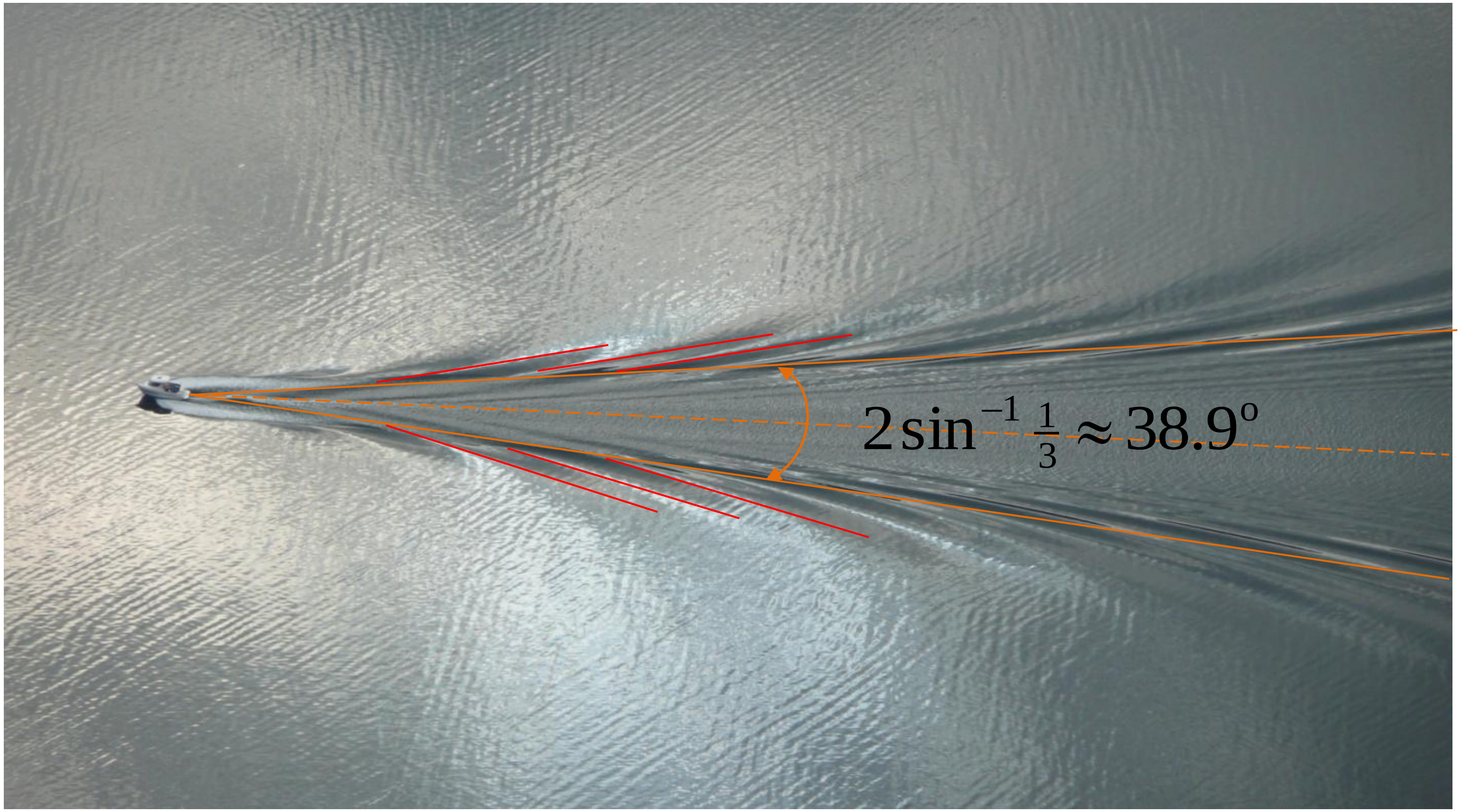
The shape of ship wakes: The Kelvin wedge



Dr Andrew French. August 2013

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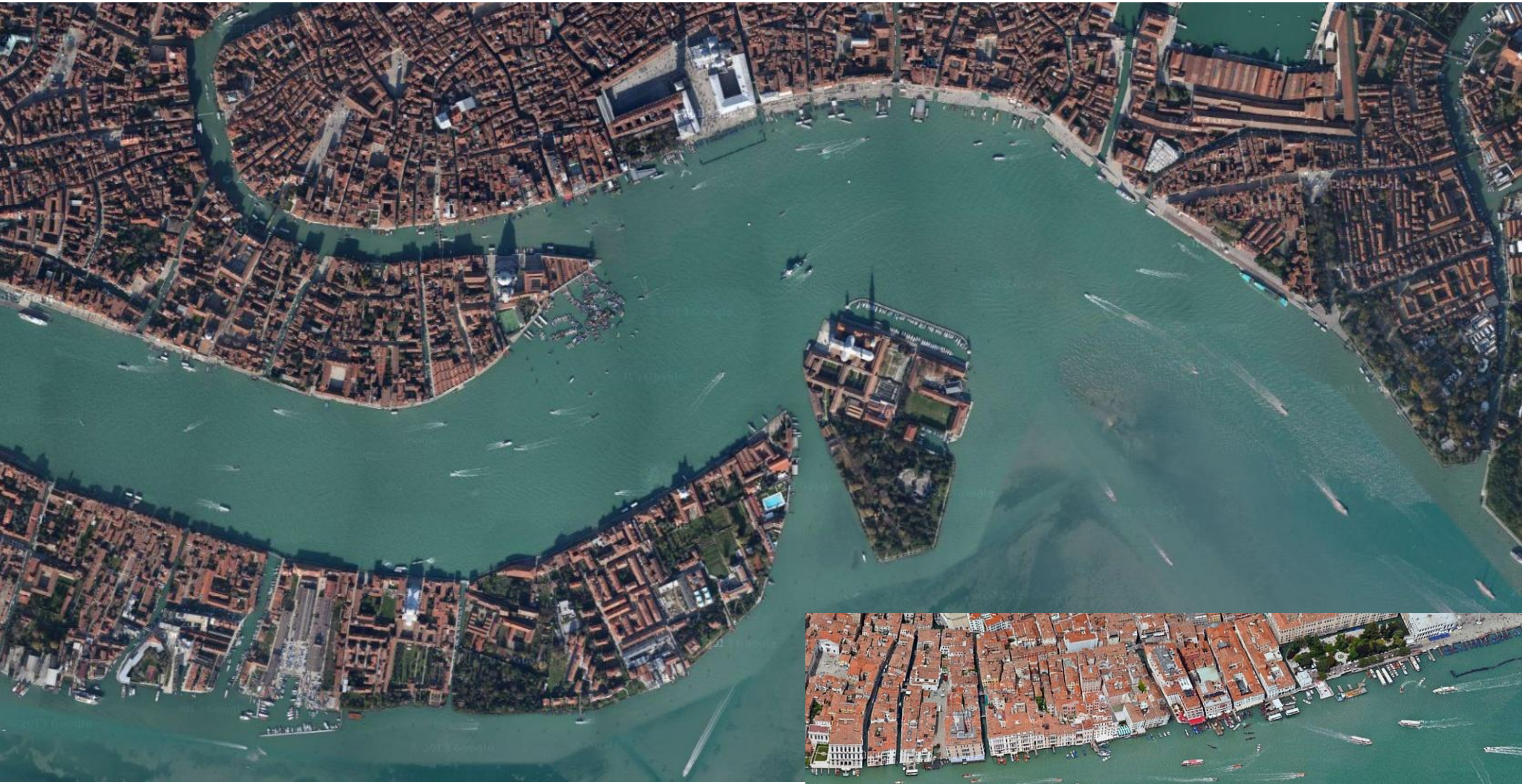
A wake is an *interference pattern* of waves formed by the motion of a body through a fluid. Intriguingly, the angular width of the wake produced by ships (and ducks!) in deep water is the same (about 38.9º). A mathematical explanation for this phenomenon was first proposed by Lord Kelvin (1824-1907). The triangular envelope of the wake pattern has since been known as the *Kelvin wedge*.

