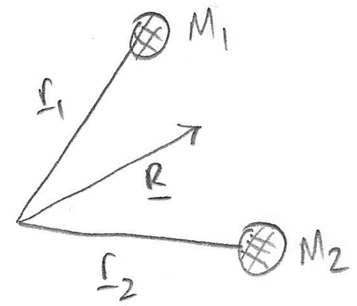
Kepler problem summary

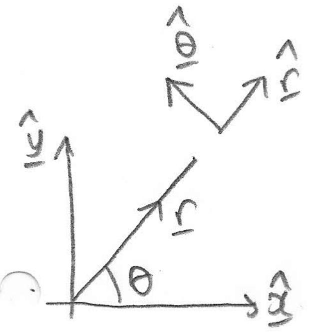
$\dot{\underline{R}} = \text{constant}$



[if $\underline{R} = 0$, centre of mass is at origin of coordinate system]

$$\underline{r}_1 = \frac{-M_2 \underline{r}}{M_1 + M_2} + \underline{R}$$

$$\underline{r}_2 = \frac{M_1 \underline{r}}{M_1 + M_2} + \underline{R}$$



$$\hat{\underline{r}} = \cos\theta\, \hat{\underline{x}} + \sin\theta\, \hat{\underline{y}}$$

$$\hat{\underline{\theta}} = -\sin\theta\, \hat{\underline{x}} + \cos\theta\, \hat{\underline{y}}$$

if $r = \dfrac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta}$

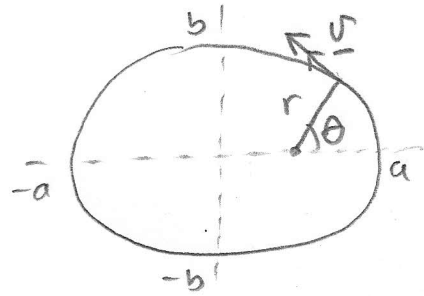
[orbits are ellipses if bound - Kepler I]

$$\underline{r} = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta}\, \hat{\underline{r}}$$

$$b = a\sqrt{1-\varepsilon^2}$$

$$\varepsilon = \sqrt{1 - b^2/a^2}$$

↑ "Eccentricity"

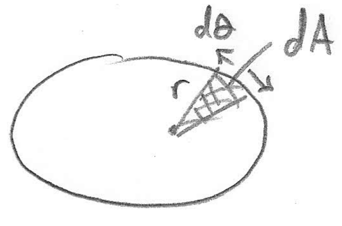


$$\underline{v} = \sqrt{\frac{G(M_1+M_2)}{a(1-\varepsilon^2)}}\,(1+\varepsilon\cos\theta)\left(\frac{\varepsilon\sin\theta}{1+\varepsilon\cos\theta}\,\hat{\underline{r}} + \hat{\underline{\theta}}\right)$$

$$\dot{\theta} = \sqrt{\frac{G(M_1+M_2)}{a^3(1-\varepsilon^2)^3}}\,(1+\varepsilon\cos\theta)^2 \quad \therefore \quad t = \int_0^\theta \sqrt{\frac{a^3(1-\varepsilon^2)^3}{G(M_1+M_2)}}\,\frac{d\theta}{(1+\varepsilon\cos\theta)^2}$$

$$E = \frac{-GM_1M_2}{2a} \qquad P^2 = \frac{4\pi^2}{G(M_1+M_2)}\,a^3 \quad \text{[Kepler III]}$$

$$\frac{dA}{dt} = \frac{1}{2}\sqrt{G(M_1+M_2)(1-\varepsilon^2)\,a} \quad \text{[Kepler II]}$$

