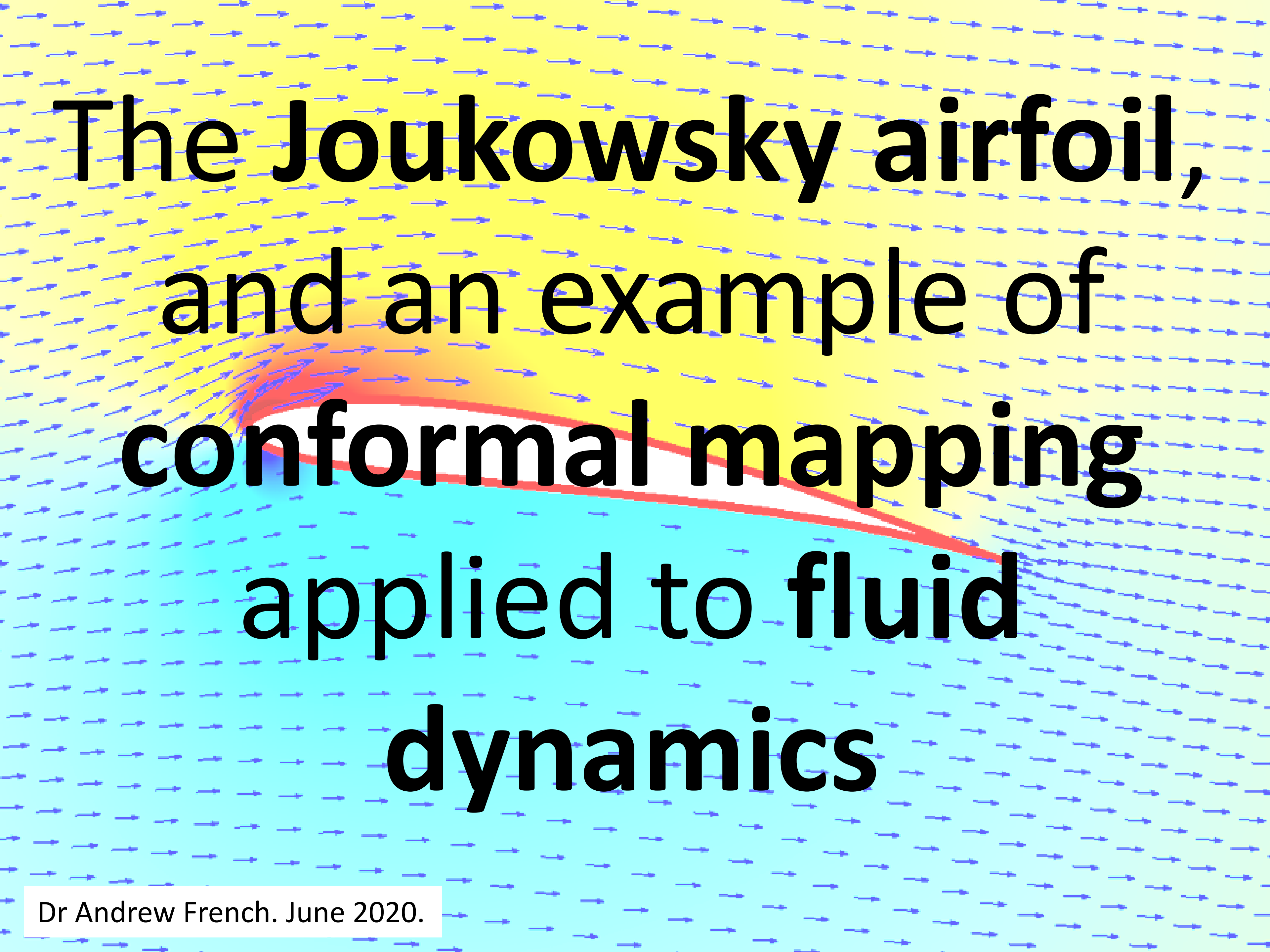
The **Joukowsky airfoil**, and an example of **conformal mapping** applied to **fluid dynamics**



Dr Andrew French. June 2020.

Cauchy-Riemann equations

$$z = x + iy$$

$$w = \phi + i\psi$$

$$\frac{dw}{dz} = \lim_{\Delta x \to 0}\left(\frac{\phi(x+\Delta x, y) + i\psi(x+\Delta x, y) - \phi(x,y) - i\psi(x,y)}{\Delta x}\right)$$

$$\boxed{\frac{dw}{dz} = \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x}}$$

$$\frac{dw}{dz} = \lim_{\Delta y \to 0}\left(\frac{\phi(x, y+\Delta y) + i\psi(x, y+\Delta y) - \phi(x,y) - i\psi(x,y)}{i\Delta y}\right)$$

$$\frac{dw}{dz} = \frac{1}{i}\left(\frac{\partial \phi}{\partial y} + i\frac{\partial \psi}{\partial y}\right) = \boxed{\frac{\partial \psi}{\partial y} - i\frac{\partial \phi}{\partial y}}$$

Now derivative should be independent of the direction in the Argand diagram that the limit is approached

$$\therefore \frac{\partial \phi}{\partial x} + i\frac{\partial \psi}{\partial x} = \frac{\partial \psi}{\partial y} - i\frac{\partial \phi}{\partial y}$$

Compare real and imaginary parts

$$\therefore \frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$\therefore \frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y}$$

Consider a complex function *w(z)*

$$z = x + iy$$

$$w(z) = \phi(x,y) + i\psi(x,y)$$

$$dw = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + i\frac{\partial \psi}{\partial x}dx + i\frac{\partial \psi}{\partial y}dy$$

$$dw = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + i\left(-\frac{\partial \phi}{\partial y}\right)dx + i\left(\frac{\partial \phi}{\partial x}\right)dy$$

