions

$$\frac{a}{b} + \frac{c}{d} = \left(\frac{a}{b}\right)\underbrace{\left(\frac{d}{d}\right)}_{1} + \left(\frac{c}{d}\right)\underbrace{\left(\frac{b}{b}\right)}_{1}$$
$$= \frac{ad + bc}{bd}$$

* Rearranging equations

Step by Step!

Find a

$$e^2 = 1 - \frac{b^2}{a^2}$$
$$\Rightarrow \frac{b^2}{a^2} = 1 - e^2$$
$$\Rightarrow b^2 = a^2(1-e^2) \Rightarrow a^2 = \frac{b^2}{1-e^2}$$
$$\Rightarrow \boxed{a = \pm\frac{b}{\sqrt{1-e^2}}}$$

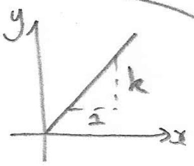
Surds

$$\sqrt[n]{x} = x^{\frac{1}{n}}$$
$$\sqrt{x}\sqrt{x} = x$$

Proportion

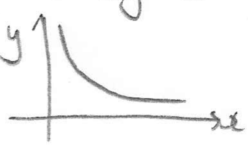
"Direct proportion"

$$y \propto x \Rightarrow y = kx$$



$$y \propto \frac{1}{x} \Rightarrow xy = k$$

"Inverse proportion"

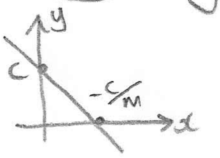


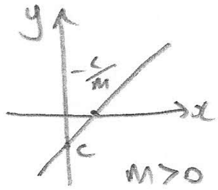
Laws of indices

$$(x^a)^b = x^{ab}$$
$$x^a x^b = x^{a+b}$$
$$\frac{1}{x^a} = x^{-a} \quad \left(\text{so } \frac{x^a}{x^b} = x^a x^{-b} = x^{a-b}\right)$$

Polynomials

linear: $y = mx + c$



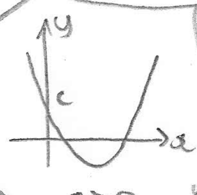


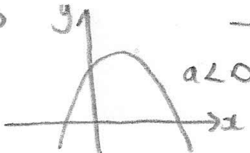
Quadratics

$$y = ax^2 + bx + c$$

when $y = 0$ [ROOTS]

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$





FACTORIZE

$$6x^2 + 8x - 14$$
$$= (3x+7)(2x-2)$$

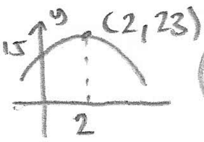
	$3x$	7
$2x$	$6x^2$	$14x$
-2	$-6x$	-14

Difference of two squares

$$(x-a)(x+a) = x^2 - a^2$$

Complete the Square

$$-2x^2 + 8x + 15$$
$$= -2\{x^2 - 4x\} + 15$$
$$= -2\{(x-2)^2 - 4\} + 15 = \boxed{-2(x-2)^2 + 23}$$



Treat like equations BUT DON'T $\times$ or $\div$ by a $-$ve.

linear inequalities

$3x + 2 \geq -\frac{1}{3}x + 7$

$\Rightarrow 3x - \frac{1}{3}x \geq 7 - 2$

$\Rightarrow \frac{9x}{3} - \frac{1}{3}x \geq 5$

$\Rightarrow \frac{8x}{3} \geq 5$

$\Rightarrow \boxed{x \geq 15/8}$

Simultaneous equations

$\leftarrow$ LABEL!

Note N equations in N unknowns MAY be solvable.

