Superposition is the (signed) addition of wave disturbances ψ . The *phase difference* between waves gives rise to the phenomena of *constructive* (waves add) or *destructive interference* (waves cancel out).

$\psi = \psi_1 + \psi_2 = A_1 \cos(k_1 x - \omega_1 t - \phi_1) + A_2 \cos(k_2 x - \omega_2 t - \phi_2)$. In most scenarios we shall look at here, $k_1 = k_2$, $\omega_1 = \omega_2$.

A is the wave ***amplitude***, $k = 2\pi/\lambda$ is the ***wavenumber***, with λ being the ***wavelength***, $\omega = 2\pi f$ where f is the ***frequency***. Note $f = 1/T$ were T is the wave ***period***. The ***speed of waves*** is: $c = f\lambda = \omega/k$.

The **phase** α of a wave $\psi(x,t) = A\cos(kx - \omega t - \phi) = A\cos\alpha$ is: $\alpha = kx - \omega t - \phi \Rightarrow \boxed{\alpha = 2\pi \frac{(x-ct)}{\lambda} - \phi}$.

Note for adding up wave contributions, a *complex representation* often results in more efficient mathematics.

$\psi(x,t) = Ae^{i(kx-\omega t-\phi)}$. This works on the basis of de-Moivre's theorem: $e^{i\alpha} = \cos\alpha + i\sin\alpha$.

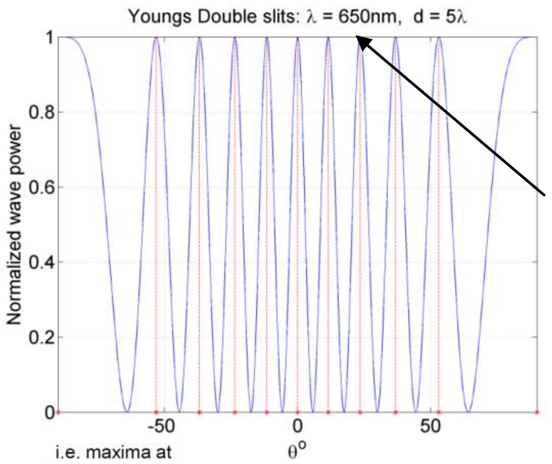
Wave power is proportional to $\omega^2 |\psi|^2$. $|\psi|$ means the amplitude of ψ . i.e. if $\psi = Ae^{i\alpha}$, $|\psi| = A$.

The ***Huygens-Fresnel Principle*** states: *"Every unobstructed point of a wavefront, at a given instant, serves as a source of spherical secondary wavelets (with the same frequency as that of the primary wave). The amplitude of the optical field at any point beyond is the superposition of all these wavelets (considering their amplitudes and relative phases)."*

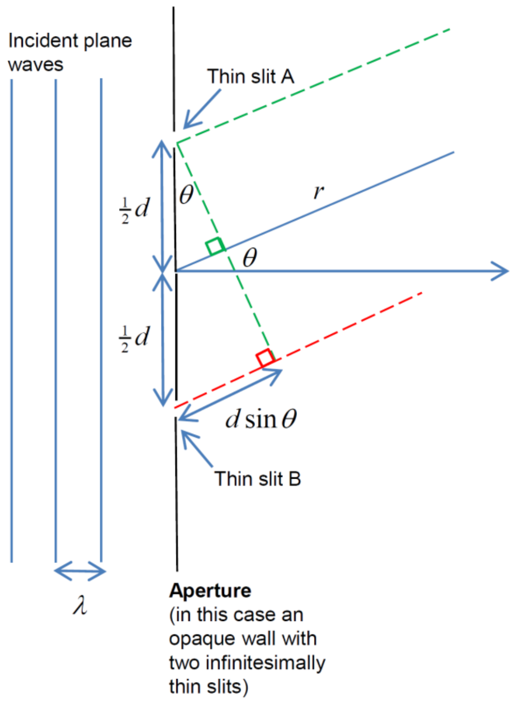
Huygen's principle can be used to explain *reflection*, *refraction* and *diffraction*, the latter being the effect of an slit, aperture or edge obstruction upon incident waves.