Cauchy-Riemann

$$dw = \frac{\partial \phi}{\partial x}(dx + idy) + \frac{\partial \phi}{\partial y}(-idx + dy)$$

$$dw = \frac{\partial \phi}{\partial x}(dx + idy) - i\frac{\partial \phi}{\partial y}(dx + idy)$$

$$dw = \left(\frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}\right)dz$$

$$\therefore \boxed{\frac{dw}{dz} = \frac{\partial \phi}{\partial x} - i\frac{\partial \phi}{\partial y}}$$

Velocity, if *potential flow* in two dimensions

$$\mathbf{v} = v_x\hat{\mathbf{x}} + v_y\hat{\mathbf{y}}$$

$$\mathbf{v} = \nabla\phi \quad \text{if} \quad \nabla \times \mathbf{v} = 0$$ No *vorticity*

$$\therefore v_x = \frac{\partial \phi}{\partial x},\ v_y = \frac{\partial \phi}{\partial y}$$

Hence:

$$z = x + iy$$

$$w = \phi + i\psi$$

$$\frac{dw}{dz} = v_x - iv_y$$

Recipe:

Find *potential* ϕ
and *stream function* ψ
and hence *w(z)*.
Differentiate and hence find velocity field $\mathbf{v}$

A note about streamline function ψ

$$\psi = \psi(x,y)$$

$$\therefore d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy$$

If $\psi = \text{constant} \Rightarrow d\psi = 0$

$$\therefore \frac{\partial \psi}{\partial x}dx = -\frac{\partial \psi}{\partial y}dy$$

Cauchy Riemann

$$\frac{\partial \phi}{\partial x} = \frac{\partial \psi}{\partial y}$$

$$\frac{\partial \psi}{\partial x} = -\frac{\partial \phi}{\partial y}$$

Since $w = \phi + i\psi$

Hence:

$$\frac{\partial \psi}{\partial x}dx = -\frac{\partial \psi}{\partial y}dy$$

$$\Rightarrow -\frac{\partial \phi}{\partial y}dx = -\frac{\partial \phi}{\partial x}dy$$

$$\Rightarrow \frac{\partial \phi}{\partial y}dx = \frac{\partial \phi}{\partial x}dy$$

If on a **streamline**, you are always *parallel to the local velocity vector*

$$d\mathbf{r} = dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}}$$ Vector displacement along streamline

$$\mathbf{v} = \nabla\phi = \frac{\partial \phi}{\partial x}\hat{\mathbf{x}} + \frac{\partial \phi}{\partial y}\hat{\mathbf{y}}$$ Assume potential flow

$$\mathbf{v} \times d\mathbf{r} = 0$$ if vector displacement along streamline is parallel to velocity

$$\therefore \left(\frac{\partial \phi}{\partial x}\hat{\mathbf{x}} + \frac{\partial \phi}{\partial y}\hat{\mathbf{y}}\right) \times \left(dx\hat{\mathbf{x}} + dy\hat{\mathbf{y}}\right) = 0$$

$$\therefore \frac{\partial \phi}{\partial x}dx\hat{\mathbf{x}} \times \hat{\mathbf{x}} + \frac{\partial \phi}{\partial x}dy\hat{\mathbf{x}} \times \hat{\mathbf{y}} + \frac{\partial \phi}{\partial y}dx\hat{\mathbf{y}} \times \hat{\mathbf{x}} + \frac{\partial \phi}{\partial y}dy\hat{\mathbf{y}} \times \hat{\mathbf{y}} = 0$$

$$\therefore \frac{\partial \phi}{\partial x}dy\hat{\mathbf{x}} \times \hat{\mathbf{y}} - \frac{\partial \phi}{\partial y}dx\hat{\mathbf{x}} \times \hat{\mathbf{y}} = 0$$

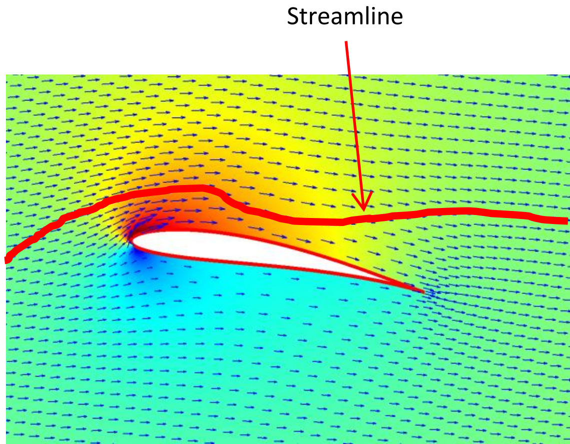
$$\therefore (\hat{\mathbf{x}} \times \hat{\mathbf{y}})\left(\frac{\partial \phi}{\partial x}dy - \frac{\partial \phi}{\partial y}dx\right) = 0$$

$$\therefore \frac{\partial \phi}{\partial x}dy = \frac{\partial \phi}{\partial y}dx$$

So if you know *w(z)*

$$\psi = \text{Im}(w)$$

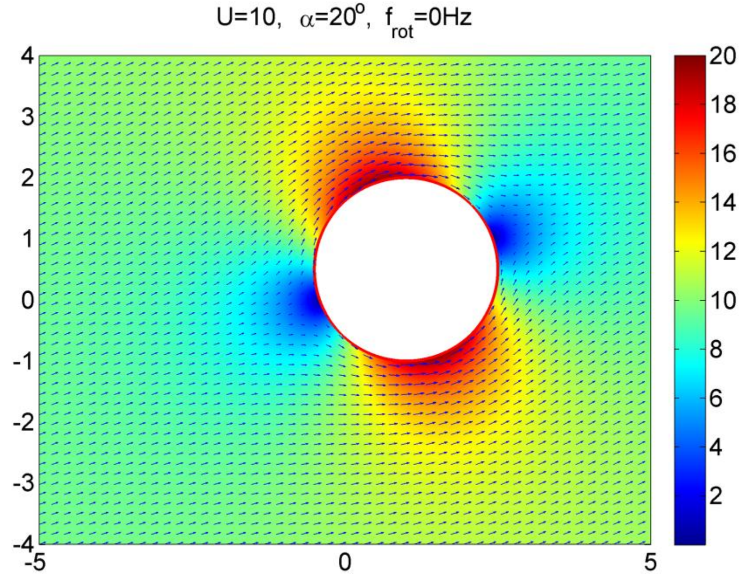
$$\phi = \text{Re}(w)$$



Complex potential for 2D flow around a cylinder of radius R, with far-field velocity U at angle α, and clockwise rotation frequency f

$$z = x + iy; \quad z_c = x_c + iy_c$$

$$z \to z - z_c$$ Shift so centre of circle is the origin



In this case *no circulation*, so **symmetric** top and bottom of the cylinder, relative to uniform flow direction.