$3 = 4x - 2y$ ①

$-4 = -x + y$ ②

$-5 = 2x$ ③ $= 2$② $+$ ①

$\Rightarrow \boxed{-\frac{5}{2} = x}$

Describe operations and combinations and label all new equations

in ②: $y = -4 + x \Rightarrow y = -4 - \frac{5}{2} = \boxed{-\frac{13}{2}}$

More algebra

Substitution

Solve: $x^2 + y^2 = 3$ ①

$y = -2x + 1$ ②

in ①: $x^2 + \{-2x + 1\}^2 = 3$

$\Rightarrow x^2 + 4x^2 - 4x + 1 - 3 = 0$

$\Rightarrow 5x^2 - 4x - 2 = 0$

$x = \frac{16 \pm \sqrt{16 - 4(5)(-2)}}{10}$

$= 1.6 \pm \frac{1}{10}\sqrt{16 + 40}$

$= \frac{16 \pm \sqrt{56}}{10}$

$\approx \boxed{0.852, 2.348}$

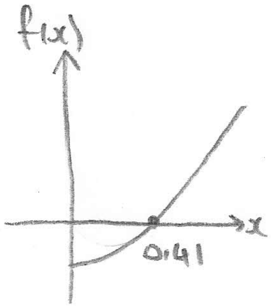
Root finding by Iteration

$f(x) = x^2 - \sqrt{x} + 3x$

when $f(x) = 0$

Write $\frac{\sqrt{x} - x^2}{3} = x$

$\Rightarrow$



Start with a guess $(x_1 = 2)$

then iterate: $x_{n+1} = \frac{\sqrt{x_n} - x_n^2}{3}$

$x_1 = 2$

$x_2 = -0.8619$

$x_3 = 0.2238$

$\vdots$

$x_{15} = 0.41423$

$x_{27} = 0.414213562373$

Functions

$x \rightarrow \boxed{f(x)} \rightarrow$ answer

If $f(x) = x^2 + \sqrt{x}$

$f(2) = 2^2 + \sqrt{2} = \boxed{4 + \sqrt{2}}$

$f(2x+1) = (2x+1)^2 + \sqrt{2x+1}$

If $g(x) = 3x + 1$

$fg(x) = (3x+1)^2 + \sqrt{3x+1}$

$gf(x) = 3(x^2 + \sqrt{x}) + 1$

Now $\frac{g-1}{3} = x$ so $g^{-1}(x) = \frac{x-1}{3}$

INVERSE FUNCTION

Quadratics cont....

$3(x+2)(2x - \frac{1}{3}) = 0 \Rightarrow \boxed{x = -2, \frac{1}{6}}$

$-2(x + \frac{1}{2})^2 - 7 = 0 \Rightarrow$ vertex $(-\frac{1}{2}, -7)$

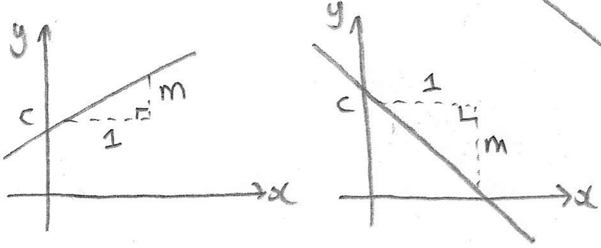
(Note this will have No real roots

Since $x + \frac{1}{2} = \pm\sqrt{-7/2}$

Graphs

Straight line graphs

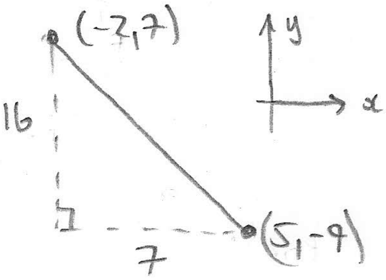
$y = mx + c$



$m = \text{gradient} = \dfrac{\Delta y}{\Delta x}$

$c = y \text{ intercept}$

"Find the straight line equation between $(-2,7)$ and $(5,-9)$"



$\therefore m = -\dfrac{16}{7}$

so $y = -\dfrac{16x}{7} + c$

using $(-2,7)$

$7 = -\dfrac{16}{7}(-2) + c$

$\Rightarrow c = 7 - \dfrac{32}{7}$

$= \dfrac{49-32}{7}$

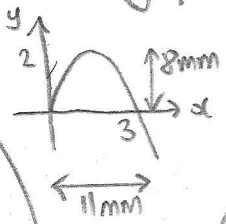
$= \boxed{\dfrac{17}{7}}$

Hence

$y = -\dfrac{16x}{7} + \dfrac{17}{7}$

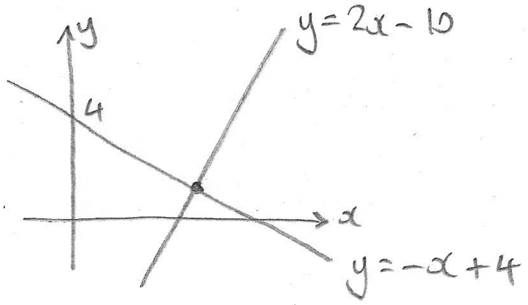
or $7y + 16 = 17$

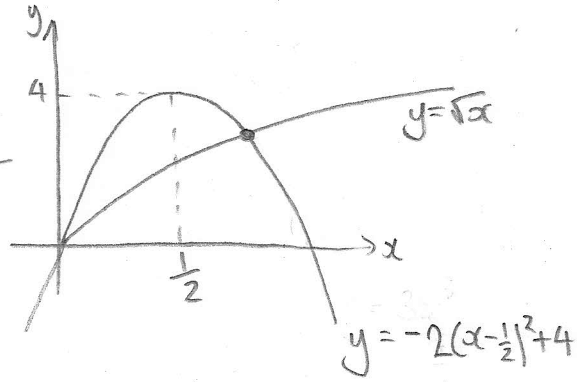
Measure accurately from graphs by finding what x and y values correspond to 1mm on the graph

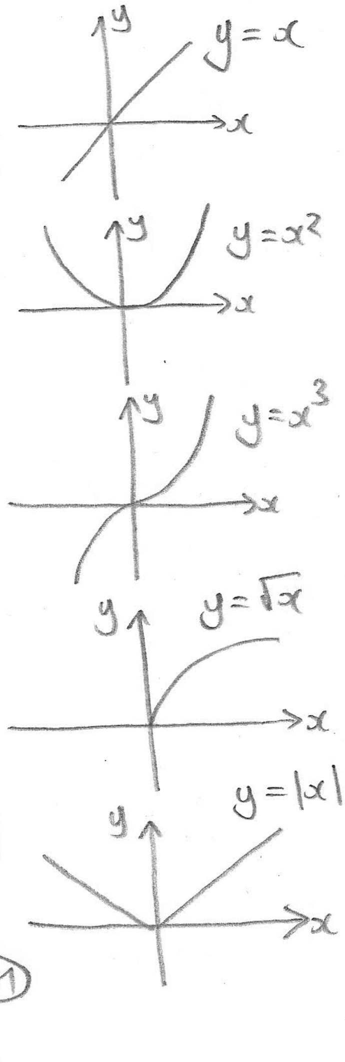


Solutions of equations graphically

Draw accurately!