In the *far-field* $(r \gg d^2/\lambda)$ from two identical spherical wave sources (e.g. "Young's Double Slits") separated by distance d , waves emanating will be approximately *plane waves* at any given angle θ . Constructive interference occurs when the phase difference $\Delta\phi$ of received waves from the sources is an integer multiple of 2π radians.

$$\Delta\phi = \frac{2\pi}{\lambda} d\sin\theta = 2\pi n \Rightarrow \boxed{\theta_n = \sin^{-1}(n\lambda/d)} .$$

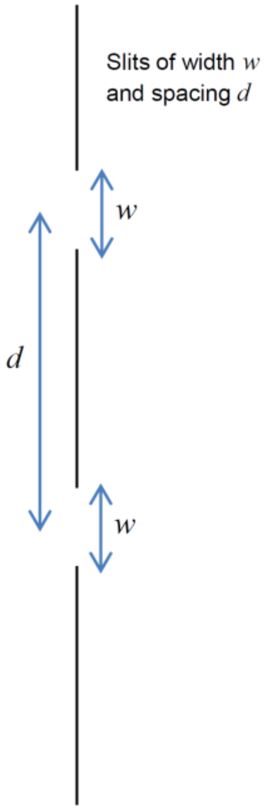


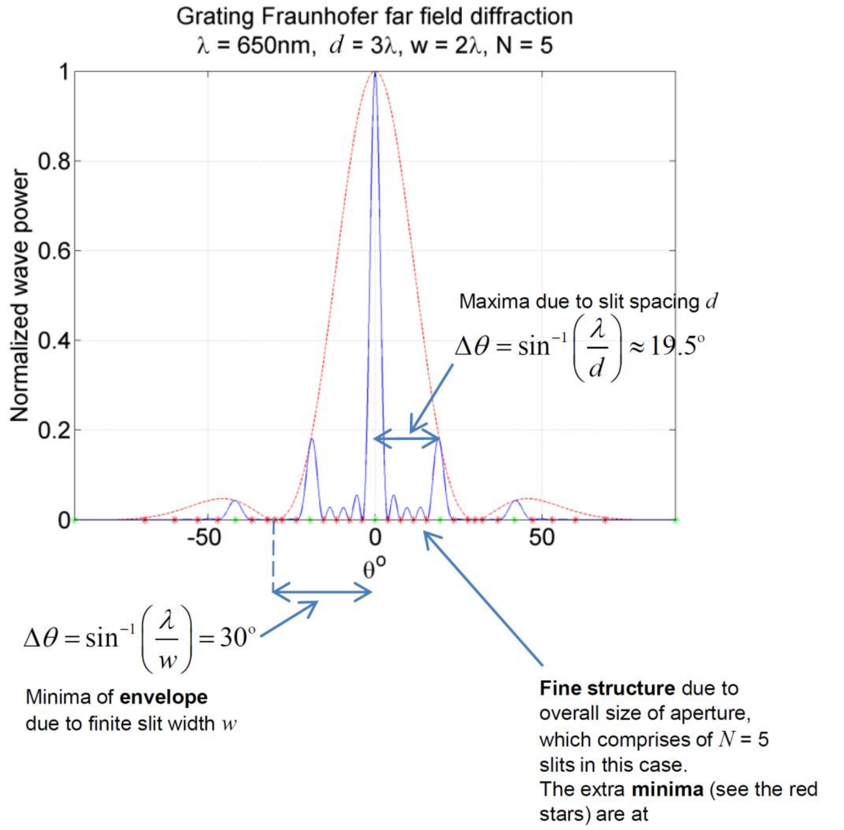
i.e. maxima at $\theta = \sin^{-1}(\frac{1}{5}n) = 0^\circ, \pm 11.5^\circ, \pm 23.6^\circ, \pm 36.9^\circ, \pm 53.1^\circ, \pm 90^\circ$



Real apertures have slightly modified interference patterns due to **(i) a *finite slit width*; (ii) *finite number of slits*; (iii) elevation as well as azimuth structure** if the aperture has two or three dimensional geometric relationships between the wave sources. The *far-field diffraction pattern* of a finite slit width w will result in an *envelope* multiplied by the 'grating pattern' produced above. Apart from a maxima at $\theta = 0^\circ$, there will be *additional zeros* when : $\boxed{\theta_m = \sin^{-1}(m\lambda/w)}$ where m is integer, but $\neq 0$.

If there are N slits, the slit-spacing 'grating pattern' will have much sharper peaks, and there will be *additional fine structure* (i.e. lots of extra little maxima) . There will be *additional zeros* when: $\boxed{\theta_p = \sin^{-1}(p\lambda/Nd)}$, although not when p/N is integer, as these correspond to the grating lobes when $\theta_n = \sin^{-1}(n\lambda/d)$.





