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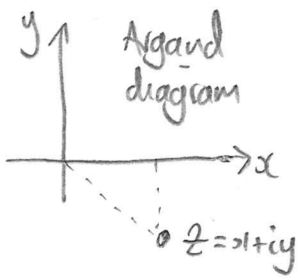
An Introduction to contour integration in the complex plane . A French May 2013

Motivational example: Prove: $$\boxed{\int_0^\infty \frac{2x\sin x}{1+x^2}\,dx = \frac{\pi}{e}} \quad [1]$$

Results needed: $\boxed{1}$ De Moivre theorem $$\boxed{e^{i\theta} = \cos\theta + i\sin\theta} \quad [2]$$

$\boxed{2}$ The residue theorem $$\boxed{\oint_C f(z)\,dz = 2\pi i \sum_j R_j} \quad [3]$$



Argand diagram

* $f(z)$ is a function of complex variable $z = x+iy$
* $f(z)$ is continuous along closed contour C
* $f(z)$ is analytic (ie has no singularities ("infinities") except from a finite # of poles, indexed by j

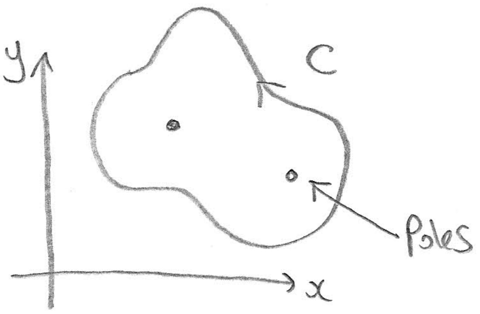
The residue R_j at pole j is the a_{-1} term of the Laurent series expansion of $f(z)$ about pole z_0 of order m

$$\boxed{f(z) = \sum_{n=-m}^{\infty} a_n (z-z_0)^n} \quad [4]$$

eg/ $f(z) = \dfrac{1}{(z-(1+i))^3}$ has a 3rd order pole at $z_0 = 1+i$

We can differentiate [4] to yield a formula for residue R_j at pole $z_0(j)$

$$[5] \quad \boxed{R_j = \lim_{z\to z_0(j)} \left\{ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left((z-z_0(j))^m f(z) \right) \right\}}$$



Note if there are no poles within C then $$[6] \quad \boxed{\oint_C f(z)\,dz = 0}$$ regardless of C

This is $\boxed{\text{Cauchy's Theorem}}$

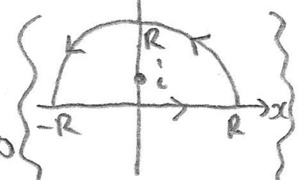
(2) Consider the complex function $f(z) = \dfrac{z e^{i\alpha z}}{1+z^2} = \dfrac{z e^{i\alpha z}}{(z+i)(z-i)}$

This has "simple" poles at $z = \pm i$ since $1+z^2 = (z+i)(z-i)$
(ie order of poles $= 1$)

$\therefore$ Residues are $R_{z_0=\pm i} = \lim\limits_{z\to z_0} \left\{ (z-z_0) f(z) \right\}$

$$= \begin{cases} \dfrac{i e^{i\alpha(i)}}{i+i} & z_0 = i \\ \dfrac{-i e^{i\alpha(-i)}}{-i-i} & z_0 = -i \end{cases}$$

$$= \begin{cases} \frac{1}{2} e^{-\alpha} & z_0 = i \\ \frac{1}{2} e^{\alpha} & z_0 = -i \end{cases}$$

Now consider the contour integral $\oint_C f(z)\,dz = \lim\limits_{R\to\infty} \left\{ \right.$  $\left. \right\}$

By the residue theorem:

$$\oint_C \frac{z e^{i\alpha z}}{1+z^2}\,dz = 2\pi i \times \frac{1}{2} e^{-\alpha} = \boxed{\pi i e^{-\alpha}}$$