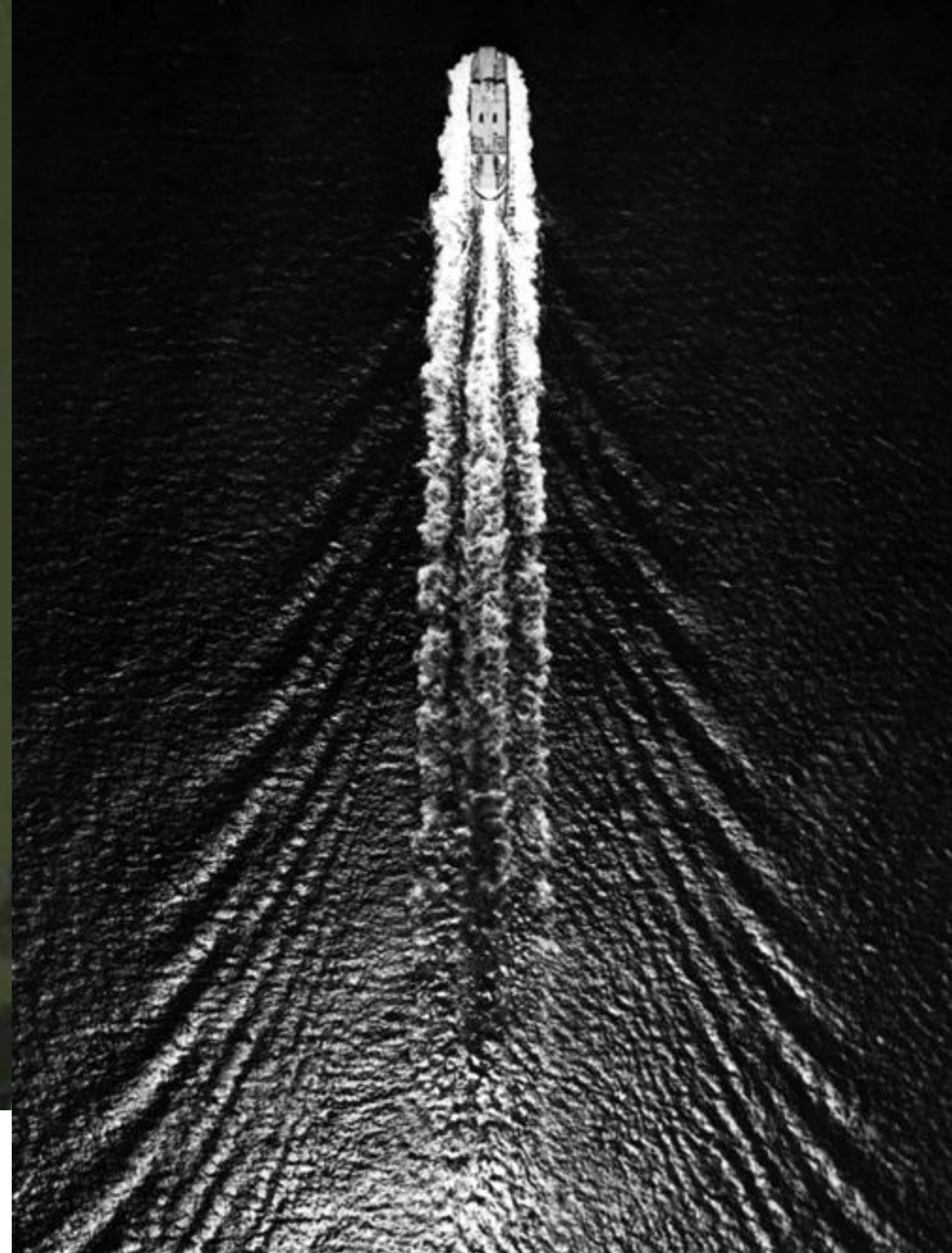
http://en.wikipedia.org/wiki/Wake

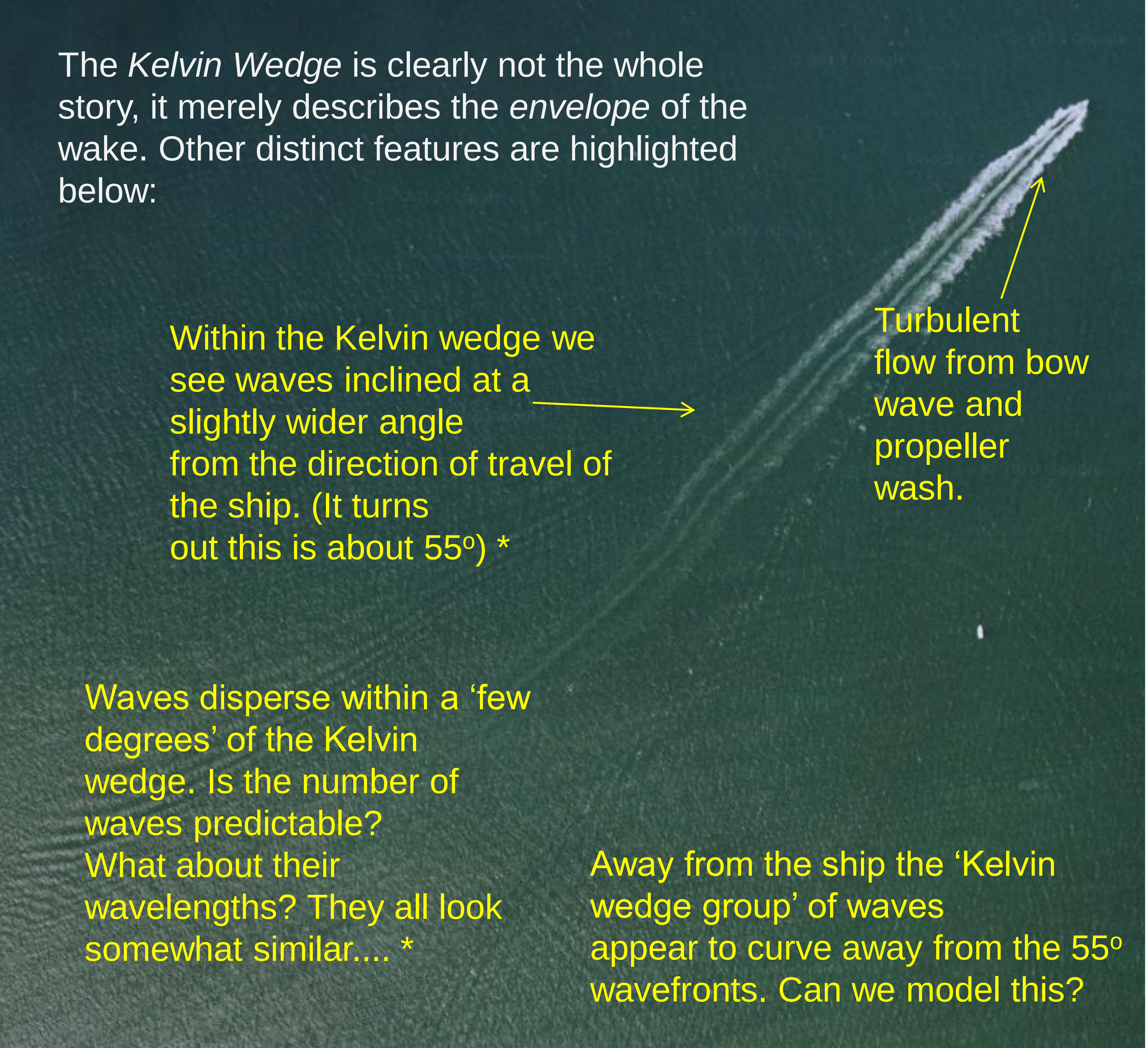
Venetian water-craft and their associated Kelvin wedges



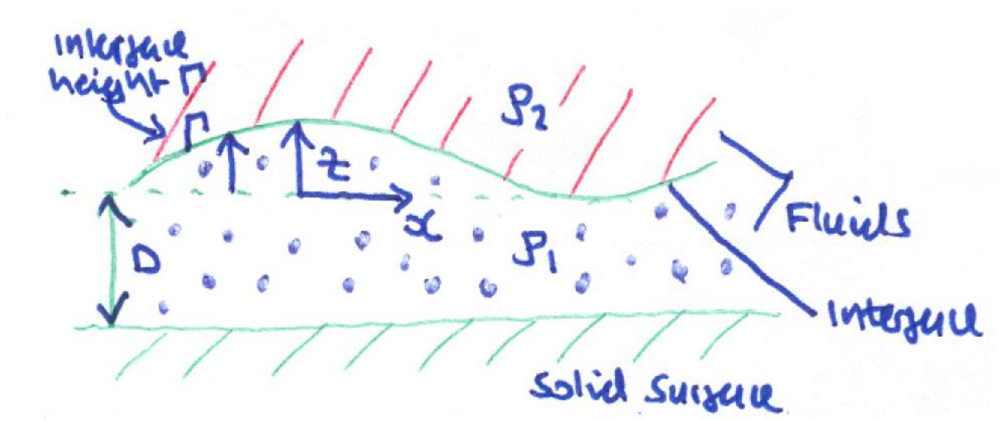
Images from Google Maps (above)
and Google Earth (right), (August 2013)

Still awake?





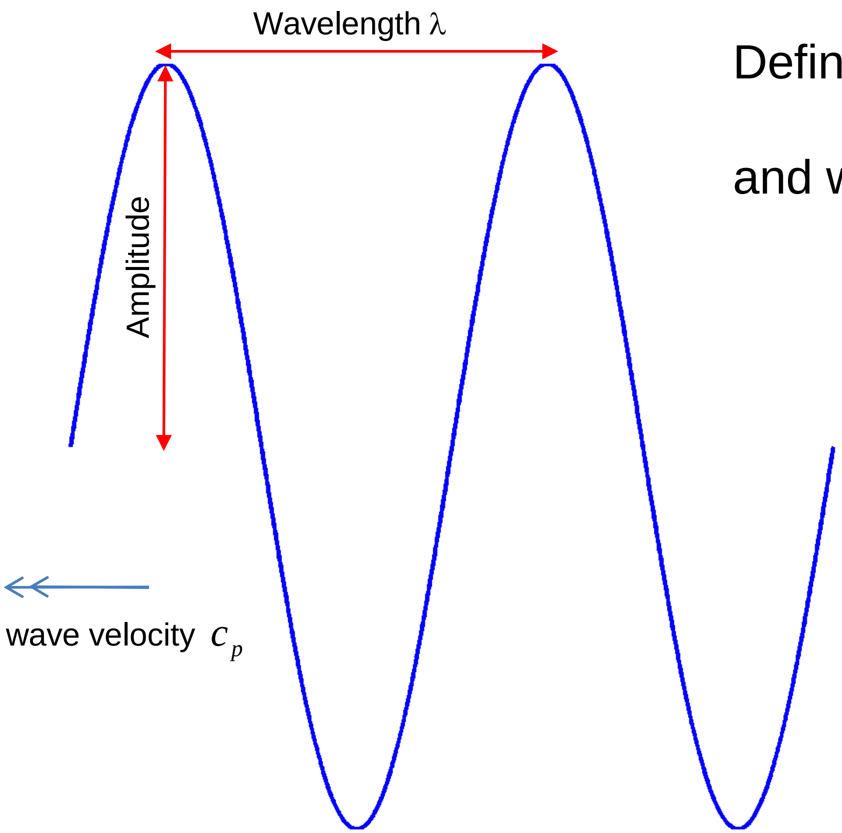
* These effects will be addressed in this presentation. The others will not!