Joseph von Fraunhofer
1787-1826

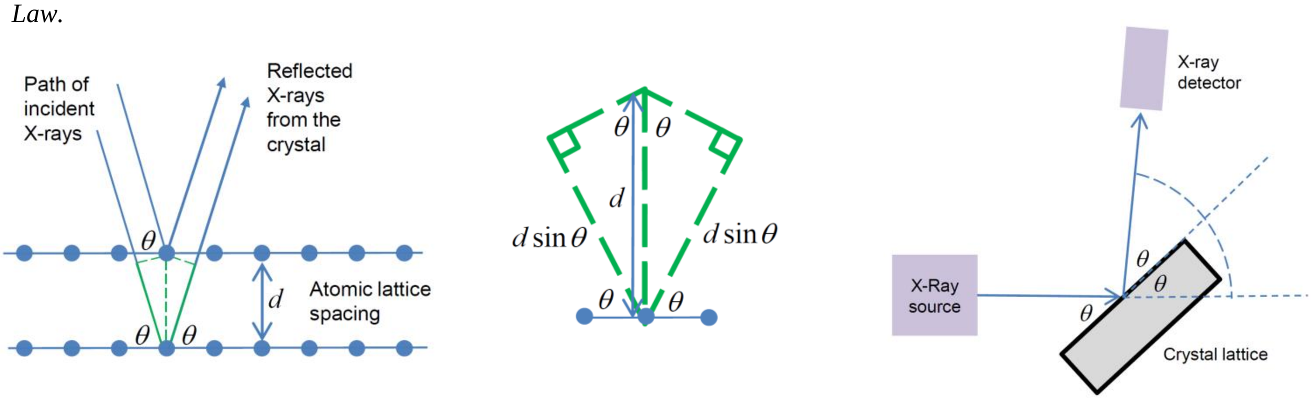
The actual formula for the full far-field *Fraunhofer* diffraction pattern power from N slits of width w spaced by distance d is:

$$\text{Power} \propto |\psi|^2 \propto \frac{1}{N^2 r^2}\left(\frac{\sin\left(\frac{\pi}{\lambda} w \sin\theta\right)}{\frac{\pi}{\lambda} w \sin\theta} \times \frac{\sin\left(\frac{\pi}{\lambda} N s \sin\theta\right)}{\sin\left(\frac{\pi}{\lambda} s \sin\theta\right)}\right)^2$$

Almost all of these diffraction effects result in a *main lobe* of angular width (in radians): $\boxed{\Delta\theta \approx \lambda/d}$ where *d* is a characteristic length of the grating, slit etc.

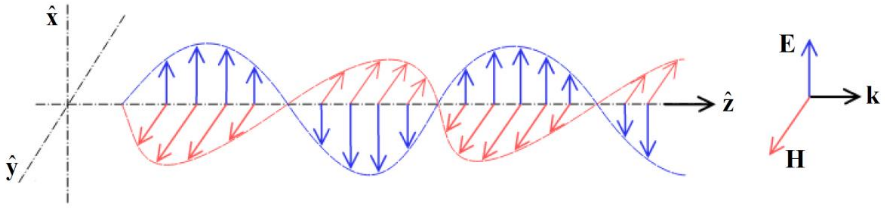
For any optical instrument, the 'resolving power' is likely to be diffraction limited. So this ratio gives *the minimum angular deviation* that two objects could be **resolved** via an optical system.

An interesting special case of wave interference is X-ray diffraction from atoms in a crystal lattice, which results in *Bragg's Law*.



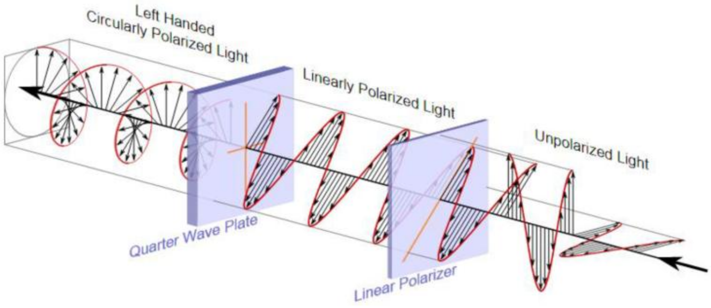
Bragg's Law: Constructive interference when: $\boxed{2d\sin\theta = n\lambda}$. Bragg's law can be used to determine atomic lattice spacing d since X-ray wavelengths λ can be determined from the properties of the source, and the diffraction angle can be measured via rotating a crystal (or detector).