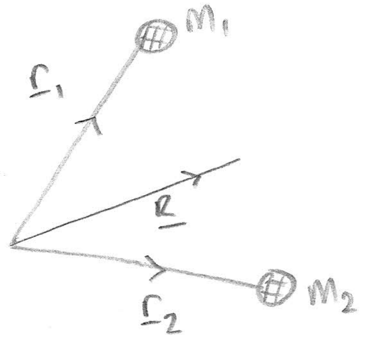


$$J^2 = \frac{G M_1^2 M_2^2 (1-\varepsilon^2)a}{M_1 + M_2}$$

↑

J is the magnitude of the angular momentum. This is a constant of the motion

Two Body KEPLER problem



Newton's Law of Gravity:

$$m_1 \ddot{\underline{r}}_1 = \frac{G m_1 m_2}{|\underline{r}_2 - \underline{r}_1|^3} (\underline{r}_2 - \underline{r}_1) \qquad (1)$$

$$m_2 \ddot{\underline{r}}_2 = \frac{-G m_1 m_2}{|\underline{r}_2 - \underline{r}_1|^3} (\underline{r}_2 - \underline{r}_1) \qquad (2)$$

Define $\boxed{\underline{r} = \underline{r}_2 - \underline{r}_1}$ $\therefore$ $\underline{r}_2 = \underline{r} + \underline{r}_1$ or $\underline{r}_1 = \underline{r}_2 - \underline{r}$

$$\boxed{\underline{R} = \frac{m_1 \underline{r}_1 + m_2 \underline{r}_2}{m_1 + m_2}}$$

Also define $\boxed{r = |\underline{r}|}$

centre of mass of system

So

$$\underline{R} = \frac{m_1 \underline{r}_1 + m_2 \underline{r} + m_2 \underline{r}_1}{m_1 + m_2}$$

$$\underline{R} = \underline{r}_1 + \frac{m_2 \underline{r}}{m_1 + m_2} \quad \therefore \quad \boxed{\underline{r}_1 = \underline{R} - \frac{m_2}{m_1 + m_2} \underline{r}} \qquad (3)$$

Also

$$\underline{R} = \frac{m_1 \underline{r}_2 - m_1 \underline{r} + m_2 \underline{r}_2}{m_1 + m_2}$$

$$\underline{R} = \underline{r}_2 - \frac{m_1 \underline{r}}{m_1 + m_2} \quad \therefore \quad \boxed{\underline{r}_2 = \underline{R} + \frac{m_1 \underline{r}}{m_1 + m_2}} \qquad (4)$$

(2) - (1):

$$\ddot{\underline{r}} = \ddot{\underline{r}}_2 - \ddot{\underline{r}}_1 = \frac{-G}{r^3} (m_2 + m_1) \underline{r}$$

ie $\boxed{\ddot{\underline{r}} = -\frac{GM}{r^3} \underline{r}}$ $\qquad \boxed{M = m_1 + m_2}$ $\qquad (5)$

ie same problem as a small particle orbiting a large mass M, in other words a one body problem.

Now $\underline{R} = \dfrac{m_1 \underline{r}_1 + m_2 \underline{r}_2}{M}$

$\ddot{\underline{R}} = \dfrac{m_1}{M} \ddot{\underline{r}}_1 + \dfrac{m_2 \ddot{\underline{r}}_2}{M}$

using ①, ②: $m_1 \ddot{\underline{r}}_1 + m_2 \ddot{\underline{r}}_2 = \underline{0}$

So $\ddot{\underline{R}} = 0 \Rightarrow \boxed{\dot{\underline{R}} = \text{Constant}}$ ⑥

ie centre of mass of system moves at a constant velocity.

let us choose a frame of reference such that $\boxed{\dot{\underline{R}} = \underline{0}}$. This is an inertial frame, so the dynamics of m_1 and m_2 will not be affected.

The angular momentum of the system is

$$\boxed{\underline{J} = m_1 \underline{r}_1 \times \dot{\underline{r}}_1 + m_2 \underline{r}_2 \times \dot{\underline{r}}_2}$$

using $\underline{r}_1 = \underline{R} - \dfrac{m_2 \underline{r}}{M}$ and $\underline{r}_2 = \underline{R} + \dfrac{m_1 \underline{r}}{M}$

$\therefore$ $\underline{J} = m_1 \left(\underline{R} - \dfrac{m_2 \underline{r}}{M} \right) \times \left(-\dfrac{m_2}{M} \dot{\underline{r}} \right) + m_2 \left(\underline{R} + \dfrac{m_1 \underline{r}}{M} \right) \times \dfrac{m_1 \dot{\underline{r}}}{M}$