Tangential velocity if no uniform flow

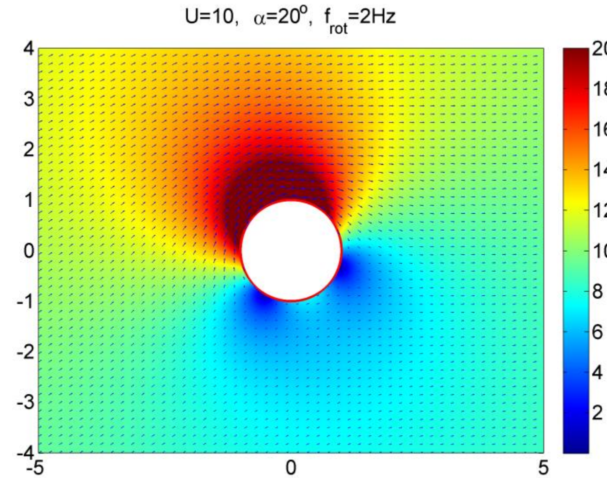
$$2\pi f_{rot} \times R = \frac{\Gamma}{2\pi R} \qquad \therefore \Gamma = 4\pi^2 R^2 f_{rot}$$ Define **circulation**

$$w = \phi + i\psi$$

$$w = Uze^{-i\alpha} + \frac{i\Gamma}{2\pi}\ln z + \frac{UR^2 e^{i\alpha}}{z}$$

Uniform flow — Vortex — Source

It is circulation which gives rise to the *Magnus effect* i.e. a **force** on a rotating object in a uniform flow. Mathematically equivalent to the force on a current in a magnetic field! Without circulation there is no net lift (or indeed drag – but that is because of the lack of *turbulence*).



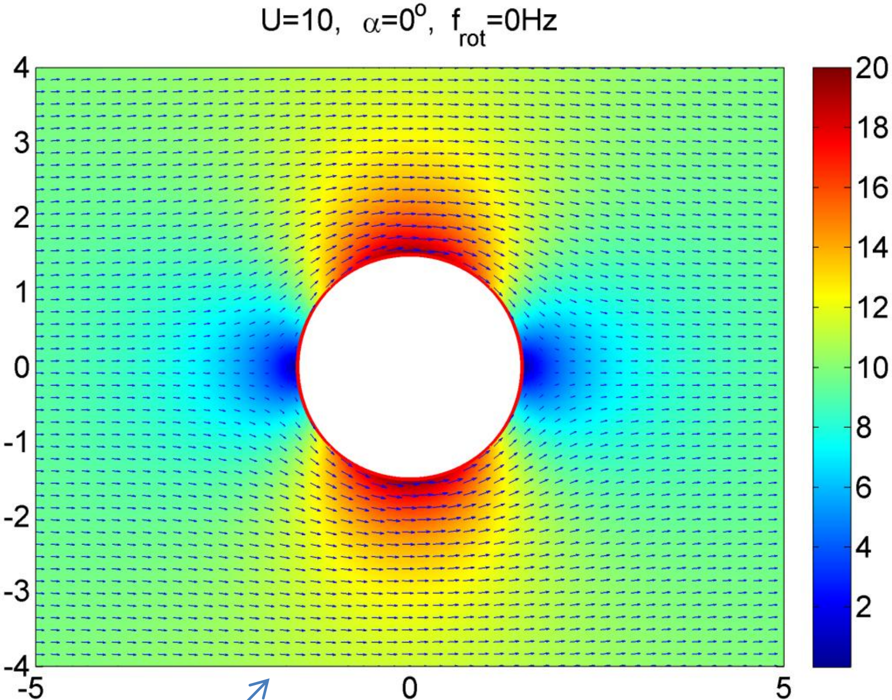
$$\therefore \frac{dw}{dz} = Ue^{-i\alpha} + \frac{i\Gamma}{2\pi z} - \frac{UR^2 e^{i\alpha}}{z^2} = v_x - iv_y$$

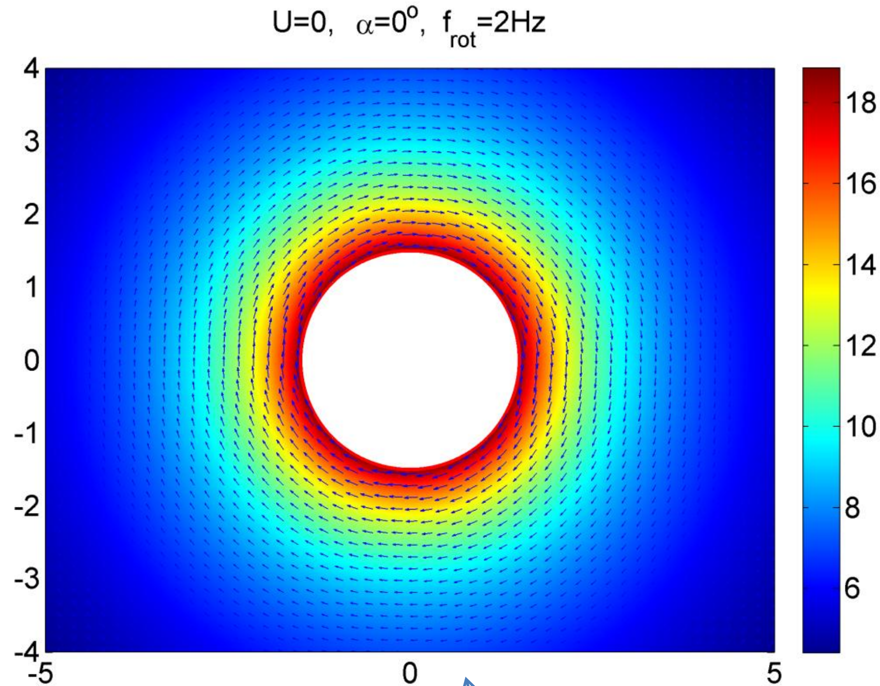
Note if computing arrays, the velocities correspond to the original (non origin shifted coordinates)