Electromagnetic (EM)waves are *transverse* oscillations of mutually perpendicular electric $\mathbf{E}$ and magnetic fields[1] $\mathbf{H}$, which travel at the speed of light, divided by the refractive index n of the medium they propagate in. For a vaccum, $n = 1$, and for air $n \approx 1$.



For water, $n \approx 1.34$ and for glass, $n \approx 1.5$.

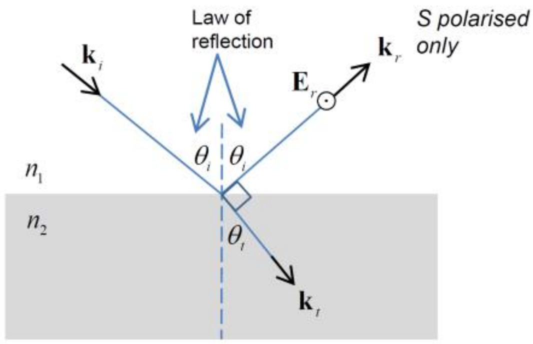
The plane of oscillation of the electric field is called the **polarization**. '*Linear polarization'* (e.g. *horizontal* or *vertical*) means this plane remains the same. However, it is possible to superimpose phase shifted EM or mutually perpendicular polarizations to form a net electric field vector which *rotates* as it propagates. If the magnitude of the electric field doesn't change this is called *circular polarization*.



Elliptical polarization is a more general situation.

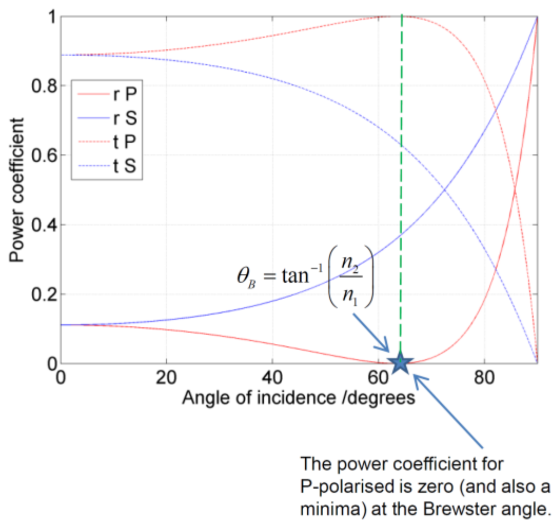
Consideration of polarization is practically useful in at least two main ways. Firstly, it may enable more information to be communicated using light at the same frequency; i.e. the polarization may give extra 'channels' given the same 'bandwidth' (i.e. range of frequencies).

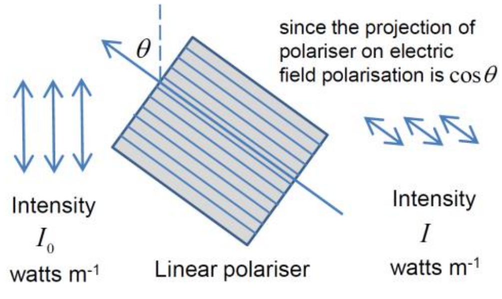
Secondly, reflection of EM waves of boundaries of differing refractive index (n_1, n_2) varies with polarization as well as angle of incidence.[2]



When the angle of incidence is the *Brewster angle* $\boxed{\theta_B = \tan^{-1}(n_2/n_1)}$,

only light with polarization *perpendicular* to the plane of the incident, reflected and transmitted wavevectors, is reflected. (This is called *S-polarized* light. *P-polarized* light, i.e. with an electric field *parallel* to the plane of the wavevector, is *not* reflected). This is very useful in *glare reduction* and is widely used in the design of sunglasses, windows and photographic lens filters.





If linear polarized light is incident upon a linear polarizer with polarization direction tilted by θ from the polarization plane of the incident light, the projection of the electric field on the polarizer is $E\cos\theta$ which means the intensity of light is proportional to $\cos^2\theta$. So for light incident upon crossed polarized rotated by angle θ, the intensity of light that makes it through is $\boxed{I = I_{\min} + (I_{\max} - I_{\min})\cos^2\theta}$. Ideally $I_{\min} = 0$. This is called ***Malus' Law.***

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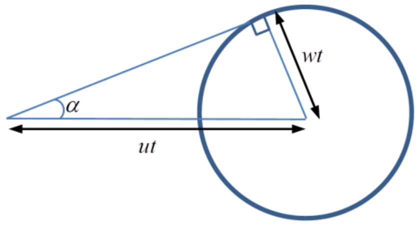
[1] Magnetic field strength is actually $\mathbf{B}$, but one must factor in magnetization of the media an electromagnetic wave passes through. For *isotropic* media (i.e the same magnetic properties in all directions), $\mathbf{H} \propto \mathbf{B}$.