Mathematical theory of surface waves

Key topics:
Frequency
Wavenumber
Dispersion relationship
Phase and group velocity
Shallow and deep water waves
Minimum velocity of deep water ripples

A means of characterizing a wave-like disturbance is via a *dispersion relation*. This is an equation which relates the frequency of the wave to its wavelength, plus other parameters such as surface tension, fluid density, depth etc.



Define angular frequency $\omega = 2\pi f$

and wavenumber $k = \dfrac{2\pi}{\lambda}$

For deep water gravity waves, the dispersion relationship is

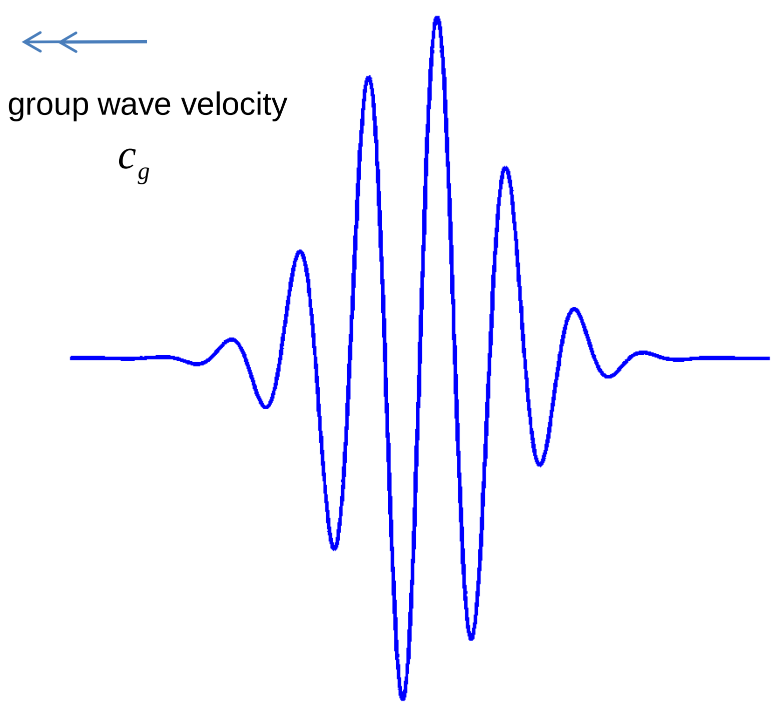
$$\omega^2 = gk$$

i.e. we ignore effects of the density of air, surface tension and assume the depth $D >> \lambda$ i.e. $kD >> 1$

The *phase velocity* (of individual wave crests) can be found from the dispersion relationship

$$c_p = f\lambda = \frac{\omega}{k}$$

The dispersion relationship allows to compute how fast groups of disturbances will travel. This *group velocity* can be *different* from the phase velocity. Relative motion of the wave crests to the overall envelope causes dispersion.



$$c_g = \frac{d\omega}{dk}$$

For deep water gravity driven waves

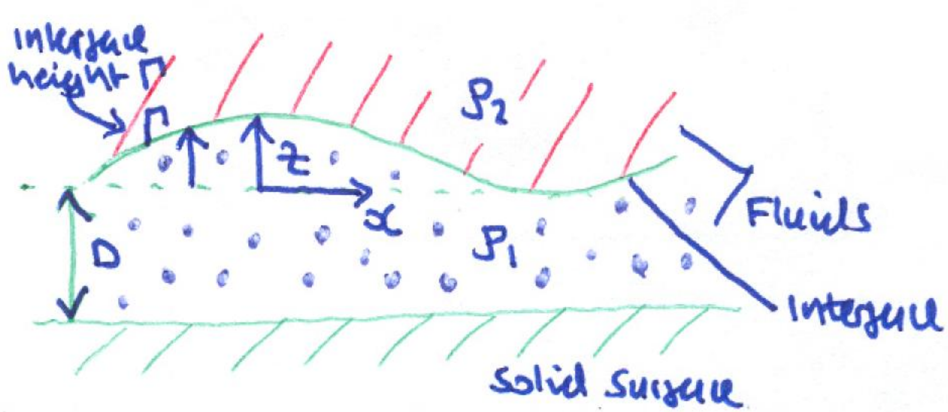
$$\omega = \sqrt{g}k^{\frac{1}{2}}$$

$$c_p = \frac{\omega}{k} = \sqrt{g}k^{-\frac{1}{2}}$$

$$c_g = \frac{d\omega}{dk} = \frac{1}{2}\sqrt{g}k^{-\frac{1}{2}}$$

$$\therefore \quad c_g = \frac{1}{2}c_p$$

In general, for *vorticity* free waves on the interface of two *incompressible, Newtonian* fluids, the dispersion relationship can be shown to be, for waves with amplitude $\Gamma << D$



surface tension

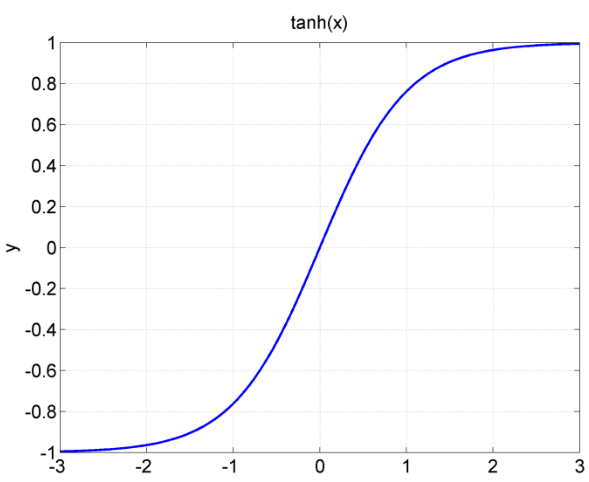
$$\omega^2 = \frac{\sigma k^3 + g(\rho_1 - \rho_2)k}{\rho_2 + \rho_1 \operatorname{cotanh}(kD)}$$

fluid densities

depth

For waves on a water, air interface $\rho_1 \approx 1000\rho_2$, so ignore ρ_2

$$\omega^2 = \left(\frac{\sigma k^3}{\rho_1} + gk\right)\tanh(kD)$$



For shallow water waves $\tanh(kD) \approx kD$

$$\omega^2 = \frac{\sigma k^4 D}{\rho_1} + gk^2 D$$

For deep water waves $\tanh(kD) \approx 1$

$$\omega^2 = \frac{\sigma k^3}{\rho_1} + gk$$

Since $k = \frac{2\pi}{\lambda}$ the higher powers of *k* will contribute less for longer wavelengths

Hence for shallow water gravity waves $\omega^2 = gk^2 D$

e.g. waves coming ashore just before they break, or waves in a shallow river or canal

Similarly for deep water waves $\omega^2 = gk$

Note for deep water *ripples* (or 'capillary waves') this approximation is invalid.

We must use the full dispersion relation

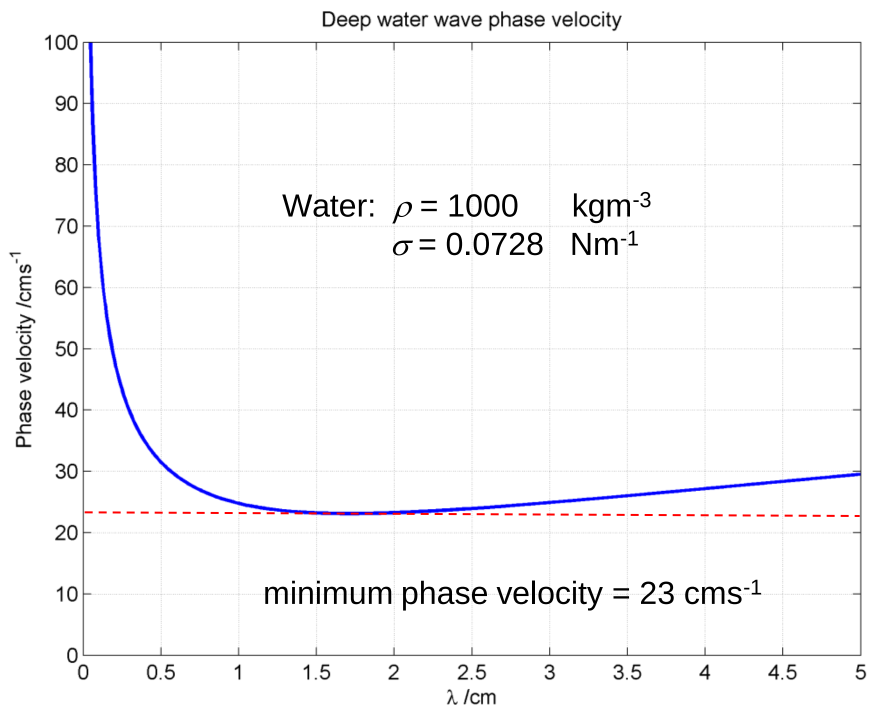
$$\omega = \sqrt{\frac{\sigma k^3}{\rho_1} + gk}$$

Ripple phase velocity is:

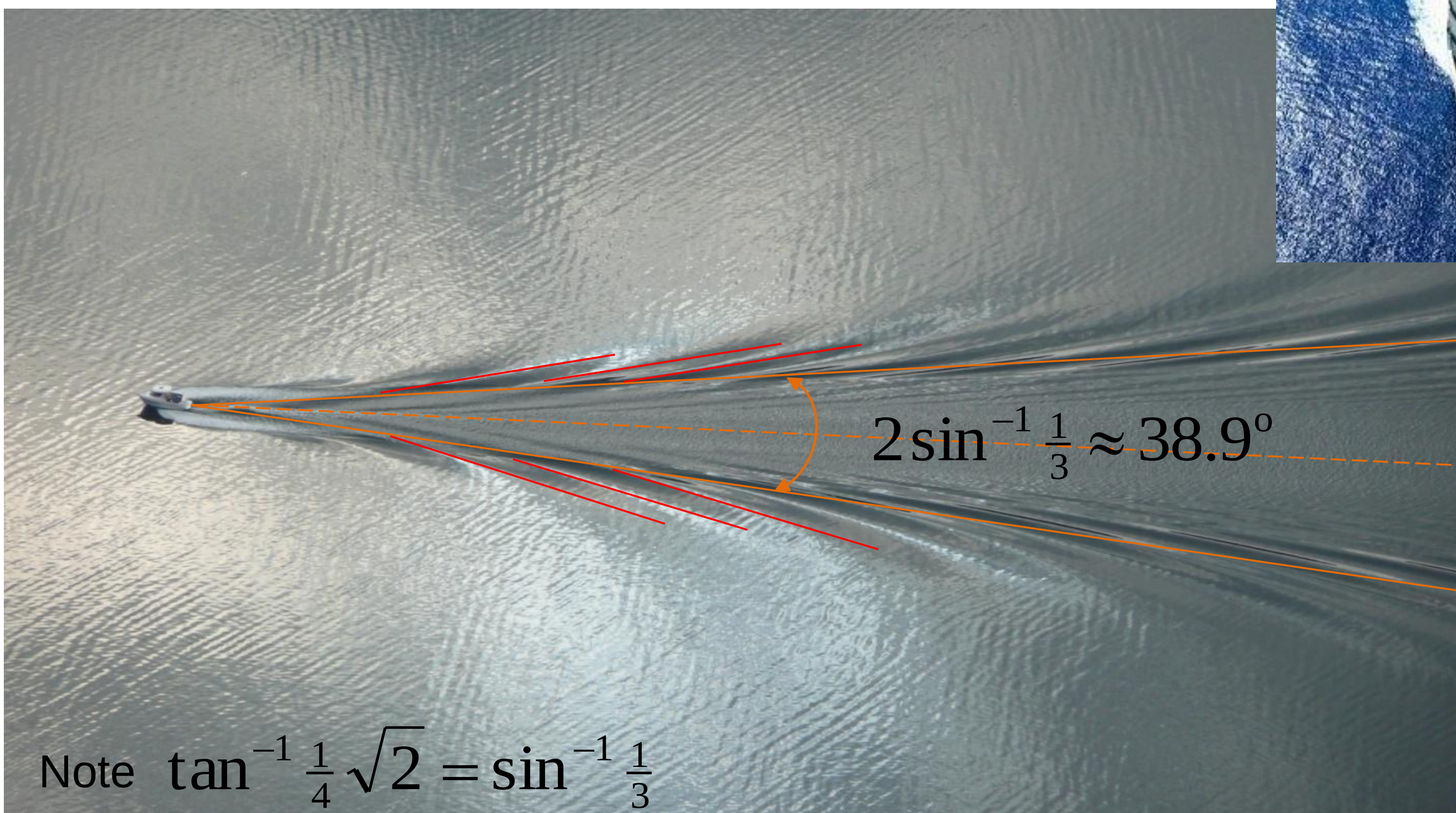
$$c_p = \frac{\omega}{k} = \sqrt{\frac{\sigma k}{\rho_1} + \frac{g}{k}} = \sqrt{\frac{2\pi\sigma}{\lambda\rho_1} + \frac{g\lambda}{2\pi}}$$

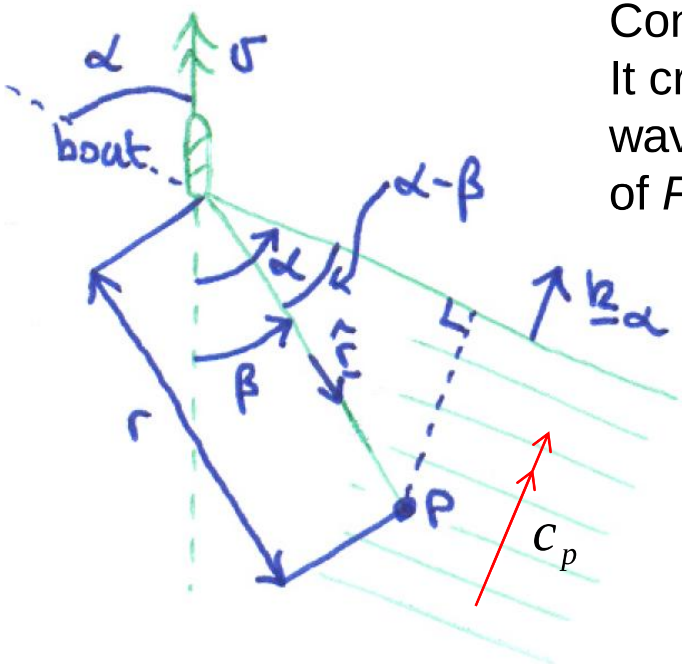
Which has a *minima* at

$$c_p = \sqrt[4]{\frac{4g\sigma}{\rho_1}}$$



Mathematical theory of the Kelvin Wedge





Consider a ship moving at velocity *v* through deep water. It creates wavelike disturbances with a variety of wavelengths. i.e. the resulting disturbance has a *spectrum* of *Fourier* components.

The waves of interest are those which are most pronounced. i.e. the Kelvin Wedge. Let these be travelling at angle α to the ship's velocity.

Unlike all other waves, which will naturally disperse (and also attenuate due to the viscosity of water) these waves continue to be sustained by the motion of the ship.

To achieve this, the phase velocity of these waves must equal the component of the ship's velocity parallel to the wavevector $\mathbf{k}_\alpha$. This is know as a 'surf riding' condition.

$$c_p = v \sin\alpha$$

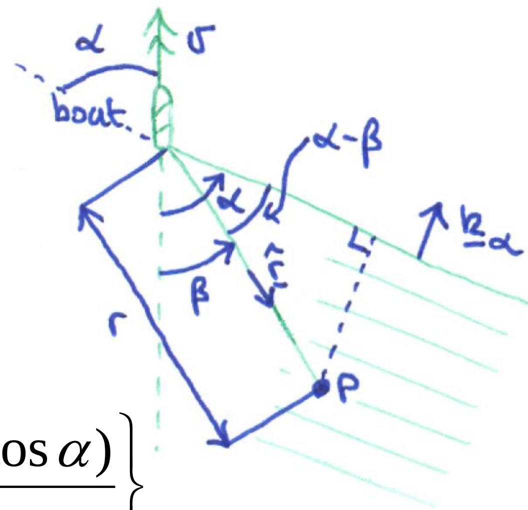
$$\omega^2 = gk$$

$$c_p = \frac{\omega}{k} = \sqrt{g}\,k^{-\frac{1}{2}}$$

Hence $$\sin\alpha = \frac{\sqrt{g}\,k^{-\frac{1}{2}}}{v}$$

$$\Rightarrow k = \frac{g}{v^2 \sin^2\alpha}$$

At point P the *phase* of the waves at angle α is $\phi = \mathbf{k}_\alpha \cdot \mathbf{r} = -kr\sin(\alpha-\beta)$



Now

$$k = \frac{g}{v^2 \sin^2 \alpha}$$

$$\therefore \quad \phi = -\frac{gr\sin(\alpha-\beta)}{v^2 \sin^2\alpha}$$

$$\therefore \quad \frac{\partial \phi}{\partial \alpha} = -\frac{gr}{v^2}\left\{\frac{\sin^2\alpha(-\cos(\alpha-\beta)) - \sin(\alpha-\beta)(-2\sin\alpha\cos\alpha)}{\sin^4\alpha}\right\}$$

$$= \frac{gr}{v^2\sin^3\alpha}\{\sin\alpha\cos(\alpha-\beta) - 2\sin(\alpha-\beta)\cos\alpha\}$$

When $\frac{\partial \phi}{\partial \alpha} = 0$ this means the wave phase doesn't change as α varies. In other words we expect *constructive interference* for angles α when $\frac{\partial \phi}{\partial \alpha} = 0$
This is known as the *principle of stationary phase.*

$$\frac{\partial \phi}{\partial \alpha} = 0 \quad \Rightarrow \quad \sin\alpha\cos(\alpha-\beta) - 2\sin(\alpha-\beta)\cos\alpha = 0$$