Use MATLAB ‘quiver’ function with a coarse grid to visualize the velocity field directions

The colour scale is a fine grid, with colour scale proportional to speed.

https://en.wikipedia.org/wiki/Joukowsky_transform



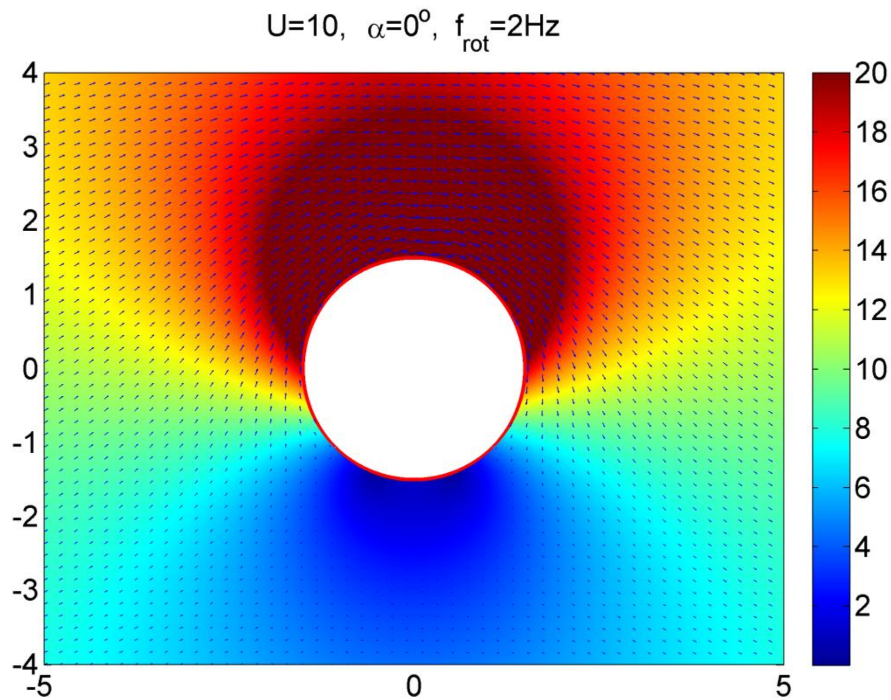


Just uniform flow

Just circulation

$$v_\theta = \frac{\Gamma}{2\pi r}$$

$$v_\theta = \frac{2\pi R^2 f_{rot}}{r}$$



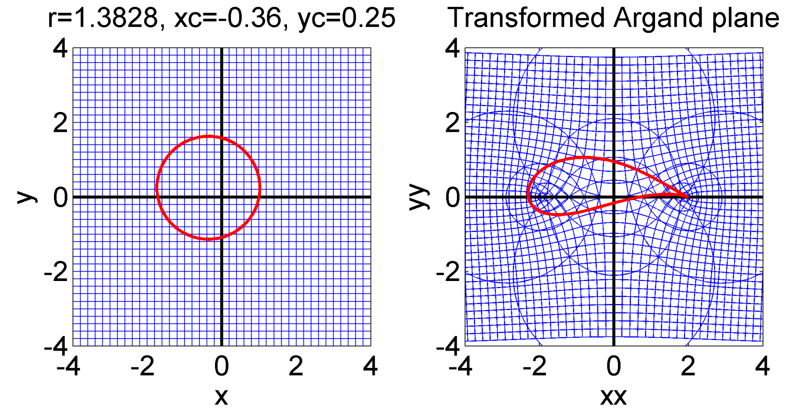
Uniform flow + circulation

Sanity check!

Turn the flow around a cylinder to flow around an airfoil using the **Joukowsky transformation** (a "conformal mapping")

$$z = x + iy$$
$$z' = z + 1/z$$

i.e. circles transform into airfoil shapes under this mapping



Chain rule

$$z' = z + 1/z$$

$$\frac{dw}{dz'} = \frac{dw}{dz} \times \frac{dz}{dz'}$$

$$= \frac{dw}{dz} \Big/ \frac{dz'}{dz}$$

$$= \frac{1}{1-z^2}\frac{dw}{dz}$$

i.e. when fluid speed is zero

In order to satisfy the Kutta criteria:
"A body with a sharp trailing edge which is moving through a fluid will create about itself a circulation of sufficient strength to hold the rear *stagnation point* at the trailing edge."