Now $\oint_C f(z)\,dz = \int_{-\infty}^{\infty} f(z)\,dz + \int_{\frown} f(z)\,dz$

$$\int_{\frown} f(z)\,dz = \lim_{R\to\infty} \left\{ \int_0^{\pi} \frac{R e^{i\theta} e^{i\alpha(R\cos\theta + iR\sin\theta)}\, iRe^{i\theta}\,d\theta}{1 + R^2 e^{2i\theta}} \right\}$$

[since $z = Re^{i\theta} = R(\cos\theta + i\sin\theta)$ $\therefore dz = iRe^{i\theta}\,d\theta$]

$$\therefore \int_{\frown} f(z)\,dz = \lim_{R\to\infty} \left\{ \int_0^{\pi} i e^{-R\alpha\sin\theta} e^{i\alpha R\cos\theta}\,d\theta \right\}$$

$$= 0$$

$\boxed{\text{NOTE}}$

$\left\{ e^{i\alpha R\cos\theta} \right.$ is a high frequency term, magnitude ≤ 1

(3)

Hence $\displaystyle\int_{-\infty}^{\infty} f(z)\,dz = \pi i e^{-\alpha}$

$$\Rightarrow \int_{-\infty}^{\infty} \frac{z\cos\alpha z}{1+z^2}\,dz + i\int_{-\infty}^{\infty} \frac{z\sin\alpha z}{1+z^2}\,dz = \pi i e^{-\alpha}$$

using $e^{i\alpha z} = \cos\alpha z + i\sin\alpha z$

$$\Rightarrow \boxed{\int_{-\infty}^{\infty} \frac{z\sin\alpha z}{1+z^2}\,dz = \pi e^{-\alpha}}$$

and $$\boxed{\int_{-\infty}^{\infty} \frac{z\cos\alpha z}{1+z^2}\,dz = 0}$$

Now using substitution $u = -z$

$$\int_{-\infty}^{0} \frac{z\sin\alpha z}{1+z^2}\,dz = \int_{\infty}^{0} \frac{(-u)\sin(-\alpha u)}{1+u^2}(-du)$$

$$= -\int_{\infty}^{0} \frac{u\sin\alpha u\,du}{1+u^2}$$

$$= \int_{0}^{\infty} \frac{u\sin\alpha u\,du}{1+u^2}$$

$$\therefore \boxed{\int_{-\infty}^{\infty} \frac{z\sin\alpha z}{1+z^2}\,dz = 2\int_{0}^{\infty} \frac{z\sin\alpha z}{1+z^2}\,dz}$$

$$\therefore \boxed{\int_{0}^{\infty} \frac{z\sin\alpha z}{1+z^2}\,dz = \frac{\pi}{2}e^{-\alpha}}$$

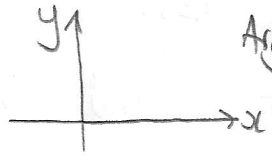
$\therefore$ if $\alpha = 1 \Rightarrow$ $$\boxed{\int_{0}^{\infty} \frac{2z\sin z}{1+z^2}\,dz = \pi/e}$$

(4) <u>Derivation of the residue theorem</u>

$$\oint_C f(z)\,dz = 2\pi i \sum_j R_j \qquad \text{where } R_j = \lim_{z \to z_0^{(j)}} \left\{ \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left((z-z_0^{(j)}) f(z) \right) \right\}$$

First we must prove the <u>Cauchy-Riemann equations</u>

let $\quad f(z) = u(x,y) + i\,v(x,y)$



Argand diagram

$\quad z = x + iy$

Now
$$\frac{df}{dz} = \lim_{\Delta x \to 0} \left\{ \frac{u(x+\Delta x, y) + i v(x+\Delta x, y) - u(x,y) - i v(x,y)}{\Delta x} \right\}$$

$$= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x}$$

But $\frac{df}{dz}$ must also equal $\lim_{\Delta y \to 0} \left\{ \frac{u(x,y+\Delta y) + i v(x,y+\Delta y) - u(x,y) - i v(x,y)}{i \Delta y} \right\}$

ie the definition of $\frac{df}{dz}$ should be independent of the direction in the Argand diagram that the limit is approached.