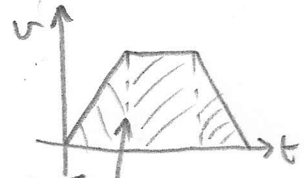


Kinematics

v = velocity

x = Displacement

a = acceleration

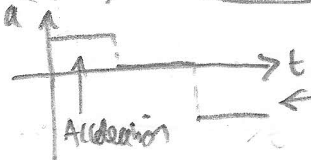


DISPLACEMENT x is area under (t, v) graph

Divide into

- rectangles
- trapezia
- triangles .. then add!

Gradient of (t, v) is acceleration



Acceleration

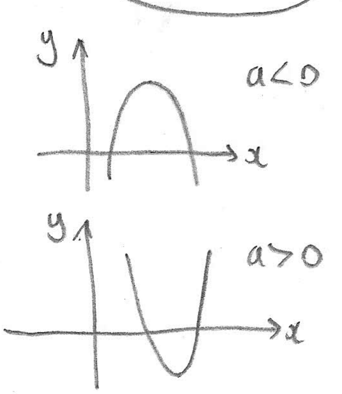
Deceleration

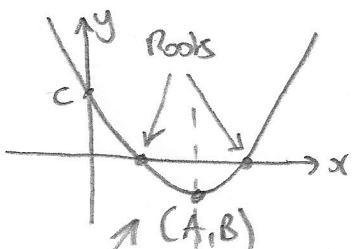
Quadratics

$y = ax^2 + bx + c$

$y = a(x - A)^2 + B$

[take $a > 0$]