Hence

$$\boxed{2\tan(\alpha-\beta) = \tan\alpha}$$

Using the addition formula for tangents

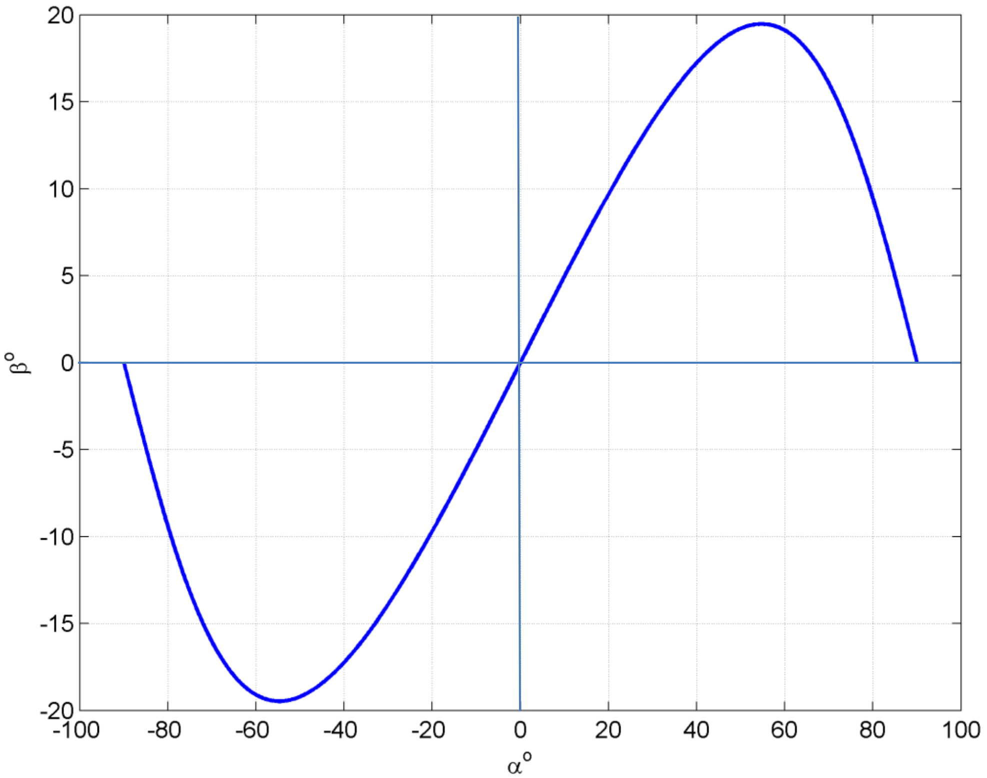
$$2\tan(\alpha-\beta)=\tan\alpha$$

$$2\left\{\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}\right\}=\tan\alpha$$

$$2\tan\alpha-2\tan\beta=\tan\alpha+\tan^2\alpha\tan\beta$$

$$\tan\alpha=\left(\tan^2\alpha+2\right)\tan\beta$$

$$\therefore\quad \boxed{\tan\beta=\frac{\tan\alpha}{\tan^2\alpha+2}}$$



$$\therefore\quad \sec^2\beta\frac{d\beta}{d\alpha}=\frac{\left(\tan^2\alpha+2\right)\sec^2\alpha-\tan\alpha\left(2\tan\alpha\sec^2\alpha\right)}{\left(\tan^2\alpha+2\right)^2}$$

$$\therefore\quad \sec^2\beta\frac{d\beta}{d\alpha}=\sec^2\alpha\frac{\left(\tan^2\alpha+2\right)-2\tan^2\alpha}{\left(\tan^2\alpha+2\right)^2}=\sec^2\alpha\frac{2-\tan^2\alpha}{\left(\tan^2\alpha+2\right)^2}$$

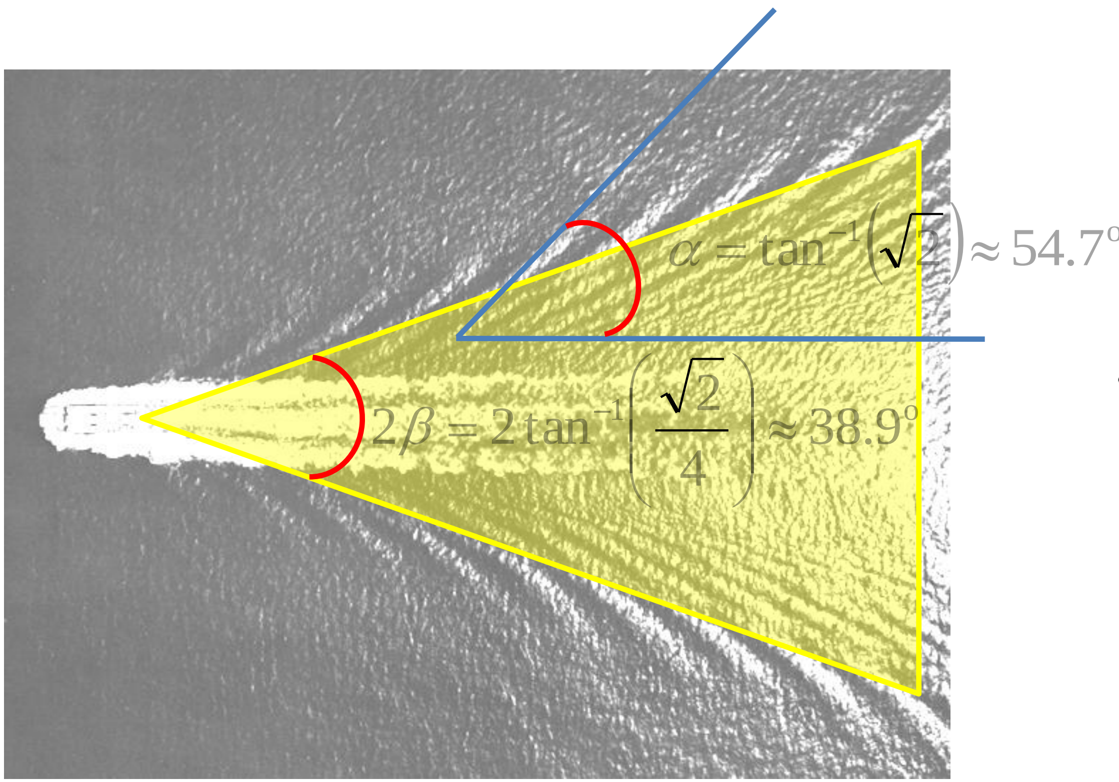
Hence $\frac{d\beta}{d\alpha}=0$ when $\boxed{\tan\alpha=\pm\sqrt{2}}$

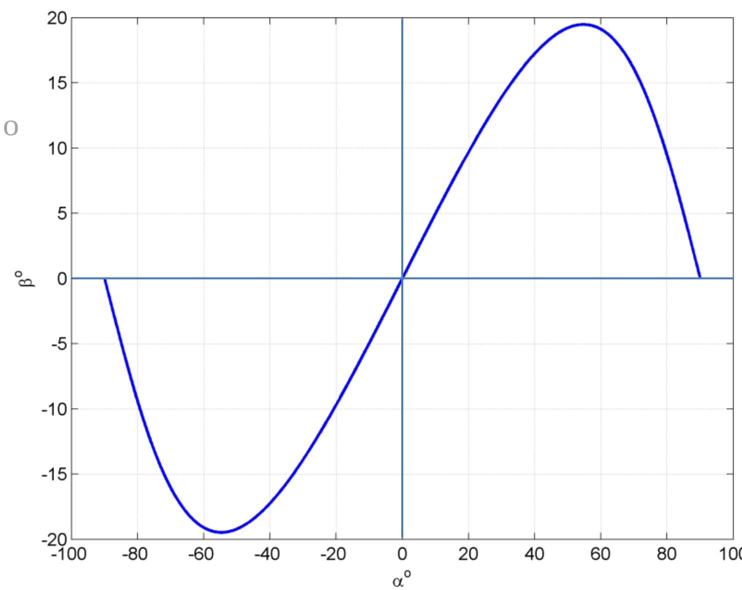
By a similar argument to ‘stationary phase’ let the Kelvin Wedge be at α when $\frac{d\beta}{d\alpha}=0$

Hence $\alpha = \tan^{-1}\left(\pm\sqrt{2}\right) = \pm\tan^{-1}\sqrt{2} \approx \boxed{\pm 54.7^{o}}$

and therefore $\beta = \tan^{-1}\left(\dfrac{\tan\alpha}{\tan^{2}\alpha + 2}\right) = \tan^{-1}\left(\dfrac{\pm\sqrt{2}}{4}\right) \approx \boxed{\pm 19.5^{o}}$

$2\beta = 38.9^{o}$ is the angular width of the **Kelvin Wedge** and $\alpha = 54.7^{o}$ is the angle of the principle waves in the ship's wake





What wavelengths do we expect for waves in the Kelvin wedge?

From our 'surf-riding condition'

$$k = \frac{g}{v^2 \sin^2 \alpha} \qquad \therefore \frac{2\pi}{\lambda} = \frac{g}{v^2 \sin^2 \alpha} \qquad \therefore \boxed{\lambda = \frac{2\pi v^2 \sin^2 \alpha}{g}}$$

Now $1 = \sin^2 \alpha + \cos^2 \alpha$ and $\tan \alpha = \frac{\sin \alpha}{\cos \alpha}$

$$\therefore \frac{1}{\sin^2 \alpha} = 1 + \frac{1}{\tan^2 \alpha} \qquad \Rightarrow \qquad \sin^2 \alpha = \frac{1}{1 + \frac{1}{\tan^2 \alpha}}$$

Hence if $\tan \alpha = \pm\sqrt{2} \quad \Rightarrow \sin^2 \alpha = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3}$

$$\therefore \boxed{\lambda = \frac{4\pi v^2}{3g}} \quad \text{or} \quad \boxed{v = \sqrt{\frac{3g\lambda}{4\pi}}}$$

This result could be used to infer ship velocities from their wake pattern

Wavelength of wake waves of the Isle of Wight Ferry (Image from Google maps August 2013) is approximately (3mm / 19mm) x 20m = 3.2m, although at this resolution it is somewhat difficult to measure accurately

Hence $v = \sqrt{\frac{3g\lambda}{4\pi}}$ $= 2.7ms^{-1}$

Now $1ms^{-1}$ = 1.944 knots

which means the model predicts the ferry is travelling at about 5.3 knots

According to the Red Funnel website the car ferry travels between 12 and 14 knots ..... However, it could have been slowing down as it enters Southampton water!