Find fluid velocities for Joukowsky airfoil

$$z = x + iy$$

$$z_c = x_c + iy_c$$

$$R = \sqrt{(1-x_c)^2 + y_c^2}$$

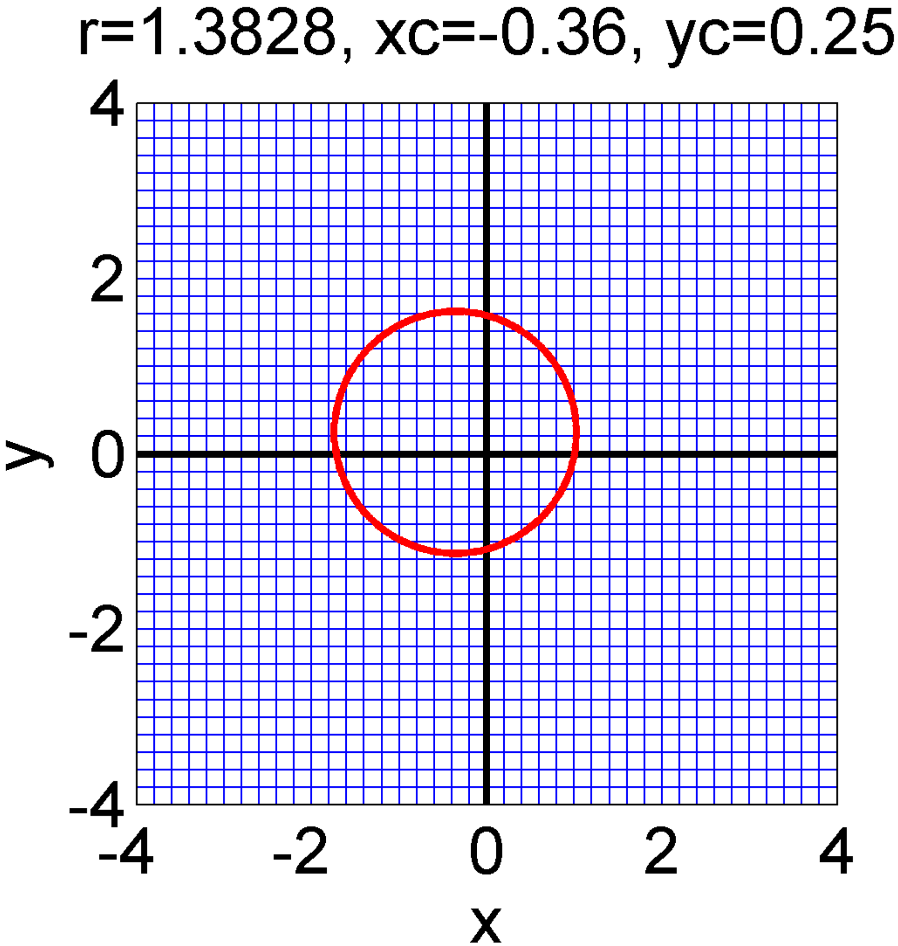
$$\Gamma = 4\pi UR \sin\left\{\alpha + \sin^{-1}\left(\tfrac{y_c}{R}\right)\right\}$$

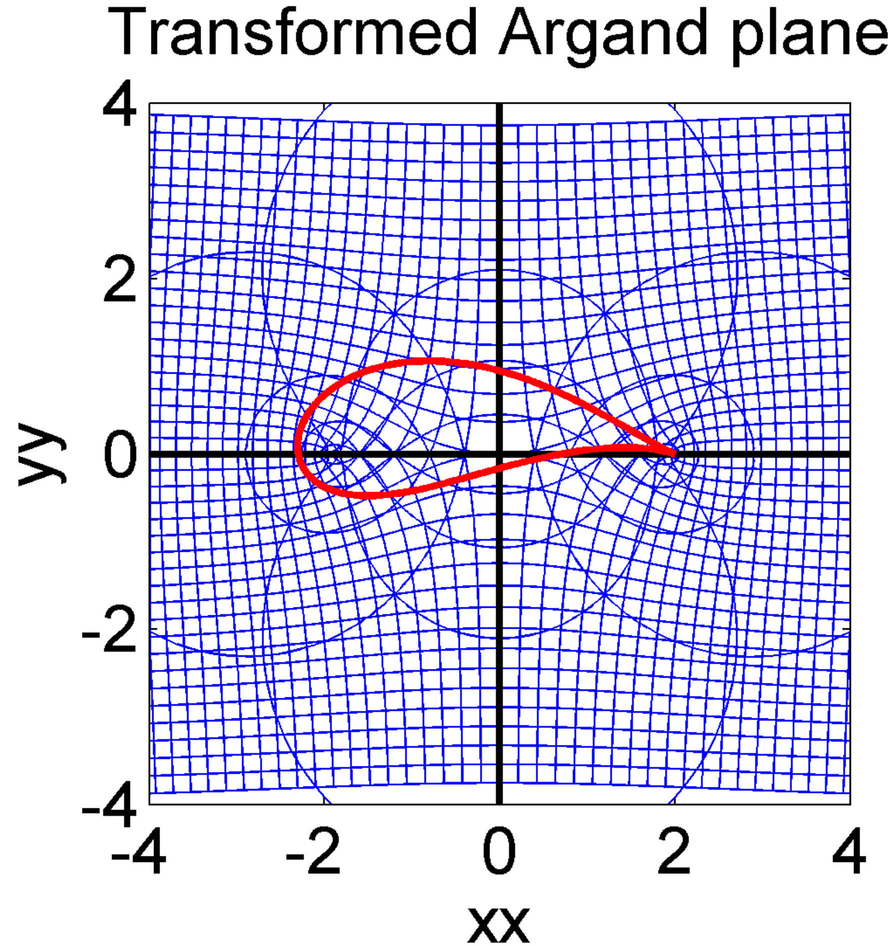
The right x axis intersection of the circle becomes the airfoil trailing edge, and this must satisfy the Kutta criteria

$$\frac{dw}{dz} = Ue^{-i\alpha} + \frac{i\Gamma}{2\pi(z-z_c)} - \frac{UR^2e^{i\alpha}}{(z-z_c)^2}$$

$$\frac{dw}{dz'} = v'_x - iv'_y = \frac{1}{1-z^2}\frac{dw}{dz}$$

i.e. cylinder until this step





$$z = x + iy$$

$$z_c = x_c + iy_c$$

$$R = \sqrt{(1 - x_c)^2 + y_c^2}$$

$$|z - z_c| = R$$

$$z = x + iy$$

$$z' = z + 1/z$$

Joukowski transformation

Complex potential for 2D flow around a cylinder of radius R, with far-field velocity U at angle α, set up for Joukowski airfoil

In order to satisfy the Kutta criteria:
"A body with a sharp trailing edge which is moving through a fluid will create about itself a circulation of sufficient strength to hold the rear *stagnation point* at the trailing edge."