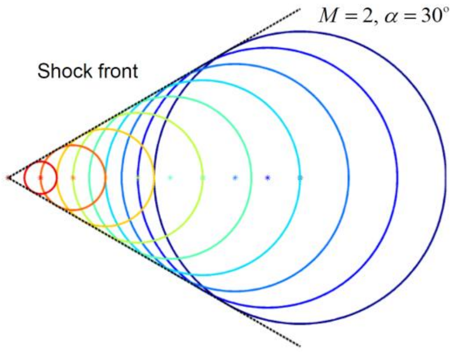
[2] The variation if known as the *Fresnel Equations*. http://www.eclecticon.info/index_htm_files/EM%20waves%20&%20polarization.pdf

The **Mach number** $M = u/w$ is the ratio between the wave source speed v and the speed of waves c. A Mach number greater than unity (e.g. for sound waves) implies a '*supersonic*' disturbance, which radiates as a **shock front**. In this scenario, the wave-medium beyond the shock front will be undisturbed by the wave source. The angle of the shock front (in 3D this will form a 'Mach cone') is given by: $\boxed{\alpha = \sin^{-1}(1/M)}$.

$$ut\sin\alpha = wt$$

$$\therefore \alpha = \sin^{-1}(w/u) = \sin^{-1}(1/M)$$

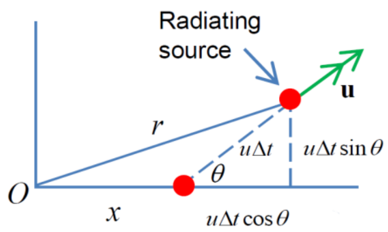


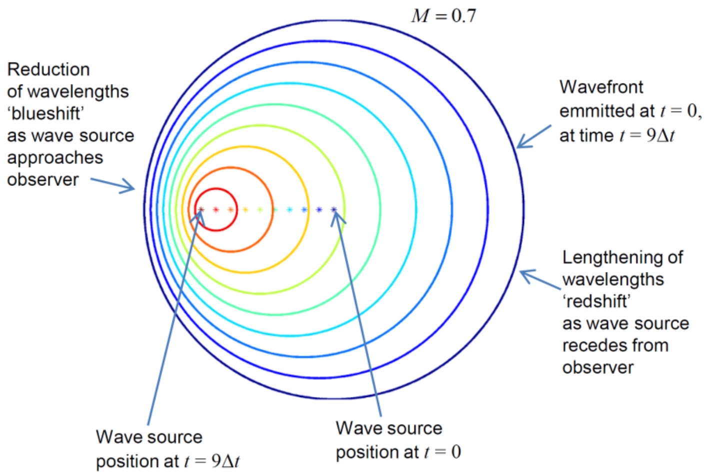


For Mach numbers < 1, there is *no* shock front. Instead, the apparent wavelengths are ***blue-shifted*** (i.e. shortened) as a wave-source approaches an observer, and wavelengths are ***red-shifted*** (i.e. lengthened) as a wave source recedes. This is called the **Doppler Effect.**

The period of waves received by an observer at O is:

$T = \Delta t + \dfrac{r - x}{w}$ where $\Delta t = 1/f$ is the time between the wave crests at the source.





$$r = \sqrt{(x + u\Delta t\cos\theta)^2 + u^2\Delta t^2\sin^2\theta} = \sqrt{x^2 + u^2\Delta t^2\cos^2\theta + 2ux\Delta t\cos\theta + u^2\Delta t^2\sin^2\theta} = \sqrt{x^2 + u^2\Delta t^2 + 2ux\Delta t\cos\theta}$$

$$\therefore x\sqrt{1 + 2\cos\theta\frac{u\Delta t}{x} + \left(\frac{u\Delta t}{x}\right)^2}$$

If $u\Delta t \ll x$; $r \approx x\sqrt{1 + 2\cos\theta\dfrac{u\Delta t}{x}} \approx x\left(1 + \cos\theta\dfrac{u\Delta t}{x}\right) = x + u\Delta t\cos\theta \therefore r - x \approx u\Delta t\cos\theta$.

Hence frequency of radiation received at O is: $F = 1/T$ where: $1/F = \Delta t + \frac{u}{w}\Delta t\cos\theta$. Hence: $F = \left(1 + \frac{u}{w}\cos\theta\right)^{-1} f$.