This sounds like an excellent experimental physics project, i.e. to test the validity of the above equation.

The idea of using Google maps to investigate the Kelvin wedge model has recently been posed by Rabaud and Moisy (arXiv: 1304.2653v1 9 April 2013)

Their thesis attempts to explain deviations to the Kelvin wedge model observed for higher ship speeds. They develop a model which characterizes the wave angle in terms of *Froude number*, although at ‘low’ Fr the fixed Kelvin wake angle of 38.9° is preserved to be consistent with observations.

Froude number is the ratio of ship’s speed to deep water wave speed. It assumes wavenumber *k* inversely scales with ship length *L*

$$Fr = \frac{v}{c_p} \qquad c_p = \sqrt{g}k^{-\frac{1}{2}} \qquad k = \frac{1}{L}$$

Hence

$$Fr = \frac{v}{\sqrt{g}\left(\frac{1}{L}\right)^{-\frac{1}{2}}} = \frac{v}{\sqrt{gL}}$$

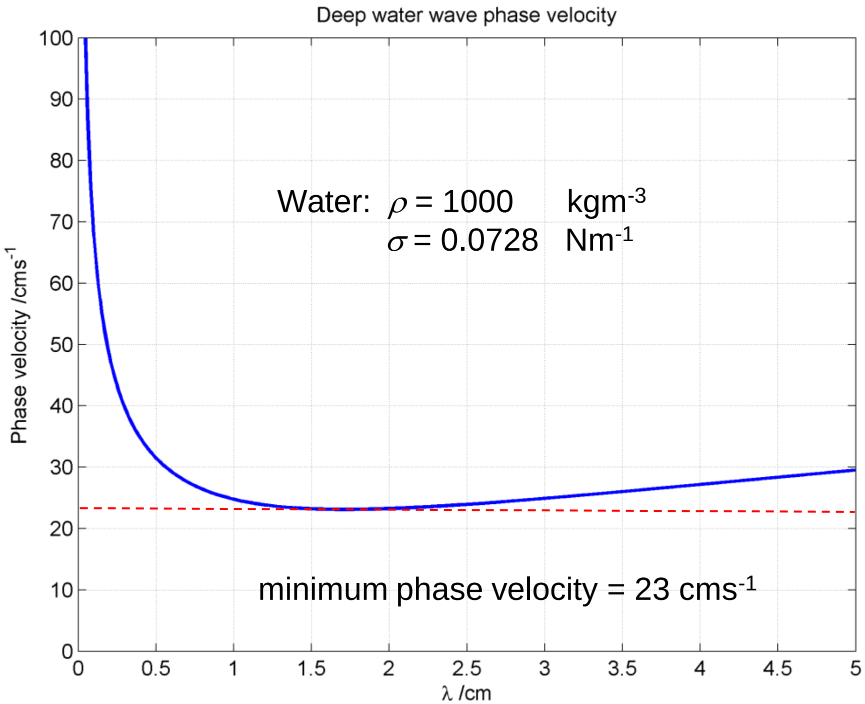
Rabaud & Moisy’s model

$$\beta = \tan^{-1}(1/\sqrt{8}) \simeq 19.47^{\circ}, \qquad Fr \leq Fr_c$$

$$\beta = \tan^{-1}\frac{\sqrt{2\pi Fr^2 - 1}}{4\pi Fr^2 - 1}, \qquad Fr \geq Fr_c,$$

$$Fr_c = \sqrt{3/4\pi} \simeq 0.49$$

Minimum ship speed needed to generate a Kelvin wedge



$$v > \sqrt[4]{\frac{9g\sigma}{4\rho_1}}$$

v > 20 cms^{-1}

What is the minimum ship speed needed to generate a wake?

This minimum phase velocity for deep water gravity waves is $c_p = \sqrt[4]{\dfrac{4g\sigma}{\rho_1}}$

Now $c_p = \sqrt{\dfrac{2\pi\sigma}{\lambda\rho_1} + \dfrac{g\lambda}{2\pi}}$ and for the Kelvin Wedge $\lambda = \dfrac{4\pi v^2}{3g}$

Hence

$$\sqrt{\frac{6g\pi\sigma}{4\pi v^2\rho_1} + \frac{g}{2\pi}\frac{4\pi v^2}{3g}} > \sqrt[4]{\frac{4g\sigma}{\rho_1}}$$

$$\sqrt{\frac{3g\sigma}{2v^2\rho_1} + \frac{2v^2}{3}} > \sqrt[4]{\frac{4g\sigma}{\rho_1}}$$

$$\frac{3g\sigma}{2v^2\rho_1} + \frac{2v^2}{3} > \sqrt{\frac{4g\sigma}{\rho_1}}$$

$$\frac{3g\sigma}{2\rho_1} + \frac{2v^4}{3} > v^2\sqrt{\frac{4g\sigma}{\rho_1}}$$

$$\frac{9g\sigma}{2\rho_1} + 2v^4 > 3v^2\sqrt{\frac{4g\sigma}{\rho_1}}$$

$$9g\sigma + 4\rho_1 v^4 > 6\rho_1 v^2\sqrt{\frac{4g\sigma}{\rho_1}}$$

$$9g\sigma + 4\rho_1 v^4 > 12v^2\sqrt{g\sigma\rho_1}$$

$$4\rho_1 v^4 - 12v^2\sqrt{g\sigma\rho_1} + 9g\sigma > 0$$

$$4\rho_1 v^4 - 12 v^2 \sqrt{g\sigma\rho_1} + 9g\sigma > 0$$

$$4\rho_1 \left\{ v^4 - 3v^2 \sqrt{\frac{g\sigma}{\rho_1}} \right\} + 9g\sigma > 0$$

$$4\rho_1 \left\{ \left(v^2 - \tfrac{3}{2}\sqrt{\frac{g\sigma}{\rho_1}} \right)^2 - \tfrac{9}{4}\frac{g\sigma}{\rho_1} \right\} + 9g\sigma > 0$$

$$4\rho_1 \left(v^2 - \tfrac{3}{2}\sqrt{\frac{g\sigma}{\rho_1}} \right)^2 - 9g\sigma + 9g\sigma > 0$$

$$v^2 - \tfrac{3}{2}\sqrt{\frac{g\sigma}{\rho_1}} > 0$$

$$v^2 > \sqrt{\frac{9g\sigma}{4\rho_1}}$$

$$v > \sqrt[4]{\frac{9g\sigma}{4\rho_1}}$$

$\sqrt[4]{4} \approx$ 1.41

$\sqrt[4]{\frac{9}{4}} \approx$ 1.22

$$v > \sqrt[4]{\frac{9g\sigma}{4\rho_1}}$$

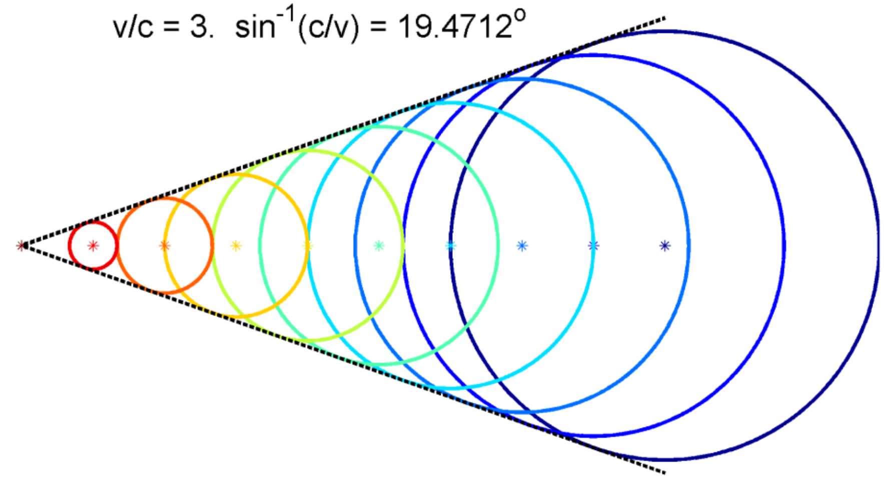
Water: ρ = 1000 kgm^{-3}
σ = 0.0728 Nm^{-1}