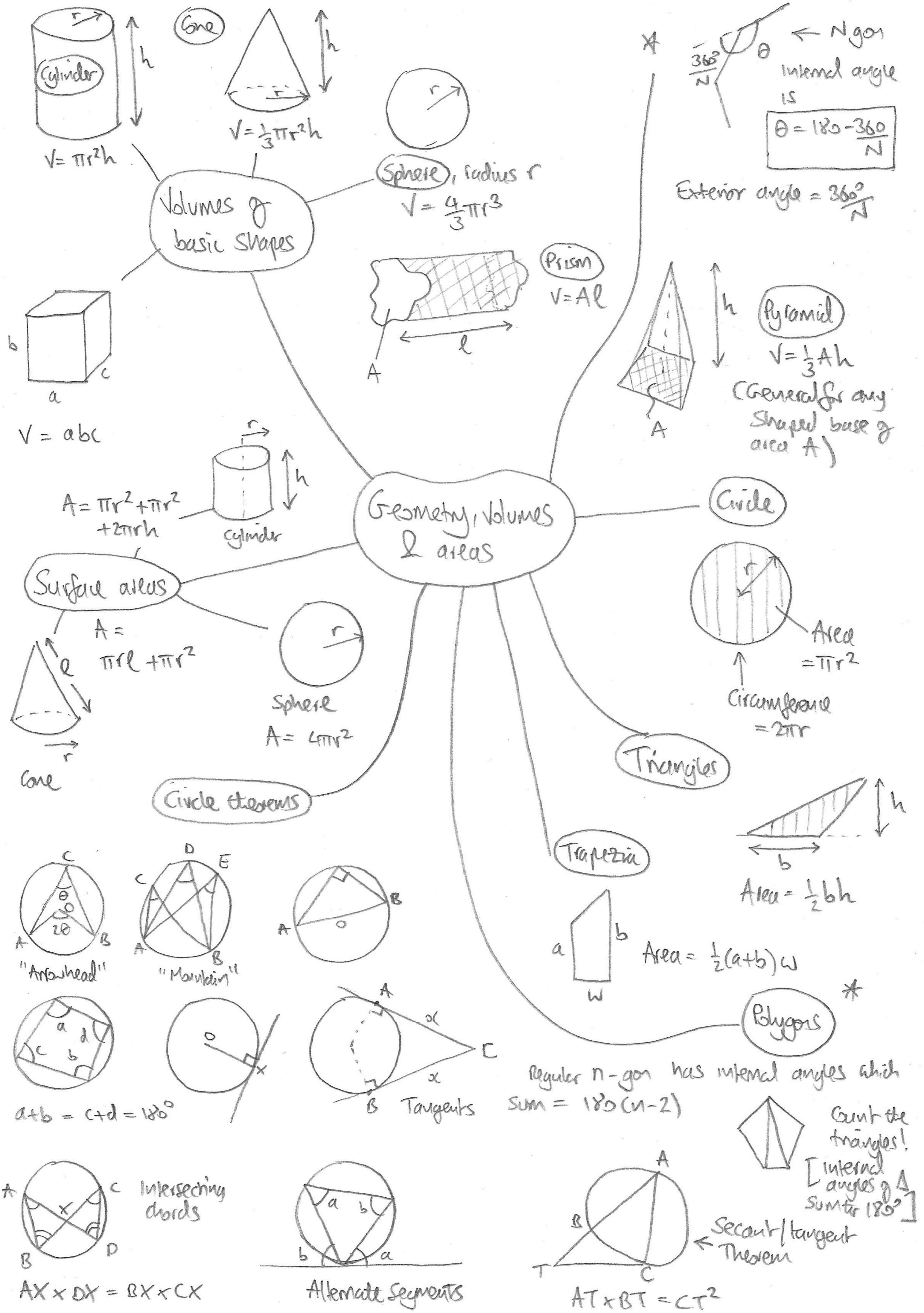




Vertex

Symmetric

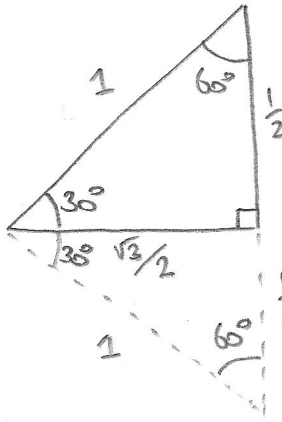
Roots are equally spaced about the vertex



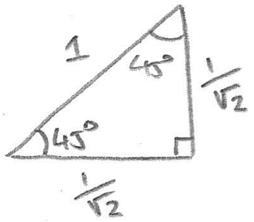
Geometry, volumes & areas
Volumes of basic shapes
Cylinder
$V = \pi r^2 h$
Cone
$V = \frac{1}{3}\pi r^2 h$
Sphere, radius r
$V = \frac{4}{3}\pi r^3$
$V = abc$
Prism
$V = Al$
Pyramid
$V = \frac{1}{3}Ah$
(General for any shaped base of area A)
N gon internal angle is
$\boxed{\theta = 180 - \frac{360}{N}}$
Exterior angle $= \frac{360°}{N}$
Surface areas
$A = \pi r^2 + \pi r^2 + 2\pi rh$
cylinder
$A = \pi rl + \pi r^2$
Cone
Sphere
$A = 4\pi r^2$
Circle
Area $= \pi r^2$
Circumference $= 2\pi r$
Triangles
Area $= \frac{1}{2}bh$
Trapezia
Area $= \frac{1}{2}(a+b)w$
Polygons
Regular n-gon has internal angles which sum $= 180(n-2)$
Count the triangles! [internal angles of Δ sum to 180°]
Circle theorems
"Arrowhead"
"Mankein"
$a+b = c+d = 180°$
Tangents
Intersecting chords
$AX \times DX = BX \times CX$
Alternate segments
Secant/tangent Theorem
$AT \times BT = CT^2$

Trigonometry

Special values

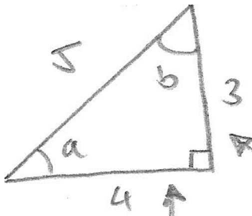


$\cos 60^\circ = \frac{1}{2}$
$\sin 60^\circ = \sqrt{3}/2$
$\cos 30^\circ = \sqrt{3}/2$
$\sin 30^\circ = \frac{1}{2}$
$\tan 30^\circ = \frac{1}{\sqrt{3}}$
$\tan 60^\circ = \sqrt{3}$



$\sin 45^\circ = \cos 45^\circ$
$= \frac{1}{\sqrt{2}}$

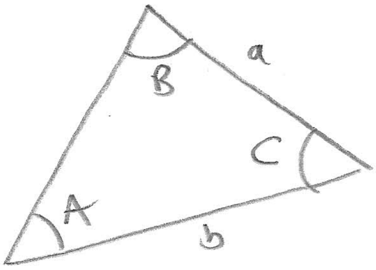
$\tan 45^\circ = 1$



"Pythagorean triple" $5^2 = 3^2 + 4^2$

$5\cos a = 4 \Rightarrow \boxed{a} = \cos^{-1}\left(\frac{4}{5}\right)$
$\simeq \boxed{36.9^\circ}$

$b = \tan^{-1}\left(\frac{4}{3}\right) \simeq \boxed{53.1^\circ}$

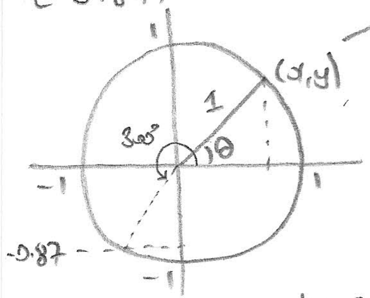


Sine rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$

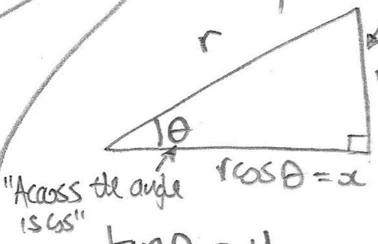
Cosine rule $a^2 = b^2 + c^2 - 2bc\cos A$

$A = \cos^{-1}\left(\frac{b^2+c^2-a^2}{2bc}\right)$

Use this to estimate SIGNS & VALUES of $\sin 300^\circ$ (−0.87)