i.e. when fluid speed is zero

$$z = x + iy$$

$$z_c = x_c + iy_c$$

$$R = \sqrt{(1-x_c)^2 + y_c^2}$$

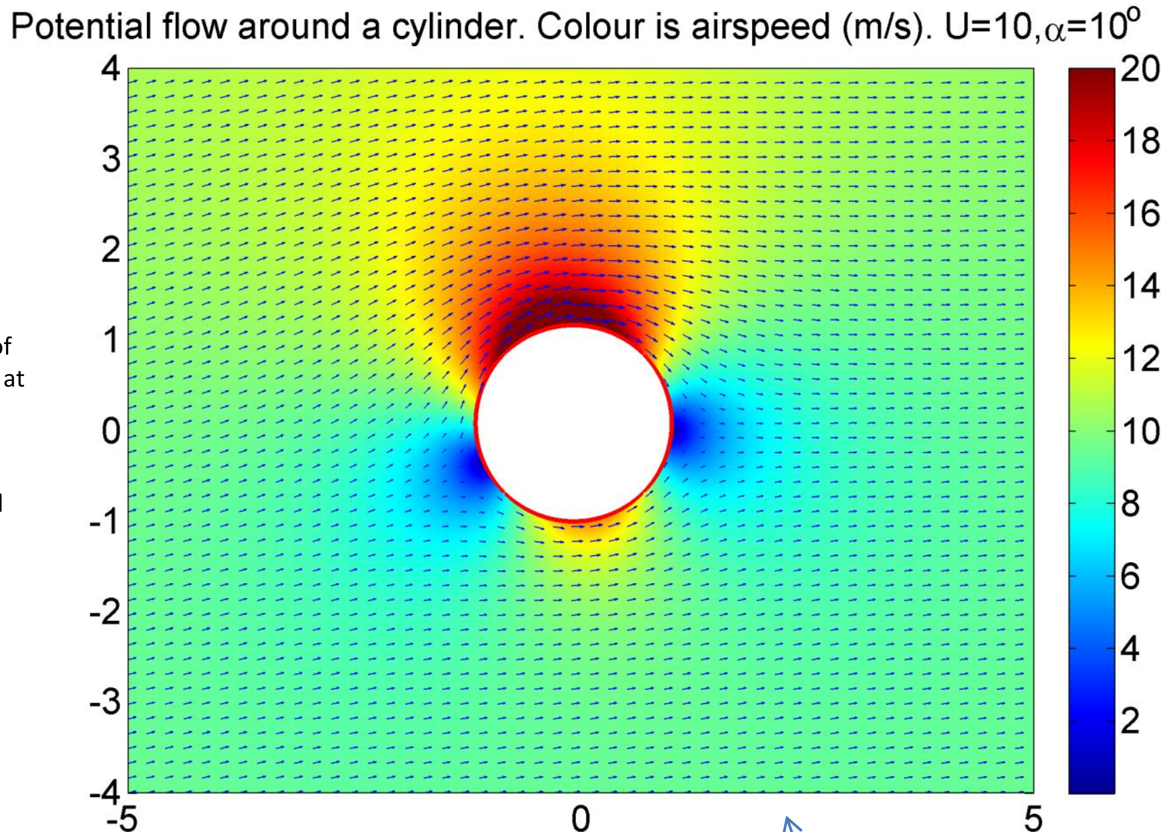
$$\Gamma = 4\pi UR\sin\left\{\alpha + \sin^{-1}\left(\frac{y_c}{R}\right)\right\}$$

$$w = \phi + i\psi$$

$$w = Uze^{-i\alpha} + \frac{i\Gamma}{2\pi}\ln(z - z_c) + \frac{UR^2e^{i\alpha}}{z - z_c}$$

i.e. uniform flow + rotation +source

$$\frac{dw}{dz} = Ue^{-i\alpha} + \frac{i\Gamma}{2\pi(z - z_c)} - \frac{UR^2e^{i\alpha}}{(z - z_c)^2} = v_x - iv_y$$