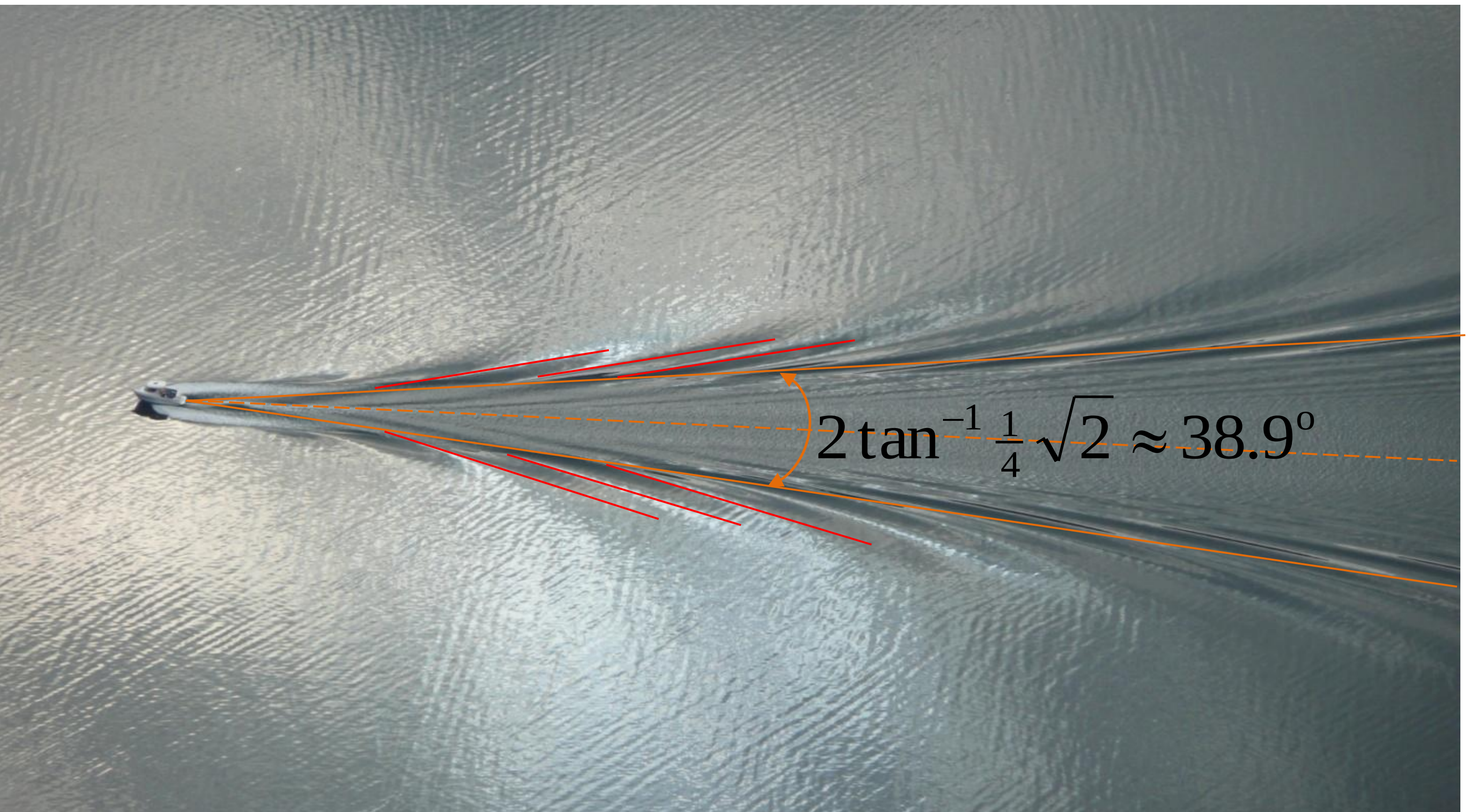
v > 20 cms^{-1}

Compare to minimum phase velocity

$$c_p = \sqrt[4]{\frac{4g\sigma}{\rho_1}} \approx \quad 23 \text{ cms}^{-1}$$

Kelvin wedge via a geometrical method?



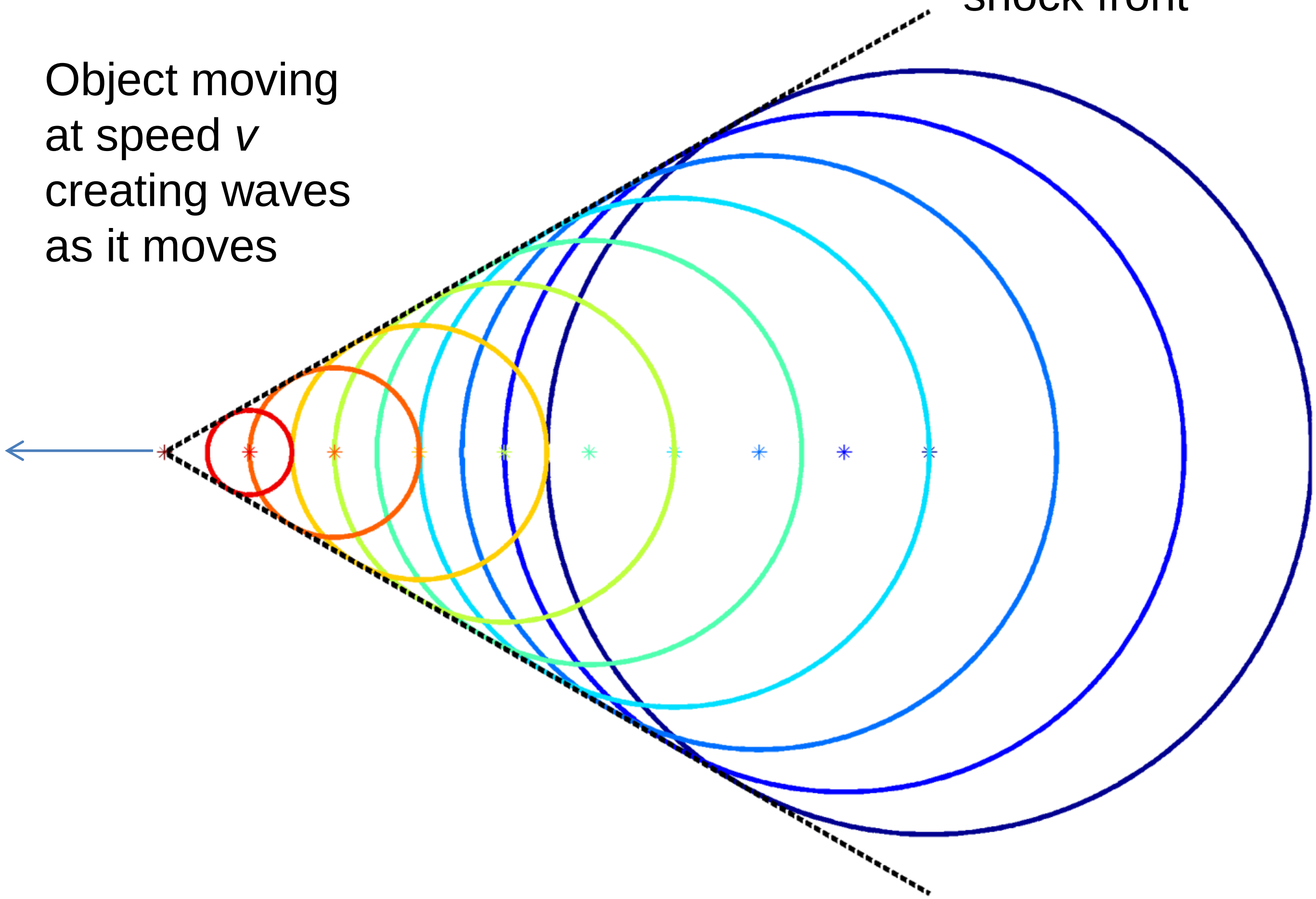


Is there an easier way to generate the Kelvin wedge pattern?
Is a deep water wave wake pattern analogous to the shock waves caused by a supersonic aircraft?

To investigate the latter let us first consider *Mach's construction*

shock front

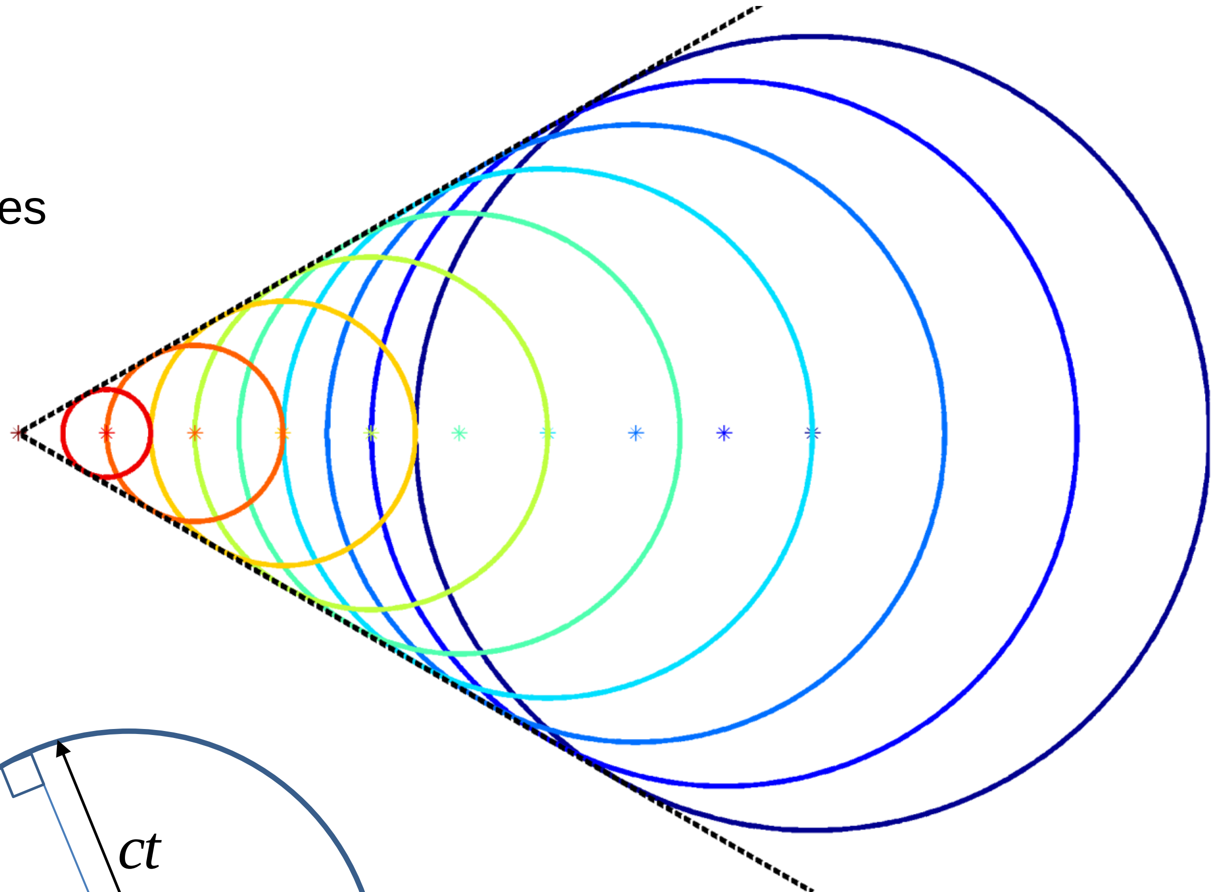
Object moving at speed *v* creating waves as it moves