UNIT CIRCLE
$x = \cos\theta$
$y = \sin\theta$



$r\sin\theta = y$

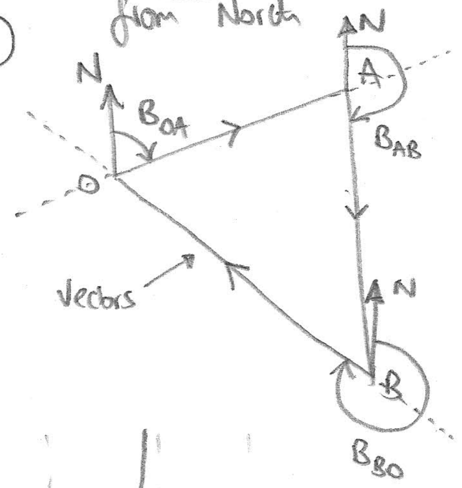
$r\cos\theta = x$

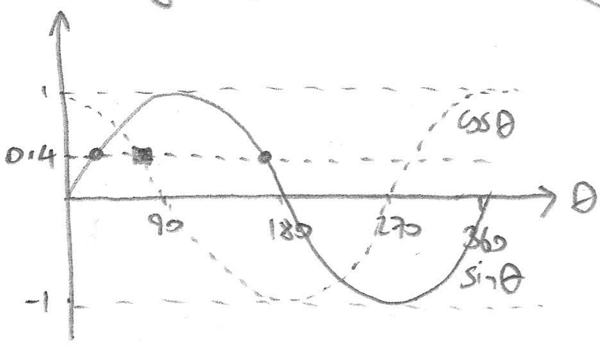
MULTIPLIERS

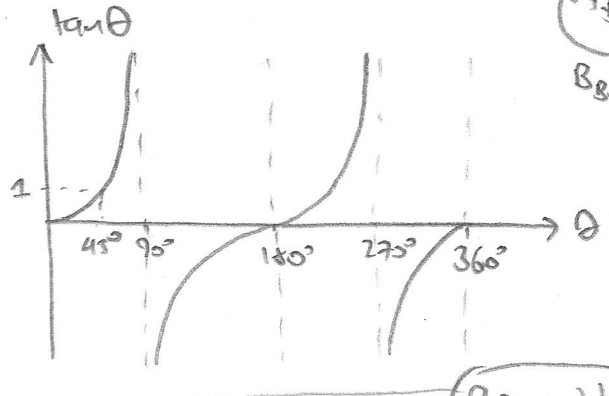
$\tan\theta = \frac{y}{x} = \frac{\sin\theta}{\cos\theta}$

$r^2 = x^2 + y^2$ Pythagoras' Theorem

Bearings are clockwise from North



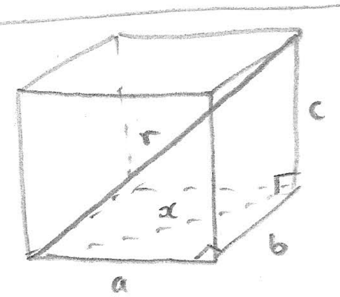


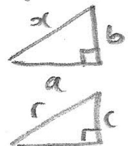


Note $\sin\theta = 0.4$ has TWO values ● in range [0,180] whereas $\cos\theta = 0.4$ has one ■. $\boxed{\text{Take care}}$ using sine rule to find OBTUSE ($\theta > 90^\circ$) angles! If in doubt, use cosine rule to find angles.

3D problems

Break up into several 2D problems





$\Rightarrow r^2 = x^2 + c^2 = a^2 + b^2 + c^2$

$\underline{x} = \{1, 2, 2, 7, -3, 5, 4, 4, 4, -1\}$ Mean of $\underline{x}$ is $\dfrac{\text{Sum}}{\#\text{elements}} = \dfrac{25}{10} = \boxed{2.5}$

Mode = Most common (4). order elements:

$\text{Sort}(\underline{x}) = \{-3, -1, 1, 2, 2, 4, 4, 4, 5, 7\}$ $\Rightarrow$ Inter-quartile range (IQR) $= \boxed{3}$

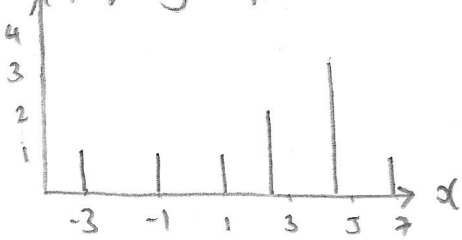
Lower quartile $= 1$ (LQ)

MEDIAN $= \dfrac{2+4}{2} = 3$

upper quartile $= 4$ (UQ)

Statistics & probability

frequency = # occurances (f)