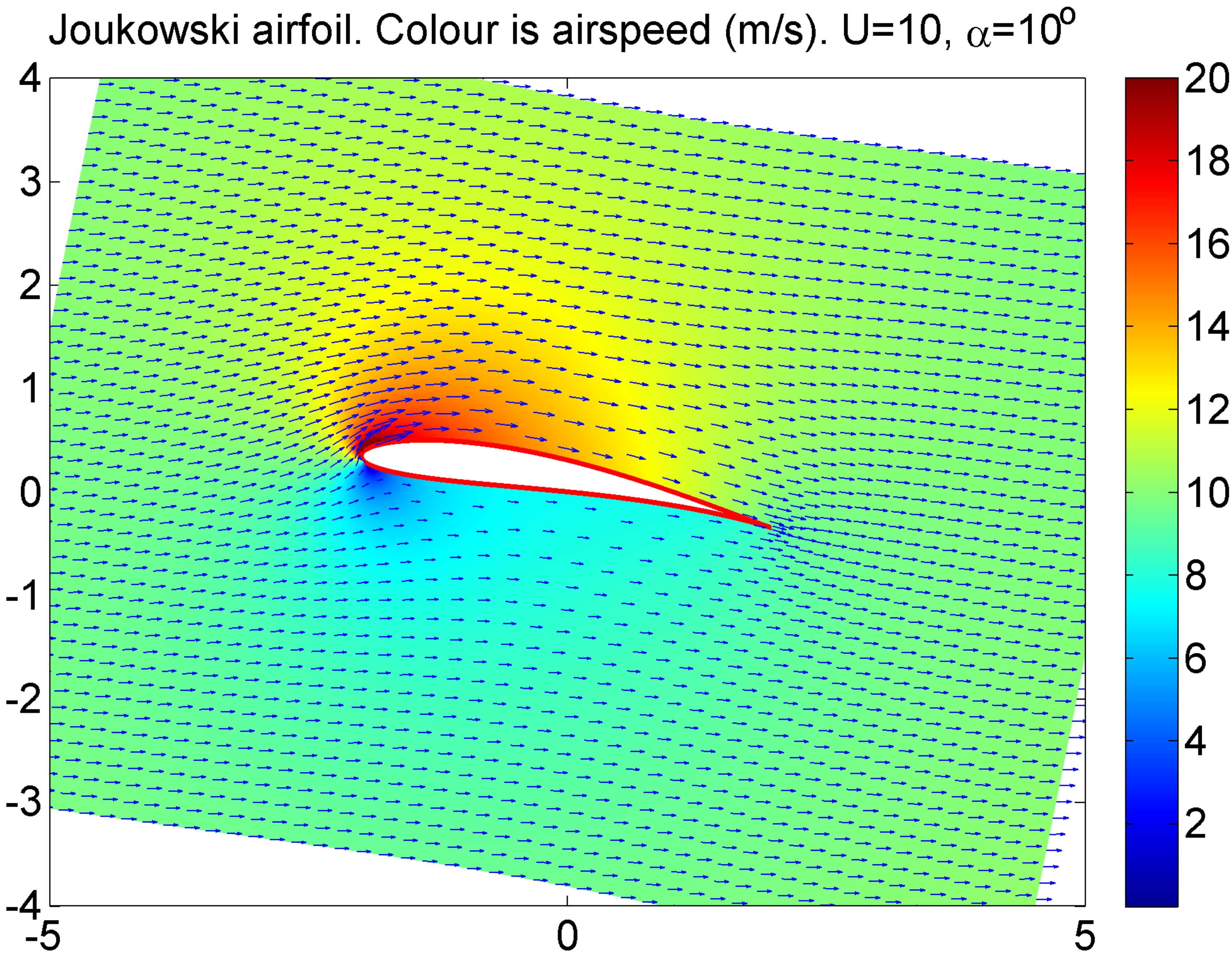


Note non-zero circulation, so flow is *asymmetric*

Use MATLAB 'quiver' function with a coarse grid to visualize the velocity field directions

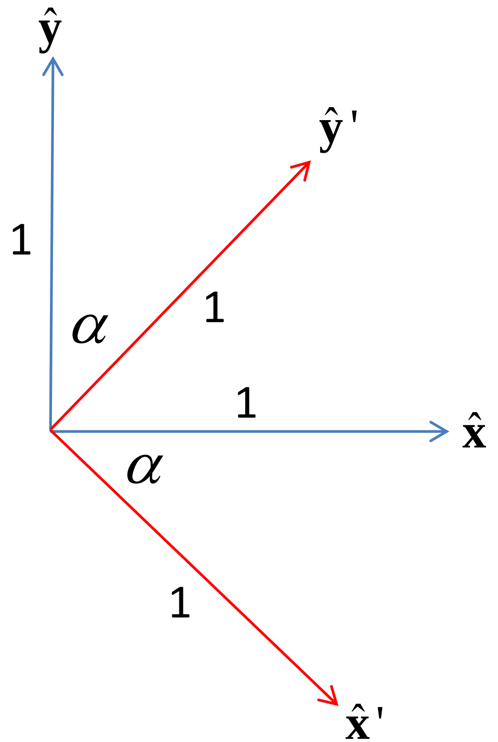
The colour scale is a fine grid, with colour scale proportional to speed.

https://en.wikipedia.org/wiki/Joukowsky_transform


Joukowski airfoil. Colour is airspeed (m/s). U=10, α=10°
4
3
2
1
0
-1
-2
-3
-4
-5
0
5
20
18
16
14
12
10
8
6
4
2

Achieve **clockwise coordinate rotation** by α about (x_r, y_r)

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x - x_r \\ y - y_r \end{pmatrix} + \begin{pmatrix} x_r \\ y_r \end{pmatrix}$$

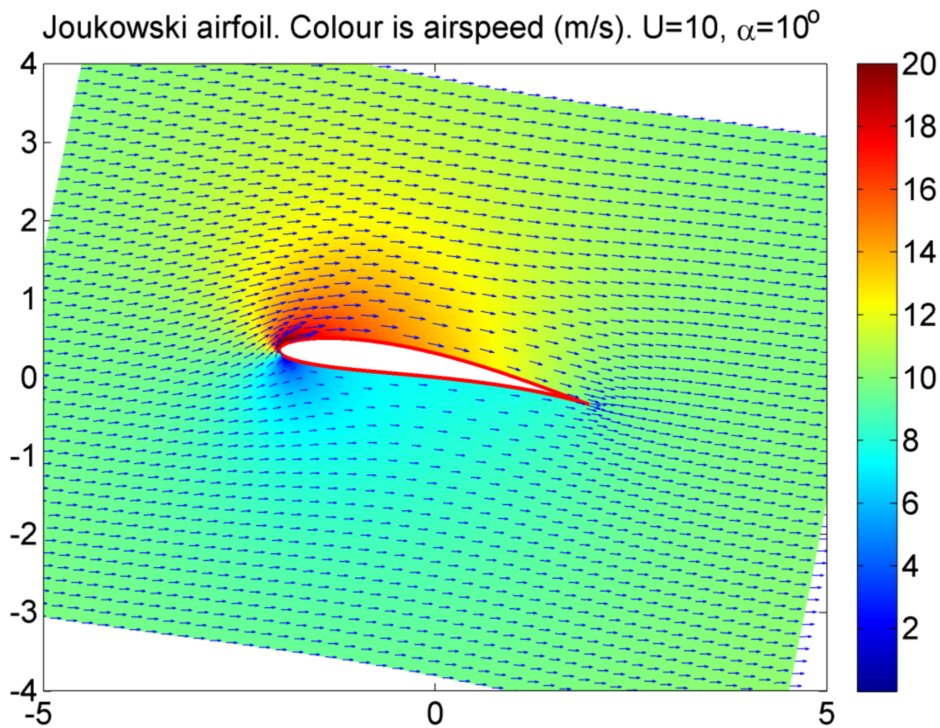


Recipe:
- Shift by centre of rotation, so this point is now the origin.
- Rotate about the origin by pre-multiplying by a **rotation matrix**
- Shift back by (x_r , y_r)