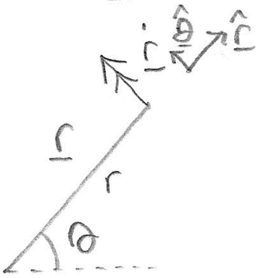
(since $\dot{\underline{R}} = 0$)

$\therefore$ $$\boxed{\underline{J} = \frac{m_1 m_2}{m_1 + m_2} \underline{r} \times \dot{\underline{r}}}$$ ⑦

$$\dot{\underline{J}} = \frac{m_1 m_2}{m_1 + m_2} \left(\dot{\underline{r}} \times \dot{\underline{r}} + \underline{r} \times \ddot{\underline{r}} \right) = \underline{0}$$

Since ⑤: $\ddot{\underline{r}} = -\dfrac{GM}{r^3} \underline{r}$ and $\dot{\underline{r}} \times \dot{\underline{r}} = \underline{0}$

$\underline{r} \times \underline{r} = \underline{0}$



$\therefore \dot{\underline{J}} = 0$ implies $\underline{J}$ is a constant. | Define $J = |\underline{J}|$

$\Rightarrow$ which means motion is $\boxed{\text{planar}}$

Now $\underline{r} = r\hat{\underline{r}}$ in plane polars

$$\dot{\underline{r}} = \dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}}$$

$$\therefore |\dot{\underline{r}}|^2 = \dot{r}^2 + r^2\dot{\theta}^2 \qquad \{|\dot{\underline{r}}|^2 = \dot{\underline{r}}\cdot\dot{\underline{r}}\}$$

For our system, the total energy is

$$\boxed{E = \tfrac{1}{2}m_1|\dot{\underline{r}}_1|^2 + \tfrac{1}{2}m_2|\dot{\underline{r}}_2|^2 - \frac{Gm_1m_2}{r}}$$

$$\dot{\underline{r}}_1 = \frac{-m_2\dot{\underline{r}}}{M} \quad \text{and} \quad \dot{\underline{r}}_2 = \frac{m_1\dot{\underline{r}}}{M}$$

$$\left[\begin{aligned} &m_1m_2^2 + m_2m_1^2 \\ &= m_1m_2(m_1+m_2) \\ &= m_1m_2M \end{aligned}\right]$$

$$\therefore E = \frac{1}{2}\frac{m_1m_2^2}{M^2}|\dot{\underline{r}}|^2 + \frac{1}{2}\frac{m_2m_1^2}{M^2}|\dot{\underline{r}}|^2 - \frac{Gm_1m_2}{r}$$

$$E = \frac{1}{2}\frac{m_1m_2}{M}|\dot{\underline{r}}|^2 - \frac{Gm_1m_2}{r}$$

$$\boxed{E = \frac{m_1m_2}{2M}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{Gm_1m_2}{r}} \qquad (8)$$

Now $$\underline{J} = \frac{m_1m_2}{M} r\hat{\underline{r}} \times (\dot{r}\hat{\underline{r}} + r\dot{\theta}\hat{\underline{\theta}})$$

$$\therefore \underline{J} = \frac{m_1m_2}{M} r^2\dot{\theta}\,(\hat{\underline{r}} \times \hat{\underline{\theta}})$$

$$\therefore J^2 = \frac{m_1^2m_2^2}{M^2} r^4\dot{\theta}^2$$

$$\therefore \boxed{\dot{\theta}^2 = \frac{M^2J^2}{m_1^2m_2^2r^4}} \qquad (8\tfrac{1}{2})$$

$$\therefore E = \frac{m_1m_2}{2M}\left(\dot{r}^2 + \frac{M^2J^2}{m_1^2m_2^2r^2}\right) - \frac{Gm_1m_2}{r}$$

Now, for reasons that will become clear later, consider a variable change

$$\boxed{u = \frac{1}{r}} \qquad \therefore \quad \dot{u} = -\frac{1}{r^2}\dot{r} \Rightarrow \dot{r} = -r^2\dot{u} = -\dot{u}/u^2$$

$$\therefore \quad E = \frac{m_1 m_2}{2M}\left(\frac{\dot{u}^2}{u^4} + \left(\frac{MJ}{m_1 m_2}\right)^2 u^2 \right) - Gm_1 m_2 u$$

Now we want the trajectory of the masses, so $u = u(\theta)$ is desired. (Rather than $u(t)$, and $\theta(t)$).