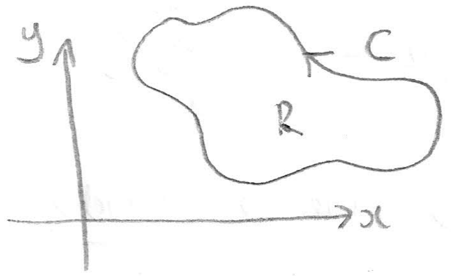
The latter version of $\frac{df}{dz} = \frac{1}{i}\left(\frac{\partial u}{\partial y} + i \frac{\partial v}{\partial y} \right) = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$

Hence
$$\frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i \frac{\partial u}{\partial y}$$

$$\Rightarrow \quad \boxed{\begin{aligned} \frac{\partial u}{\partial x} &= \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= -\frac{\partial u}{\partial y} \end{aligned}}$$

These are the Cauchy-Riemann equations for a <u>differentiable</u> function of complex variable z
$f(z) = u + iv$

We can now use <u>Green's theorem in a plane</u> to prove <u>Cauchy's theorem</u>



(R is bounded by C)

Consider $\oint_C f(z)dz$, a closed contour integral of $f(z)$.

$f(z) = u + iv$ and $dz = dx + i\,dy$ since $z = x + iy$

$$\therefore \oint_C f(z)dz = \oint_C (u+iv)(dx+i\,dy) = \oint_C (u\,dx - v\,dy) + i\oint_C (v\,dx + u\,dy)$$

Now $\boxed{\text{Green's theorem in a plane}}$ states for functions p and q with <u>continuous first derivatives</u> in and on C, (which bounds R)

$$\iint_R \left(\frac{\partial p}{\partial x} + \frac{\partial q}{\partial y}\right) dx\,dy = \oint_C (p\,dy - q\,dx)$$

Hence
$$\oint_C (u\,dx - v\,dy) = -\oint_C (v\,dy - u\,dx) = -\iint_R \left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) dx\,dy$$

$$\oint_C (v\,dx + u\,dy) = \oint_C (u\,dy - (-v)\,dx) = \iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right) dx\,dy$$

Now from the Cauchy-Riemann equations:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \Rightarrow \quad \frac{\partial u}{\partial x} - \frac{\partial v}{\partial y} = 0$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \quad \Rightarrow \quad \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} = 0$$

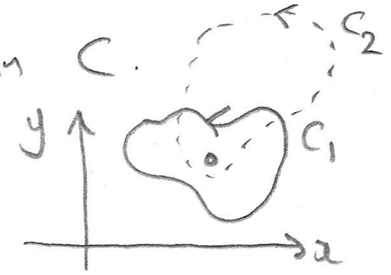
Hence if $\oint_C f(z)dz = \iint_R \left\{ -\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) + i\left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\right\} dx\,dy$

$$\Rightarrow \quad \boxed{\oint_C f(z)dz = 0} \qquad \underline{\text{Cauchy's theorem}}$$

i.e. regardless of the shape of C!

(6) So what happens if there are poles, (infinities) lurking within C?

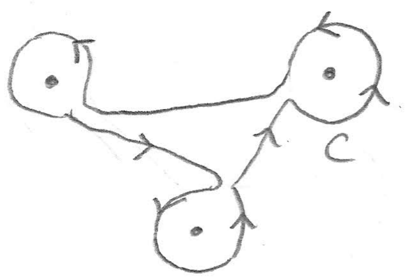
Well, it turns out* $\oint_C f(z)\,dz$ is the same as long as the poles are contained within C.



eg if • is a pole

$$\oint_{C_1} f(z)\,dz = \oint_{C_2} f(z)\,dz.$$

$\therefore$ we can always define C as a circle centred on each pole + the connections between the circles.



