

An Introduction to contour integration in the complex plane. A French May 2013

Motivational example:

Prove:

$$\int_0^{\infty} \frac{x \cos x}{1+x^2} dx = \frac{\pi}{e} \quad [1]$$

Results needed:

[1]

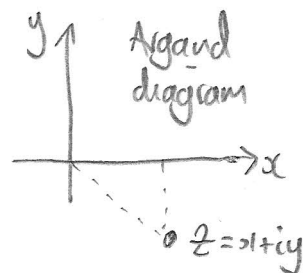
De Moivre's theorem

$$e^{i\theta} = \cos\theta + i\sin\theta \quad [2]$$

[2] The residue theorem

$$\oint_C f(z) dz = 2\pi i \sum_j R_j \quad [3]$$

- * $f(z)$ is a function of complex variable $z = x + iy$
- * $f(z)$ is continuous along closed contour C
- * $f(z)$ is analytic (i.e. has no singularities ("infinities")) except from a finite # of poles, indexed by j



The residue R_j at pole j is the a_{-1} term of the Laurent series expansion of $f(z)$ about pole z_0 of order m

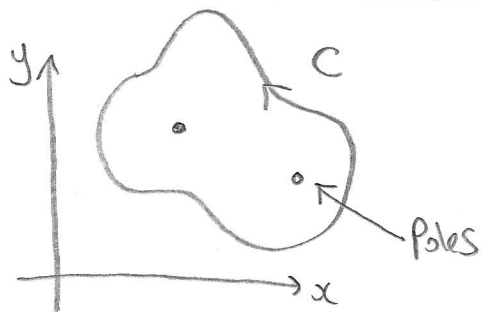
$$f(z) = \sum_{n=-m}^{\infty} a_n (z-z_0)^n \quad [4]$$

eg/ $f(z) = \frac{1}{(z-(1+i))^3}$ has a 3rd order pole at $z_0 = 1+i$

We can differentiate [4] to yield a formula for residue R_j at pole $z_0(j)$

[5]

$$R_j = \lim_{z \rightarrow z_0(j)} \left\{ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left((z-z_0(j))^m f(z) \right) \right\}$$



Note if there are no poles within C then

[6]

$$\oint_C f(z) dz = 0 \quad \text{regardless of } C$$

This is

Cauchy's Theorem

2) Consider the complex function $f(z) = \frac{ze^{iz}}{1+z^2} = \frac{ze^{iz}}{(z+i)(z-i)}$

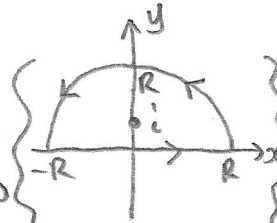
This has "simple" poles at $z = \pm i$ since $1+z^2 = (z+i)(z-i)$
(i.e. order of poles = 1)

\therefore Residues are $R_{z_0 = \pm i} = \lim_{z \rightarrow z_0} \left\{ (z-z_0)f(z) \right\}$

$$= \begin{cases} \frac{ie^{i(i)}}{i+i} & z_0 = i \\ \frac{-ie^{i(-i)}}{-i-i} & z_0 = -i \end{cases}$$

$$= \begin{cases} \frac{1}{2}e^{-1} & z_0 = i \\ \frac{1}{2}e^1 & z_0 = -i \end{cases}$$

Now consider the contour integral

$$\oint_C f(z) dz = \lim_{R \rightarrow \infty} \left\{ \int_{-R}^R f(z) dz + \int_{\text{arc}} f(z) dz \right\}$$


By the residue theorem:

$$\oint_C \frac{ze^{iz}}{1+z^2} dz = 2\pi i \times \frac{1}{2}e^{-1} = \boxed{\pi i e^{-1}}$$

Now $\oint_C f(z) dz = \int_{-\infty}^{\infty} f(z) dz + \int_{\text{arc}} f(z) dz$

$$\int_{\text{arc}} f(z) dz = \lim_{R \rightarrow \infty} \left\{ \int_0^\pi \frac{Re^{i\theta} e^{i(R\cos\theta + iR\sin\theta)} \cdot iRe^{i\theta} d\theta}{1+R^2e^{2i\theta}} \right\}$$

[since $z = Re^{i\theta} = R(\cos\theta + i\sin\theta) \therefore dz = iRe^{i\theta} d\theta$]

$$\int_{\text{arc}} f(z) dz = \lim_{R \rightarrow \infty} \left\{ \int_0^\pi i e^{-R\sin\theta} e^{iR\cos\theta} d\theta \right\}$$

$$= 0$$

NOTE
 $e^{iR\cos\theta}$ is a high frequency term, magnitude ≤ 1