Now $$\frac{du}{d\theta} = \frac{du}{dt} \times \frac{dt}{d\theta} = \frac{\dot{u}}{\dot{\theta}}$$

$$\therefore \quad \dot{u}^2 = \dot{\theta}^2 \left(\frac{du}{d\theta}\right)^2$$

Eqn $8^1/_2$

using $$\dot{\theta}^2 = \left(\frac{MJ}{m_1 m_2 r^2}\right)^2 = \left(\frac{MJ}{m_1 m_2}\right)^2 u^4$$

$$\Rightarrow \quad \dot{u}^2 = \left(\frac{du}{d\theta}\right)^2 \left(\frac{MJ}{m_1 m_2}\right)^2 u^4$$

$$\therefore \quad E = \frac{M}{2m_1 m_2} J^2 \left(\left(\frac{du}{d\theta}\right)^2 + u^2 \right) - Gm_1 m_2 u$$

Now conservation of energy $\Rightarrow \dfrac{dE}{dt} = \dfrac{dE}{d\theta} = 0$

$$\therefore \quad 0 = \frac{M}{2m_1 m_2} J^2 \left(2\frac{du}{d\theta}\frac{d^2u}{d\theta^2} + 2u\frac{du}{d\theta} \right) - Gm_1 m_2 \frac{du}{d\theta}$$

Since $\dfrac{du}{d\theta} \neq 0 \quad \forall \ \theta$

$$\Rightarrow \quad \frac{M}{m_1 m_2} J^2 \left(\frac{d^2u}{d\theta^2} + u \right) = Gm_1 m_2$$

$$\boxed{\frac{d^2u}{d\theta^2} + u = \frac{Gm_1^2 m_2^2}{(m_1 + m_2) J^2}} \qquad (9)$$

Now, an <u>ellipse</u> has polar equation

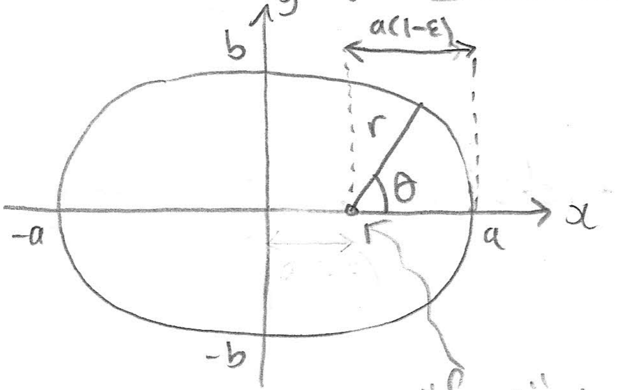
when $\theta = 0$, $r = \frac{a(1+\varepsilon)(1-\varepsilon)}{1+\varepsilon}$
$= a(1-\varepsilon)$

$$\boxed{r = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta}}$$

a is the "semi-major axis"

ε is the <u>eccentricity</u>

$$\boxed{\varepsilon = \sqrt{1 - b^2/a^2}}$$



Note $\boxed{b = a\sqrt{1-\varepsilon^2}}$

ie circular orbit
$\Rightarrow \varepsilon = 0$.
ie $a = b$

$\therefore$ $u(\theta) = \frac{1+\varepsilon\cos\theta}{a(1-\varepsilon^2)}$ substituting into (9)

$$\frac{-\varepsilon\cos\theta}{a(1-\varepsilon^2)} + \frac{1+\varepsilon\cos\theta}{a(1-\varepsilon^2)} = \frac{GM_1^2M_2^2}{(M_1+M_2)J^2}$$

$\frac{d^2u}{d\theta^2}$

$$\Rightarrow \frac{1}{a(1-\varepsilon^2)} = \frac{GM_1^2M_2^2}{(M_1+M_2)J^2}$$

$$\therefore \quad \boxed{J^2 = \frac{GM_1^2M_2^2(1-\varepsilon^2)a}{M_1+M_2}} \quad (10)$$

ie (9) accepts ellipses as solutions THUS PROVING KEPLER'S FIRST LAW, that is

"Bound orbits of a two-mass system driven by an inverse square law of gravitational attraction are ellipses".