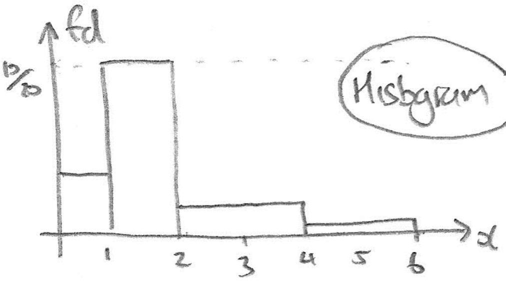


FREQUENCY TABLE

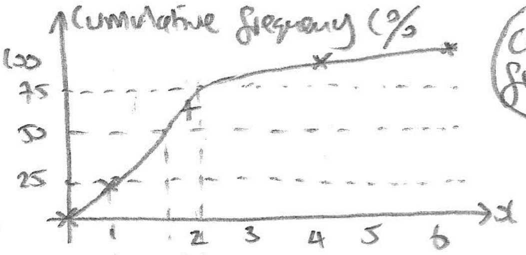
Frequency density $= \dfrac{\text{freq}}{\text{range}}$

x range	f	f/total frequency	fd
$0 \le x < 1$	4	4/20	4/20
$1 \le x < 2$	10	10/20	10/20
$2 \le x < 4$	4	4/20	2/20
$4 \le x < 6$	2	2/20	1/20

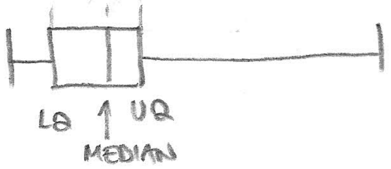
Histogram



Cumulative frequency



Box & whisker



frequency $\le x$	x	cumulative frequency
4	1	4/20 = 20%
14	2	14/20 = 70%
18	4	18/20 = 90%
20	6	20/20 = 100%

TABLES

Sum total of a 6 and 7 sided fair dice. $P(7) = 6/42 = 1/7$

DICE 1 \ DICE 2	1	2	3	4	5	6	7
1	2	3	4	5	6	7	8
2	3	4	5	6	7	8	9
3	4	5	6	7	8	9	10
4	5	6	7	8	9	10	11
5	6	7	8	9	10	11	12
6	7	8	9	10	11	12	13

$6 \times 7 = 42$ possibilities

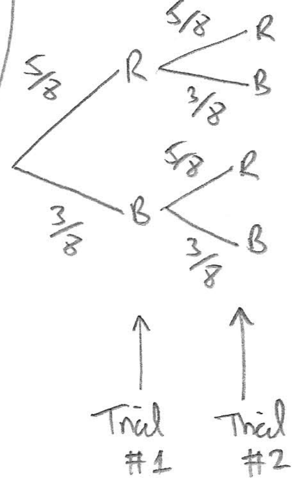
Probability of desired outcome $= \dfrac{\#\text{desired outcomes}}{\text{Total possible }\#\text{outcomes}}$

BINARY TRIAL "RED could be event happens" "BLUE could mean event doesn't happen"

Bag of balls. 5 red (R), 3 blue (B)

TREE DIAGRAMS

independent trials → i.e. replace balls each draw (or Trial)

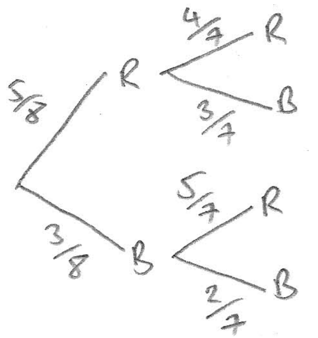


"Multiply along branches"

$P(RR) = \left(\frac{5}{8}\right)^2$

$P(R, B) = P(RB) + P(BR)$ (Red or Blue) $= 2\left(\frac{5}{8}\right)\left(\frac{3}{8}\right)$

Conditional trials → i.e. Don't replace balls each draw. Probabilities conditional on previous trial.



$P(BB) = \left(\frac{3}{8}\right)\left(\frac{2}{7}\right)$

Strictly $P(B|B)$ "B given B before"

More numbers

Sequences

n	1	2	3	4
U_n	2	5	8	11

+3 +3 +3

$U_n = 1 + 3(n-1)$ ARITHMETIC

$U_{n+1} = U_n + 3$

Suffix notation

n	1	2	3	4
U_n	5	$-\frac{5}{2}$	$\frac{5}{4}$	$-\frac{5}{8}$

$$U_n = \frac{5(-1)^{n-1}}{2^{n-1}}$$

GEOMETRIC

$$(-1)^n = \begin{cases} -1 & n \text{ odd} \\ 1 & n \text{ even} \end{cases}$$

Distance and time

$v = \frac{x}{t}$ ← Distance, ← time

Speed (or velocity if direction is included)

$73\,\text{km}h^{-1}$

$= 73 \times \frac{\text{miles}}{1.609} h^{-1}$

$= \boxed{45.4\,\text{mph}}$

Units conversions

1 hrs = 60 min

1 day = 24 hrs

1 min = 60 s

1 km = 1000 m

1 mile ≅ 1609 m = 1.609 km

∴ 64 mph =

$64 \times \frac{\text{mile}}{\text{hrs}}$ } Substitute for the unit. Treat like algebra.