Let $\boxed{F = f + \Delta f}$, so define **Doppler shift:** $\boxed{\Delta f = -\dfrac{\frac{u}{w}\cos\theta}{1 + \frac{u}{w}\cos\theta} f}$.

Also define **redshift**: $\boxed{z = \dfrac{f - F}{F}} = \dfrac{-\Delta f}{f + \Delta f} \Rightarrow z = \frac{u}{w}\cos\theta \Rightarrow \boxed{z = \dfrac{\lambda_o}{\lambda_e} - 1}$, where wave speed $w = F\lambda_o = f\lambda_e$.

If $u \ll w,\ \theta = 0$: $\boxed{\Delta f \approx -\frac{u}{w} f}$.

Note in radar scenarios, a transmitted EM wave ($w = c$ i.e. the speed of light) is backscattered of a moving target and then received by an antenna, typically co-located at the same location as the transmitter. (Often they share the *same* antenna). In this case, the 'there-and-back' path means: $F = \left(1 + \frac{u}{w}\cos\theta\right)^{-2} f$ so if $u \ll w,\ \theta = 0$: $\boxed{\Delta f_{\text{radar}} \approx -\frac{2u\cos\theta}{c} f}$.

Question 1

(i) Two thin (width $w \ll \lambda$) slits of separation $d = 4\lambda$ are illuminated with laser light of wavelength 532nm in a darkened room. A flat wall is 3.00m away and three bright green spots are seen in a horizontal pattern either side of a central bright spot. Calculate the separation (in m) of the spots from the central bright spot.

(ii) Explain what would happen to the spot pattern in (i) (i.e. sketch and explain the *light intensity vs angle* graph) if:

(a) A violet laser (445nm) is used, but the double slits and their spacing kept the same.

(b) The slit widths are still thin, and 532nm light is still used, but a *diffraction grating* is used instead of a double slit, with 470 lines per mm.

(c) The double slit is swapped for a single slit of width $w = 3\lambda$, and the 532nm laser kept.

(iii) A diffraction grating labeled 250 lines per mm is illuminated with red light of frequency 650nm. The slits are about 100nm wide. A bright spot is formed on a wall 5.0m away, and several slightly less bright spots extend to the left and right of this. Calculate the distance from the central spot to place a pen so it is precisely illuminated by the 3rd order (i.e. $n = 3$) maxima. The grating is exchanged for one labeled 500 lines per mm. Where should the pen be moved to now so it covers the $n = 2$ maxima?

(iv) Calculate the angular resolution (in arc seconds) of (a) the human eye (max pupil diameter is about 7mm) and (b) the Hubble Space telescope (telescope diameter 2.4m), both using $\lambda = 500\text{nm}$. 3600 arc-seconds = 1 degree.

(v) Calculate the angular resolution (in arc-seconds) for The Five-Hundred-metre Aperture Spherical Telescope (FAST) in China, for its operational frequency range of 70MHz to 3.0GHz.

(vi) A crystal lattice of inter-atomic separation $d = 4.56 \times 10^{-10}\,\text{m}$ is illuminated with X-rays. A detector is placed on a turntable and rotated by angle θ from the direction of the incident X-rays. If the wavelength of the X-rays is 0.123nm, calculate the angles that correspond to peaks in the detector response vs θ.

(vii) Light from a lamp is shone through a pair of linear polarizers. When their respective planes of polarization are aligned, 100W/m^2 passes through and is detected via a lux meter. When they are rotated by 90^o from this alignment, 10W/m^2 passes through. Calculate the expected light intensity when the rotation is 30^o from the point of maximum light intensity received. What is the angle between the polarizers which results in a received intensity of 42W/m^2?

(viii) Derive and evaluate the Brewster angle (in degrees) for an air-glass interface. Assume $n_{air} = 1.00$ and $n_{glass} = 1.50$.

(ix) A siren for an emergency vehicle has an average pitch of 2,000Hz. If the vehicle is *approaching* at 30mph (13.4m/s), calculate the Doppler shift, assuming the speed of sound is 340m/s. Calculate the wavelength of the emitted sound from the siren, and compare this to the wavelength of the received sound by a stationary observer.

(x) The Andromeda galaxy is approaching the Milky Way at a velocity of about 110km/s. Calculate the redshift of 500nm light emitted by Andromeda, and hence work out the wavelength (in nm) observed on Earth.