Now $E = \frac{M}{2m_1 m_2} J^2 \left(\left(\frac{-\sin\theta\, \varepsilon}{a(1-\varepsilon^2)} \right)^2 + \left(\frac{1+\varepsilon\cos\theta}{a(1-\varepsilon^2)} \right)^2 \right) + \dots$

$\left(\frac{dy}{d\theta}\right)^2$

$\dots - \frac{G m_1 m_2 (1+\varepsilon\cos\theta)}{a(1-\varepsilon^2)}$

$$E = \frac{M}{2m_1 m_2} \times \frac{G m_1^2 m_2^2 (1-\varepsilon^2) a}{(m_1+m_2)} \left\{ \frac{\varepsilon^2 \sin^2\theta + 1 + \varepsilon^2\cos^2\theta + 2\varepsilon\cos\theta}{a^2(1-\varepsilon^2)^2} \right\}$$

$[M = m_1 + m_2]$

$$- \frac{G m_1 m_2 (1+\varepsilon\cos\theta)}{a(1-\varepsilon^2)}$$

$$\Rightarrow \quad E = \frac{G m_1 m_2}{2a} \left\{ \frac{\varepsilon^2}{1-\varepsilon^2} + \frac{1+2\varepsilon\cos\theta}{1-\varepsilon^2} \right\} - \frac{G m_1 m_2 (1+\varepsilon\cos\theta)}{a(1-\varepsilon^2)}$$

$$= \frac{G m_1 m_2}{2a} \left\{ \frac{\varepsilon^2 + 1 + 2\varepsilon\cos\theta - 2 - 2\varepsilon\cos\theta}{1-\varepsilon^2} \right\}$$

$$\boxed{E = -\frac{G m_1 m_2}{2a}} \quad (11)$$

In summary so far:

$r(\theta) = \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta}$

$E = -\frac{G m_1 m_2}{2a}$

$\underline{r} = r(\theta)\,\hat{\underline{r}}$

$\underline{r} = r\cos\theta\,\hat{\underline{x}} + r\sin\theta\,\hat{\underline{y}}$

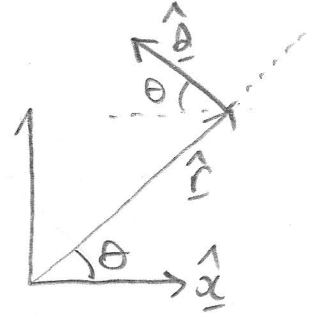
$J^2 = \frac{G m_1^2 m_2^2 (1-\varepsilon^2) a}{m_1+m_2}$

$\underline{r}_1 = -\frac{m_2}{m_1+m_2}\underline{r}$

(6) $\dot{\theta}^2 = \frac{(m_1+m_2)^2 J^2}{m_1^2 m_2^2 r^4}$

$\underline{r}_2 = \frac{m_1}{m_1+m_2}\underline{r}$

$$\underline{v} = \dot{r}\,\hat{\underline{r}} + r\dot{\theta}\,\hat{\underline{\theta}} = \dot{\underline{r}}$$



$$\boxed{\hat{\underline{\theta}} = -\sin\theta\,\hat{\underline{x}} + \cos\theta\,\hat{\underline{y}}}$$

$$\dot{\underline{r}} = -\frac{a(1-\varepsilon^2)}{(1+\varepsilon\cos\theta)^2}(-\varepsilon\sin\theta)\,\dot{\theta}\,\hat{\underline{r}} + \frac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta}\,\dot{\theta}\,\hat{\underline{\theta}}$$

[Since $r = \dfrac{a(1-\varepsilon^2)}{1+\varepsilon\cos\theta}$]

$$\therefore\quad \underline{v} = \frac{a(1-\varepsilon^2)\,\dot{\theta}}{1+\varepsilon\cos\theta}\left(\frac{\varepsilon\sin\theta}{1+\varepsilon\cos\theta}\,\hat{\underline{r}} + \hat{\underline{\theta}}\right)$$