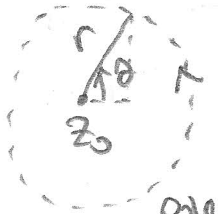
In otherwords $\oint_C f(z)\,dz$

is $\underbrace{\oint_C f(z)\,dz}_{\text{no poles}} + \sum_{\text{poles}} \oint_{\text{circle round pole}}$

The first contour is zero by Cauchy's theorem

$$\therefore \oint_C f(z)\,dz = \sum_{\substack{\text{poles}\\ j}} \int_0^{2\pi} f(z_0^{(j)} + re^{i\theta})\, ire^{i\theta}\,d\theta$$

$$\left[\; z = z_0^{(j)} + re^{i\theta} \quad \therefore dz = ire^{i\theta}\,d\theta \right]$$



pole contour

Now let $f(z) = \sum_{n=-m}^{\infty} a_n (z - z_0^{(j)})^n$ (Laurent expansion about pole z_0)

$$\therefore \oint_C f(z)\,dz = \sum_j \sum_{n=-m}^{\infty} a_n \int_0^{2\pi} (re^{i\theta})^n\, ire^{i\theta}\,d\theta$$

$$= \sum_j \sum_{n=-m}^{\infty} a_n i r^{n+1} \int_0^{2\pi} e^{i(n+1)\theta}\,d\theta$$

Assumes pole of order m

eg $f(z) = \frac{1}{(z-i)^m}$

$z_0 = i$

* I've never proved this. It might be quite hard!

Now $\displaystyle\int_0^{2\pi} e^{i(n+1)\theta}\, d\theta = \begin{cases} 2\pi & n = -1 \\ 0 & n \neq -1 \end{cases}$

$$\left[\text{if } n \neq -1 : \int_0^{2\pi} e^{i(n+1)\theta}\, d\theta = \left[\frac{1}{i(n+1)} e^{i(n+1)\theta} \right]_0^{2\pi} \right.$$

$$= \frac{1}{i(n+1)} \left\{ \left(e^{2\pi i}\right)^{n+1} - e^0 \right\}$$

$$\left. = 0 \right]$$

Hence $\boxed{\oint_C f(z)\, dz = \sum_j 2\pi i\, a_{-1}}$

Now if $\displaystyle f(z) = \sum_{n=-M}^{\infty} a_n \left(z - z_0^{(j)}\right)^n$

$$\left(z - z_0^{(j)}\right)^M f(z) = \sum_{n=-M}^{\infty} a_n \left(z - z_0^{(j)}\right)^{n+M}$$

$$= a_{-M} + a_{-M+1}\left(z - z_0^{(j)}\right) + a_{-M+2}\left(z - z_0^{(j)}\right)^2 + \dots a_{-1}\left(z - z_0^{(j)}\right)^{M-1}$$
$$+ \dots\dots$$

$$\therefore \frac{d^{M-1}}{dz^{M-1}} \left\{ \left(z - z_0^{(j)}\right)^M f(z) \right\} = (M-1)!\, a_{-1} + \sum_{n=1}^{\infty} b_n \left(z - z_0^{(j)}\right)^n$$

This $\to 0$ as $z \to z_0^{(j)}$

who cares what these are!

$$\therefore \lim_{z \to z_0^{(j)}} \left\{ \frac{d^{M-1}}{dz^{M-1}} \left(\left(z - z_0^{(j)}\right)^M f(z) \right) \right\} = (M-1)!\, a_{-1}$$

$$\Rightarrow \boxed{a_{-1} = \lim_{z \to z_0^{(j)}} \left\{ \frac{1}{(M-1)!} \frac{d^{M-1}}{dz^{M-1}} \left(\left(z - z_0^{(j)}\right)^M f(z) \right) \right\}}$$

This is called the <u>residue</u> of $f(z)$ at pole $z_0^{(j)}$