'Infinitesimally thin' spherical shells of disturbance are created continuously as the object moves. They radiate out at the wave speed *c*

Mach's construction

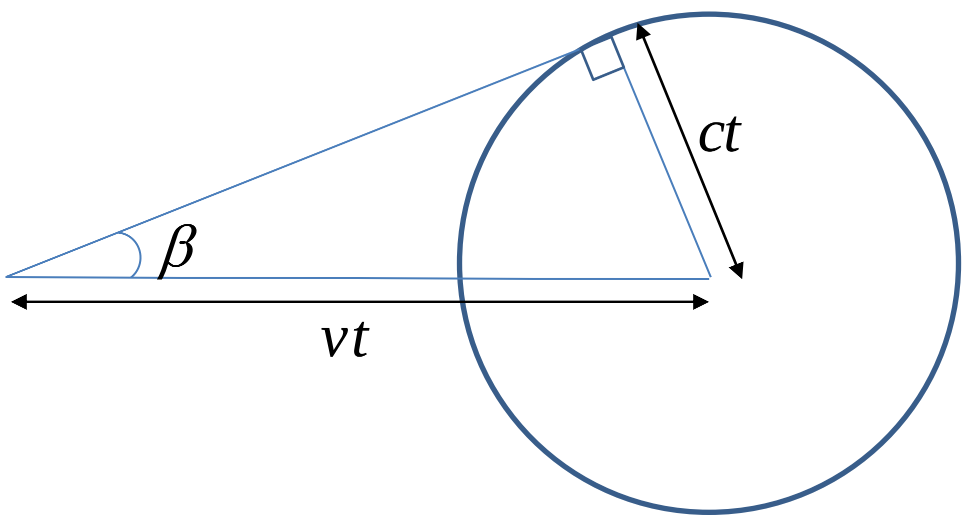
$v/c = 2.\ \ \sin^{-1}(c/v) = 30^{\circ}$



c is the wave speed
v is the velocity of the object making the waves

Mach number

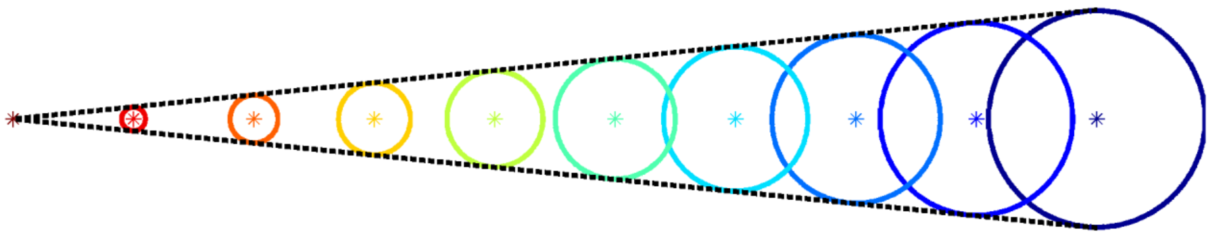
$$M = \frac{v}{c}$$



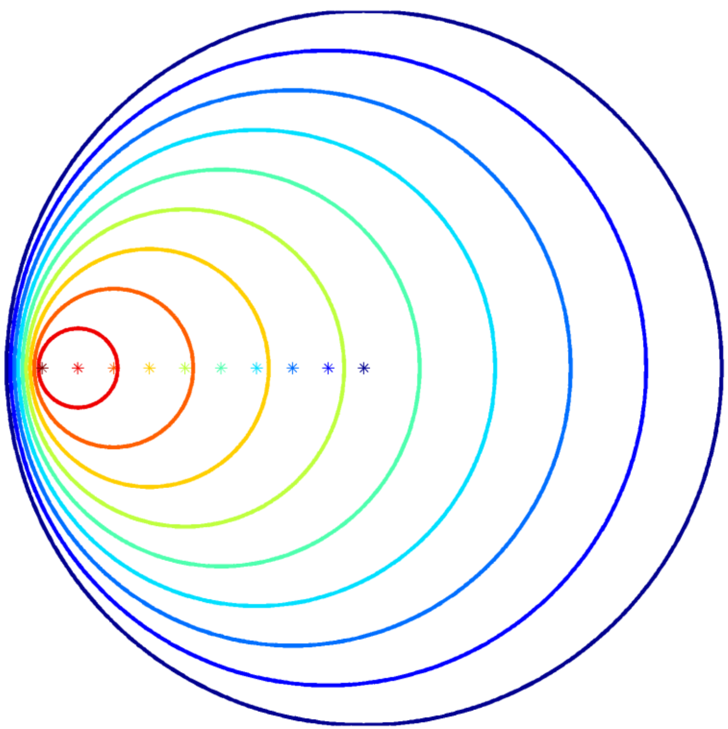
$$vt \sin\beta = ct$$

$$\therefore \ \beta = \sin^{-1}\left(\frac{c}{v}\right) = \sin^{-1}\tfrac{1}{M}$$

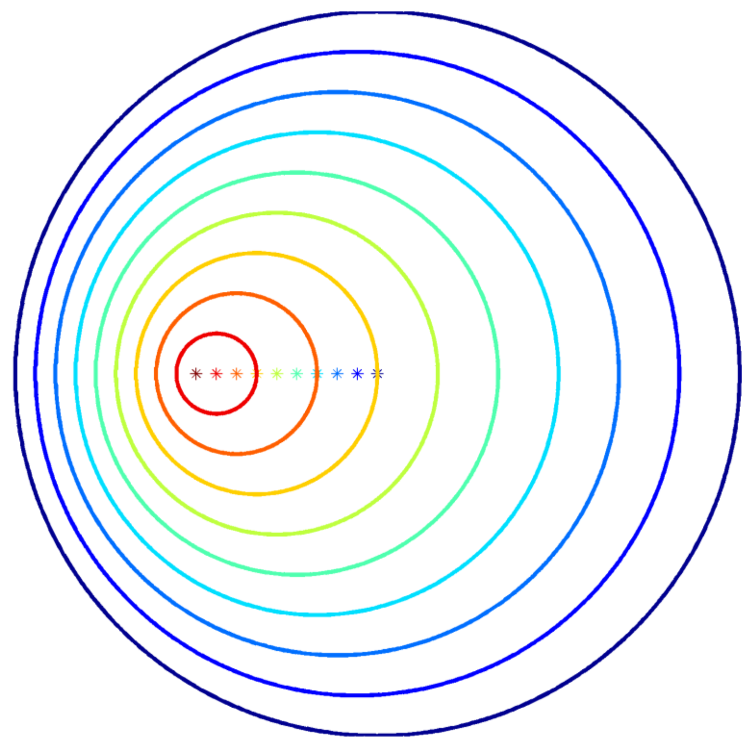
v/c = 10. $\sin^{-1}(c/v) = 5.7392^{o}$



v/c = 0.9. $\sin^{-1}(c/v) = \text{NaN}^{o}$



v/c = 0.5. $\sin^{-1}(c/v) = \text{NaN}^{o}$



For the Kelvin Wedge we know $\tan\beta = \frac{\sqrt{2}}{4}$

$$\sin^2\beta = \frac{1}{1+\frac{1}{\tan^2\beta}} \qquad \text{hence} \qquad \sin^2\beta = \frac{1}{1+\frac{1}{\frac{2}{16}}} = \frac{1}{9} \qquad \Rightarrow \qquad \sin\beta = \tfrac{1}{3}$$

Hence Mach's construction gives the Kelvin Wedge result if the corresponding Mach number is $M = 3$**.**

Is this just serendipity? Is it at all relevant? What does $M = 3$ *mean* for a ship moving at velocity v, generating deep water, gravity driven surface waves? (Which of course have wave speeds which *vary* with wavelength).

T.E. Faber attempts to discuss this possible connection in *Fluid Dynamics for Physicists* (p194). He makes a argument involving the fact that the group velocity of deep water gravity waves is half their phase velocity. However, he admits the whole discussion is on somewhat shaky foundations!

Further reading

Simulation of the Kelvin Wedge
http://www.math.ubc.ca/~cass/courses/m309-01a/carmen/Mainpage.htm

Rabaud and Moisy's model. "Ship wakes: Kelvin or Mach angle?" (arXiv: 1304.2653v1 9 April 2013)
http://arxiv.org/pdf/1304.2653.pdf

http://blog.physicsworld.com/2009/07/27/google-earth-physics/
http://physicsworld.com/cws/article/news/2013/may/30/physicists-rethink-celebrated-kelvin-wake-pattern-for-ships

Faber, T.E. Fluid Dynamics for Physicists. Cambridge University Press. 1995. p188-194

http://en.wikipedia.org/wiki/Wake

Perkinson, J. "Gravity waves: The Kevin wedge and related problems"
http://web.mit.edu/joyp/Public/HarvardWork/AM201FinalPaper.pdf