Now $$\dot{\theta}^2 = \frac{(m_1+m_2)^{\cancel{2}}}{\cancel{m_1^2 m_2^2}\, r^4}\;\frac{G\,\cancel{m_1^2 m_2^2}\,(1-\varepsilon^2)\,a}{\cancel{m_1+m_2}}$$

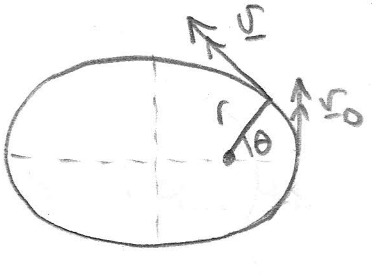
$\dfrac{1}{r^4}$

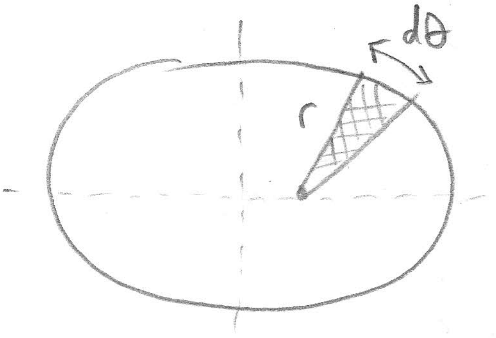
$$\Rightarrow\quad \dot{\theta}^2 = G(m_1+m_2)(1-\varepsilon^2)\,a \times \frac{(1+\varepsilon\cos\theta)^4}{a^4(1-\varepsilon^2)^4}$$

$$\boxed{\dot{\theta} = \sqrt{\frac{G(m_1+m_2)}{a^3(1-\varepsilon^2)^3}}\;(1+\varepsilon\cos\theta)^2} \qquad (12)$$

$$\therefore\quad \boxed{\underline{v} = \sqrt{\frac{G(m_1+m_2)}{a(1-\varepsilon^2)}}\;(1+\varepsilon\cos\theta)\left(\frac{\varepsilon\sin\theta}{1+\varepsilon\cos\theta}\,\hat{\underline{r}} + \hat{\underline{\theta}}\right)} \qquad (13)$$

Note when $\theta = 0$ $\quad \underline{v}_{\theta=0} = \sqrt{\dfrac{G(m_1+m_2)}{a(1-\varepsilon^2)}}\;(1+\varepsilon)\,\hat{\underline{\theta}}$





Area swept out by orbit occurs at rate s.t $dA = \frac{1}{2} r^2 d\theta$

$$\therefore \frac{dA}{dt} = \frac{1}{2} r^2 \dot{\theta}$$

Now $$r^2 \dot{\theta} = \frac{a^2(1-\varepsilon^2)}{(1+\varepsilon\cos\theta)^2} \sqrt{\frac{G(m_1+m_2)}{a^3(1-\varepsilon^2)^3}} (1+\varepsilon\cos\theta)^2$$

$$= \sqrt{G(m_1+m_2)a(1-\varepsilon^2)}$$

So $$\boxed{\frac{dA}{dt} = \frac{1}{2}\sqrt{G(m_1+m_2)(1-\varepsilon^2)a}}$$

KEPLER'S SECOND LAW: "The rate of area $\frac{dA}{dt}$ swept out by the elliptical orbit from the focus of the ellipse is a constant"

Now since $\frac{dA}{dt}$ is a constant, the orbital period

$$P = \frac{\pi ab}{\frac{dA}{dt}}$$ Now $b = a\sqrt{1-\varepsilon^2}$

$$\therefore P = \frac{\pi a^2 \sqrt{1-\varepsilon^2}}{\frac{1}{2}\sqrt{G(m_1+m_2)(1-\varepsilon^2)a}}$$

$$\Rightarrow \boxed{P^2 = \frac{4\pi^2}{G(m_1+m_2)} a^3}$$

KEPLER'S THIRD LAW

$P^2 \propto a^3$

AH. 12/9/14