Rotation matrix comprises where 'basis vectors go under the transformation'

```
% Model of the flow of air around an airfoil, ignoring
% turbulence effects etc. This is achieved via the
% Joukowski complex number transformation.
% https://en.wikipedia.org/wiki/Joukowsky_transform
function joukowski

%Airfoil 'centre' coordinates
xc = -0.08; yc = 0.08;

%Angle of attack of aerofoil /radians
a = 10*pi/180;

%Airspeed away from aerofoil /ms^-1
U = 10;

%Argand diagram limits
xlimits = [-5,5]; ylimits = [-4,4];

%Fontsize for graphs
fsize = 16;

%

%Define circle of radius R, centre xy,yc
theta = linspace(0,2*pi,300);
R = sqrt( (1-xc)^2 + yc^2 );
xx = R*cos(theta)+ xc; yy = R*sin(theta) + yc;

%Apply Joukowski transformation to circle to define
airfoil shape
[xx,yy] = jtrans(xx,yy); [xx,yy] = rot( a,xx,yy,0,0 );

%Determine velocity field around airfoil
[xq,yq,vxq,vyq] = airfoil( xc,yc, a, U,
50,xlimits,ylimits);
[x,y,vx,vy] = airfoil( xc,yc, a, U,
1000,xlimits,ylimits);
```

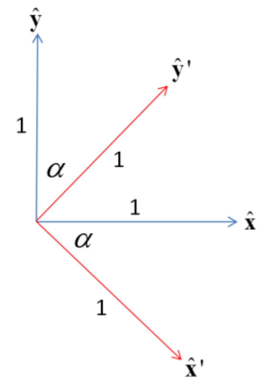
```
%Plot velocities
pcolor(x,y,sqrt(vx.^2 + vy.^2 ) ); shading interp; hold
on; axis tight; box on;
plot( xx,yy,'r-','linewidth',2);
set(gca,'fontsize',fsize); set(
gcf,'units','normalized','position',[0 0 1 1] );
caxis([0,2*U]); colorbar('fontsize',fsize);
colormap('jet'); icmap;
quiver( xq,yq,vxq,vyq );
axis equal; xlim(xlimits); ylim(ylimits);
title(['Joukowski airfoil. Colour is airspeed (m/s).
U=',num2str(U),...
    ',   \alpha=',num2str(a*180/pi),'^o']);

%Print PNG output
print( gcf,'joukowski.png','-dpng','-r300');
```



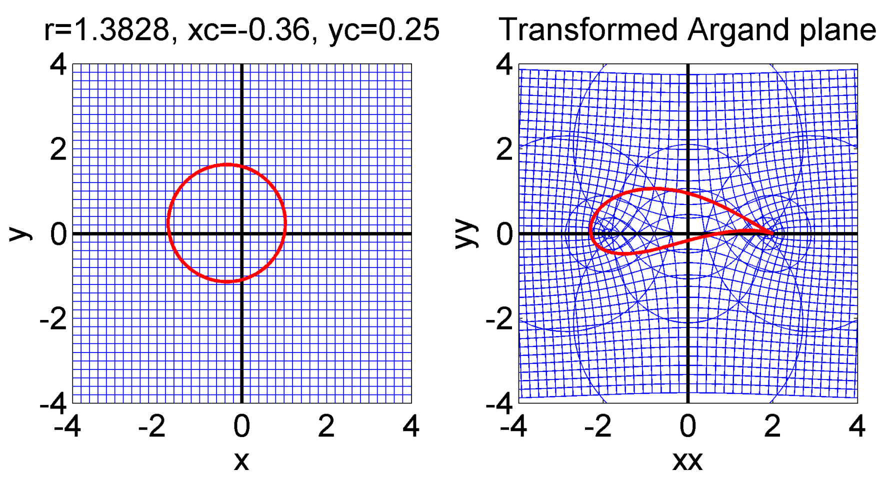
MATLAB code

```
%Rotate clockwise by theta (radians) about points (xr,yr)
function [xx,yy] = rot( theta, x,y,xr,yr )
x = x - xr; y = y - yr;
dim = size(x); x = reshape( x, [1,numel(x)] );
y = reshape( y, [1,numel(y)] );
xy = [ cos(theta), sin(theta); -sin(theta), cos(theta) ] * [x;y];
xx = xy(1,:); yy = xy(2,:);
xx = reshape( xx,dim ) + xr; yy = reshape( yy,dim ) + yr;
```

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x - x_r \\ y - y_r \end{pmatrix} + \begin{pmatrix} x_r \\ y_r \end{pmatrix}$$




```
%Joukowski conformal mapping
function [x,y] = jtrans(x,y)
z = x + 1i*y; z = z + 1./z;
x = real(z); y  = imag(z);
```



```
%Joukowsky airfoil function
function [x,y,vx,vy] = airfoil( xc,yc, a, U, N,xlimits,ylimits)

%Define circle radius and ciculation, which satisfies the Kutta criteria
R = sqrt( (1-xc)^2 + yc^2 );
F = 4*pi*U*R*sin( a + asin( yc/R ) );
zc = xc + 1i*yc;

%Define Argand diagram
x = linspace( xlimits(1), xlimits(2), N );
y = linspace( ylimits(1), ylimits(2), N);
[x,y] = meshgrid(x,y); z = x + 1i*y;

%Set as NaN the velocities within the circle
z = z - xc -1i*yc; z( abs(z) <=R ) = NaN; z = z + xc + 1i*yc;

%Define complex velocity
W = U*exp(-1i*a) + 1i*( F/(2*pi) )./(z - zc) -...
    U*(R^2)*exp(1i*a)./( (z - zc).^2 );

%Apply Joukowski conformal mapping
W = W./( 1 - z.^(-2) ); [x,y] = jtrans(x,y);

%Define velocities
vx = real(W); vy = -imag(W);

%Rotate entire space and velocities by a clockwise (i.e. so far-field
%velocity is horizontal left to right)
[x,y] = rot( a,x,y,0,0 ); [vx,vy] = rot( a,vx,vy,x,y );
```