(xi) The North American X-15, piloted by William J. Knight in 1967, achieved a record-breaking supersonic speed of Mach 6.70. Calculate the angle of the Mach cone (in degrees). Explain why an aircraft breaking the sound barrier over an urban area is more likely to be heard than a futuristic hypersonic aircaft passing over at a cruising speed of Mach 5.0.

(xii) The propeller rotation frequency for the WWII *Supermarine Spitfire* aircraft[3] is about 1431RPM (revolutions per minute). If *four times this* is the frequency of sound emitted, (since the Spitfire propeller has four blades) calculate the frequency heard by a *stationary observer on the ground* if: (a) the elevation angle of the spitfire is 30^o and the Spitfire is approaching; (b) the spitfire is directly overhead; (c) the elevation of the spitfire is 45^o and the spitfire is flying away. Assume the speed of sound is 343m/s, and the airspeed of the Spitfire is 169.9m/s (380mph).

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[3] http://www.mnealon.eosc.edu/SpitfirePropSpeed.htm

Question 2

(a) Draw an appropriate diagram to show why grating lobes for the diffraction pattern of a laser-illuminated grating of slit spacing s occur at angles: $\theta = \sin^{-1}\left(\frac{n\lambda}{s}\right)$

(b) A double-slit is illuminated by laser light of wavelength 650nm. The slit spacing is 3900nm.

Calculate the angles of the maxima and hence sketch the diffraction pattern (i.e. beam intensity vs angle in degrees) assuming the slit widths are about 50nm.

(c) A single slit of width 1950nm is illuminated by the same laser. Determine the angles of the minima, and hence sketch the diffraction pattern.

(d) Using the calculations and diffraction patterns from (b) and (c), determine the diffraction pattern of a double slit of slit spacing 3900nm, of slit width 1950nm.

(e) A diffraction grating is labeled '256 lines per mm'. Sketch the expected diffraction pattern when illuminated with 650nm light, and comment on the major difference between your diagram for (d).

(f) Explain how the diffraction pattern of the grating will change if green light is used.

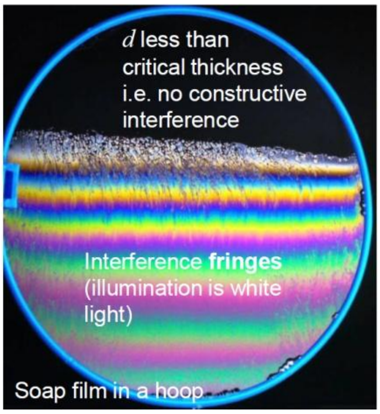
Question 3

A *Culture* General Service Vehicle [4]*AU Finger* is in orbit about our Sun diametrically opposite the Earth. To experience the day-night pattern of Earth without having to rotate the spacecraft, engineers propose slowly rotating polarizers in the windows.

(a) Explain why a single rotating polarizer won't be sufficient. Two are required, and only one rotates.

(b) Sketch a graph of light intensity through the polarizers vs angle, and rotation angle vs time /hours. What is the rotation rate (in degrees per hour) of the polarizer?

Question 4



If a soap film in a metal hoop is illuminated by a diffuse white light source, one will observe coloured fringes in the light reflected off the film. After a short time a dark region will appear above the fringes. This will typically grow until the film pops.

(a) By considering light perpendicular to the, explain why constructive interference occurs when the phase difference between light reflecting off the near and far soap bubble surfaces is given by: $\Delta\phi = 2\pi m = 2\pi \times \frac{2d}{\lambda_{film}} + \pi$ where m is an integer, d is the thickness of the soap bubble and λ_{film} is the wavelength of the light inside the film

(b) Explain why $\lambda_{film} = \frac{\lambda}{n}$, where λ is the wavelength of light in air, and n is the refractive index of the soap film.

(c) Determine an expression for the minimum thickness $d_{\min}$ of the soap film for constructive interference, and evaluate it for violet light (380nm) and red light (680nm). Assume the refractive index of the soap film is $n = 1.33$ i.e. ignore any variations of n with light frequency.

(d) How thin must glass of refractive index $n = 1.50$ be to look a purple-blue* colour in daylight?

(* Blue is 380nm to 500nm)

(e) In a soap bubble demonstration, $m = 5$ red fringes are observed. What is the maximum thickness of the soap film?