$= 64 \times \frac{1609\,\text{m}}{60^2\,\text{s}}$

$= 64 \times 0.447\ \text{ms}^{-1}$

$\simeq \boxed{28.6\,\text{ms}^{-1}}$

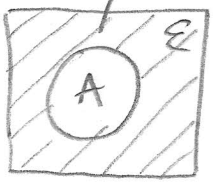
Sets & logic

$n(A) \Rightarrow$ # of elements in set A. eg $\{1,2,5,7\} = A$

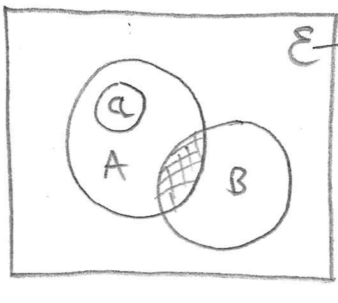
$n(A) = 4$

$\mathcal{E}' = \phi$

A' "NOT A"



ϕ $\boxed{\text{Nothing}}$



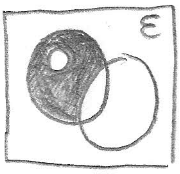
$\mathcal{E}$ $\boxed{\text{Everything}}$

$A \cap B$ $\boxed{\text{Intersection}}$ of A and B

$a \subset A$ $\boxed{\text{Subset}}$ of A

$A \cup B$ $\boxed{\text{Union}}$

Venn diagram



$(a' \cap A) \cap B'$

Note:

$\{3,7,5\} \cup \{2,5,7\}$

$= \{2,3,5,7\}$ Don't include common elements twice

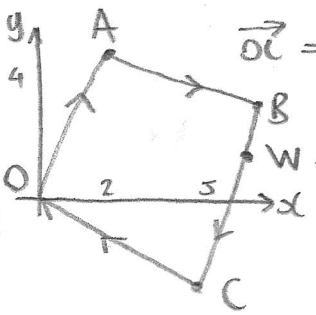
Vectors

Add tip to tail!

$\vec{OA} = \begin{pmatrix} 2 \\ 4 \end{pmatrix}$

$\vec{OB} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$

$\vec{OC} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}$



W ← $\frac{1}{3}$ along $\vec{BC}$

$\vec{AB} = -\vec{OA} + \vec{OB}$

$= \begin{pmatrix} 5 \\ -1 \end{pmatrix}$

$\vec{OW} = \vec{OB} + \frac{1}{3}\vec{BC}$

$= \vec{OB} + \frac{1}{3}(-\vec{OB} + \vec{OC}) = \begin{pmatrix} 6\frac{1}{3} \\ 1 \end{pmatrix}$

Enlargement, Scale factor k — $\begin{pmatrix} k & 0 \\ 0 & k \end{pmatrix}$

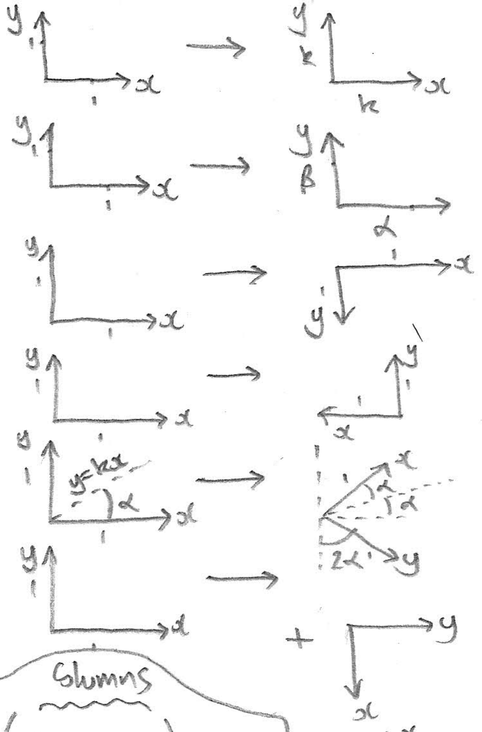
Stretch $\alpha \parallel x$, $\beta \parallel y$ — $\begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$

Reflection in x axis — $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

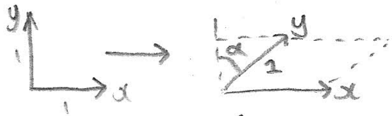
Reflection in y axis — $\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$

Reflection in line $y = kx$ — $\begin{pmatrix} \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & -\cos 2\alpha \end{pmatrix}$, $\alpha = \tan^{-1}(k)$

Rotation $\pm 90°$ clockwise — $\begin{pmatrix} 0 & \pm 1 \\ \mp 1 & 0 \end{pmatrix}$



Shear $\parallel x$, angle α [Note area preserved]

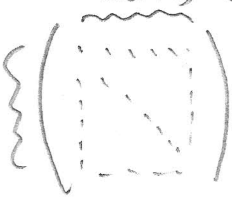


$\begin{pmatrix} 1 & \tan\alpha \\ 0 & 1 \end{pmatrix}$ $\tan\alpha$ = "Shear factor"

Shear $\parallel y$, angle α

$\begin{pmatrix} 1 & 0 \\ \tan\alpha & 1 \end{pmatrix}$ *

Transforms about (0,0) using matrices



columns

rows

Size or dimensions are rows × columns

2×2: $\underline{\underline{A}}^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$\boxed{\text{Determinant}}$ $\det(\underline{\underline{A}})$

Matrices & transformations

$\underline{\underline{A}}^{-1}$ is the inverse. Only defined for square matrices

since $\underline{\underline{A}}^{-1}\underline{\underline{A}} = \underline{\underline{I}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ldots$