[4] Inspired by the excellent science fiction books by Ian M. Banks

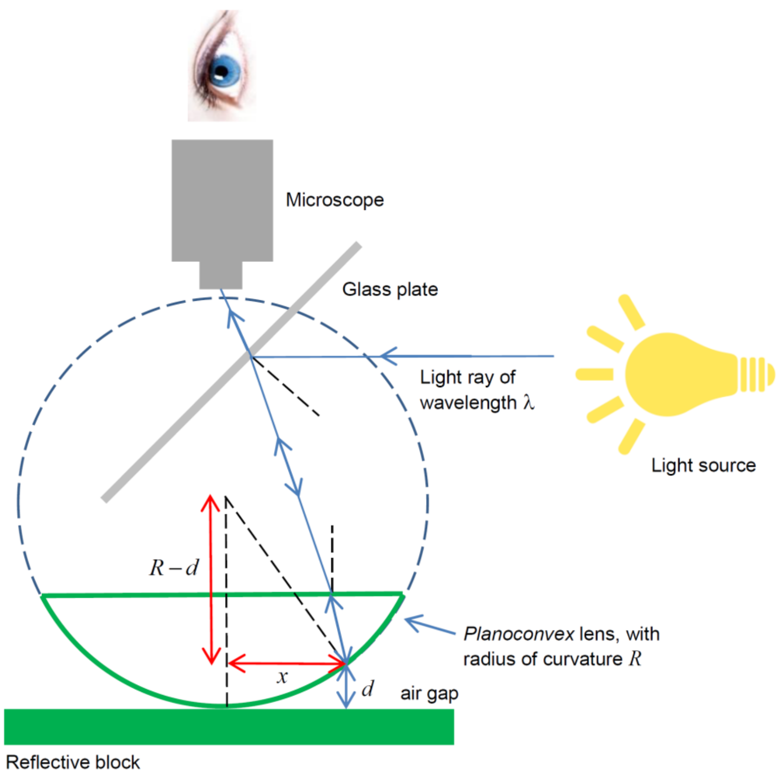
Question 5 Wolfgang has a friend, Johann, who likes to sing loudly (always the same note) while driving his effectively silent new Tesla cabriolet with the top down. Johann is driving round a circuit at constant speed v km/h.

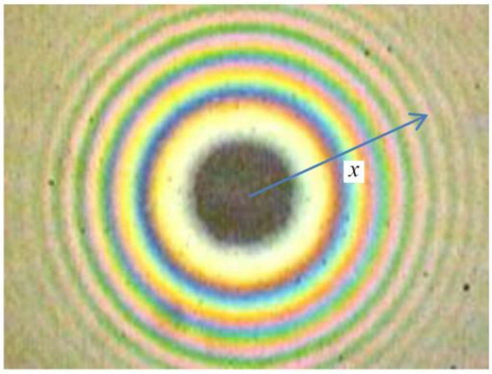
When Johann is approaching, Wolfgang (who has perfect pitch) determines the note to be F, whereas when he is receding, the note is C.

F is $440\times 2^{\frac{3}{12}}$ Hz whereas $440\times 2^{\frac{9}{12}}$ Hz. Assume A is 440Hz, and *semitones* A, Bb, B, C, C#, D, Eb, E, F, F#, G, G# are in a *geometric* series, with multiplier $\times\sqrt[12]{2}$.

How fast is Johannes travelling, and what is the nearest note that Johannes is singing?. You can assume the speed of sound is: $c = 342\text{ms}^{-1}$.

Question 6 A spherical lens of radius of curvature R (think a horizontal slice from the top of a sphere of radius R) is placed on a reflective block as show in the diagram below. When observed from above using a microscope, a pattern of concentric ring are observed. (These are called '*Newton's Rings*'). If a white-light source is used, the rings have coloured fringes, with purple-blue being at slightly smaller radii than the red rings. Use the diagram below to show that the radii of rings is given by $x = \sqrt{R\lambda\left(n - \tfrac{1}{2}\right)}$ where $n = 1,2,3,4,5...$ If $\lambda = 400\text{nm}$ and the 5th ring has radius $x = 3.14\text{mm}$, calculate the radius of curvature of the lens. Assume $R \gg d$.





Question 7

The wavefield from the superposition of identical spherical wave sources is given by: $\psi(r,t) = \dfrac{Ae^{i(kp-\omega t)}}{p} + \dfrac{Ae^{i(kq-\omega t)}}{q}$.

Use the *cosine rule* and apply *binomial expansion*(s) in the far-field limit of $r \gg d, \lambda$

to show that: $|\psi|^2 \approx \dfrac{4A^2}{r^2}\cos^2\left(\tfrac{1}{2}kd\sin\theta\right)$.

Explain why the resulting graph of $|\psi|^2$ vs θ will have *maxima* when: $\theta = \sin^{-1}\left(n\lambda/d\right)$.