IDENTITY

Algebra

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \underline{\underline{A}}$

$\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \underline{\underline{B}}$

$\underline{\underline{A}} \pm \underline{\underline{B}} = \begin{pmatrix} a\pm e & b\pm f \\ c\pm g & d\pm h \end{pmatrix}$ "Addition elementwise"

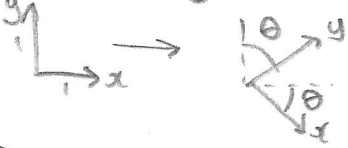
$\underline{\underline{A}}\,\underline{\underline{B}} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}$

$\underline{\underline{B}}\,\underline{\underline{A}} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} ea+fc & eb+fd \\ ga+hc & gb+hd \end{pmatrix}$

So "matrix multiplication is <u>non commutative</u>" i.e. $\underline{\underline{A}}\,\underline{\underline{B}} \neq \underline{\underline{B}}\,\underline{\underline{A}}$ in general

Note $\underline{\underline{R}}^n \Rightarrow n \times \theta$ rotation which means one can calculate $\underline{\underline{R}}^{0.7}\ldots$

* Rotation by θ clockwise



$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$

For a multiplication of matrices $\underline{\underline{X}}$ and $\underline{\underline{Y}}$

"$(\text{rows}_X \times \text{cols}_X) \times (\text{rows}_Y \times \text{cols}_Y)$"

describes the dimensions. $\boxed{\text{cols}_X = \text{rows}_Y}$

If this is the case, resulting matrix will have dimensions $\text{rows}_X \times \text{cols}_Y$

e.g. $\underset{2\times3}{\begin{pmatrix} 1 & 2 & 4 \\ 7 & 8 & 9 \end{pmatrix}}\underset{3\times③}{\begin{pmatrix} 3 & 4 & 7 \\ 2 & 9 & -1 \\ 1 & 6 & 0 \end{pmatrix}} = \underset{②\times③}{\begin{pmatrix} 10 & 46 & 5 \\ 46 & 154 & 41 \end{pmatrix}}$

Extra

Logarithms

$\log_b x^a = a \log_b x$

$b^{\log_b x} = x$

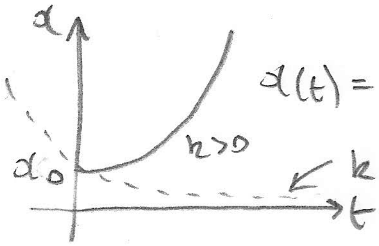
$\ln x = \log_e x$

$\boxed{\log_b x = y \leftrightarrow x = b^y}$

$\log_b x + \log_b y = \log_b xy$

$\log_b x - \log_b y = \log_b (x/y)$

Growth Curves



$x(t) = x_0 b^{kt}$

eg
$$\left.\begin{aligned} x &= 2^t \\ x &= 2^{-t} \\ x &= 10^t \\ x &= e^{at} \end{aligned}\right\} \text{when } t=0,\ x=1$$

[This is special! gradient is $a \times x$]

Arithmetic Sequence

$(n \geq 1)$

e.g. $2, 5, 8, 11, 14$ (+3 +3 +3 +3) $\Rightarrow d = 3$

$U_n = U_1 + (n-1)d$

$S_n = U_1 + U_2 + \ldots U_n$

$= \frac{1}{2} n (U_1 + U_n)$

Geometric Sequence

$U_n = U_1 r^{n-1}$

eg $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$

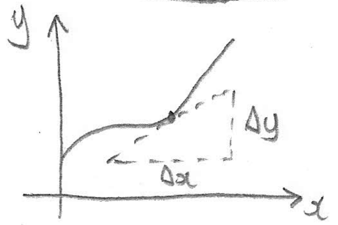
$(U_1 = 1$

$r = \frac{1}{2})$

$S_n = U_1 + U_2 + \ldots U_n$

$= \dfrac{U_1(1-r^n)}{1-r}$

Intro Calculus



$\dfrac{dy}{dx} = \lim_{\Delta x \to 0} \left(\dfrac{\Delta y}{\Delta x}\right)$ "Derivative"

If $y = ax^n$

$\dfrac{dy}{dx} = nax^{n-1}$

so if $y = x^2$, $\dfrac{dy}{dx} = 2x$

Pascal's triangle and the Binomial Expansion

```
            1                n →
          1   1
        1   2   1            (n=2)
      1   3   3   1          (n=3)
    1   4   6   4   1        (n=4)
  1   5   10  (10)  5   1    (n=5)
  ↑   ↑                 ↑
 r=0 r=2               r=5
      → r
```

$\begin{pmatrix} n \\ r \end{pmatrix} = \begin{pmatrix} 5 \\ 3 \end{pmatrix}$

$(a+b)^n$

$= \binom{n}{0} a^n b^0 + \binom{n}{1} a^{n-1} b^1 + \binom{n}{2} a^{n-2} b^2$

$+ \ldots \binom{n}{n} a^0 b^n$

eg $(1+x)^5 = 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5$

$\binom{n}{r} = \dfrac{n!}{(n-r)!\,r!}$ or ${}^nC_r$

"n choose r"