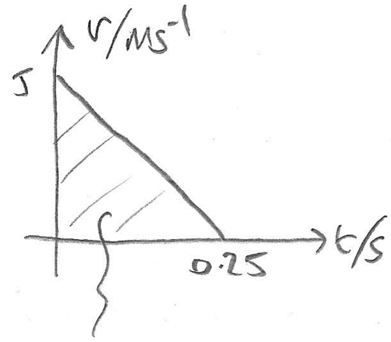
KINEMATICS

1/ (i)



$$\boxed{v = 5 - 20t}$$

$v = 0$ when $20t = 5$

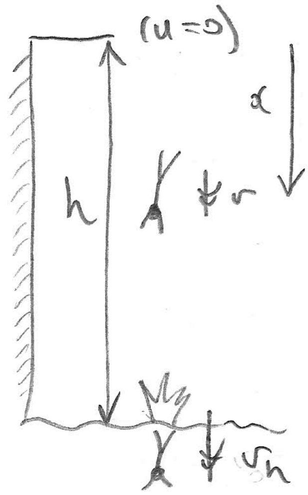
$t = 5/20$

$\boxed{t = 0.25s}$

Displacement $x = \frac{1}{2}(0.25)(5)$

$= \boxed{0.625\ m}$ (62.5 cm)

(ii)



$\downarrow g = 9.81\ m/s^2$

$v^2 = 2gx$ $\therefore$ $v_h = \sqrt{2gh}$

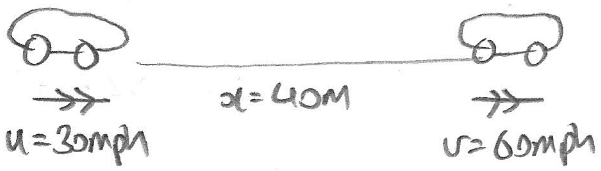
$v_h = \sqrt{2 \times 9.81 \times 27}$

$= \boxed{23.0\ m/s}$

$h = \frac{1}{2}gt^2$ $\therefore$ $\sqrt{\frac{2h}{g}} = t$

$\therefore t = \sqrt{\frac{2 \times 27}{9.81}} = \boxed{2.35s}$

(iii)



$v^2 = u^2 + 2ax$

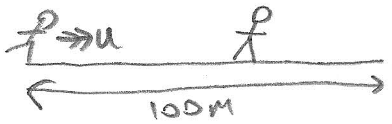
$\therefore \frac{v^2 - u^2}{2x} = a$

$\therefore$ Since 1 mph $= \frac{1}{2.24}$ ms⁻¹

$\Rightarrow a = \frac{\left(\frac{60}{2.24}\right)^2 - \left(\frac{30}{2.24}\right)^2}{2 \times 40}$ (m/s^2)

$\Rightarrow \boxed{a = 6.73\ m/s^2}$

(iv)



$$x = ut + \tfrac{1}{2}at^2$$

$$\therefore \frac{x - \frac{1}{2}at^2}{t} = u$$

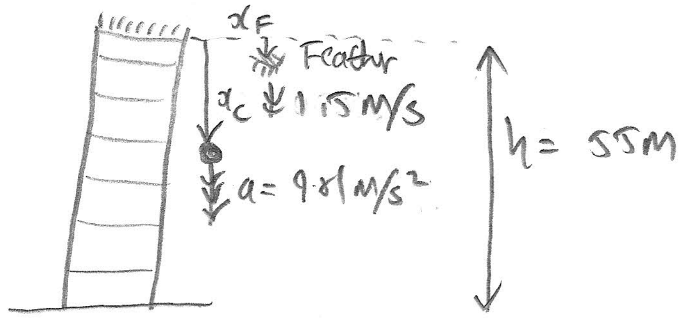
$$\therefore \frac{100 - \frac{1}{2}\times 0.1\times 12.0^2}{12.0} = u$$

$$\therefore \boxed{u = 7.73\ \text{M/s}}$$

[Note he crosses the line at $v = 7.73 + 0.1\times 12.0$

$\boxed{v = 8.93\text{M/s}}$]

v)



Leaning tower of Pisa.

a) For cannon ball impact speed

$$v = \sqrt{2ah}$$

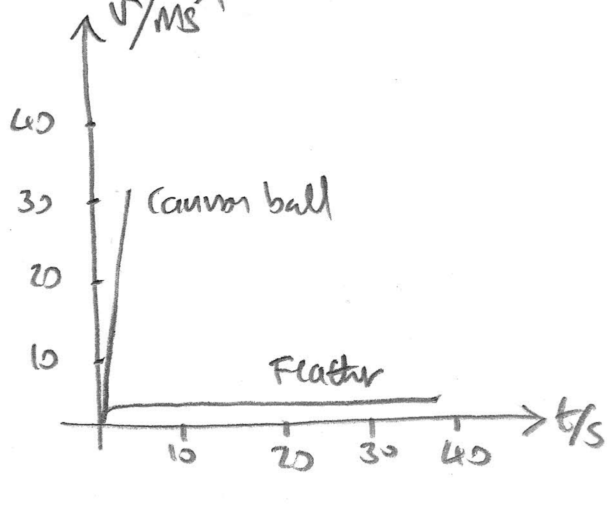
$$[\ v^2 = u^2 + 2ah\ ]$$

zero

$$\Rightarrow v = \sqrt{2\times 9.81\times 55}$$

$$\boxed{v = 32.8\text{M/s}}$$

b)



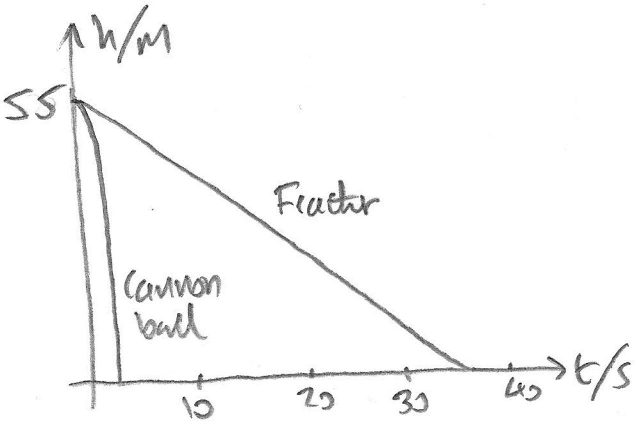
For cannon ball:

$$h = \tfrac{1}{2}at^2 \Rightarrow t = \sqrt{\frac{2h}{a}}$$

$$t = \sqrt{2\times 55/9.81} = \boxed{3.34\text{s}}$$

For feather:

$$h = 1.5t \therefore t = \frac{55}{1.5} = \boxed{36.7\text{s}}$$



c) At time t, feather is height:

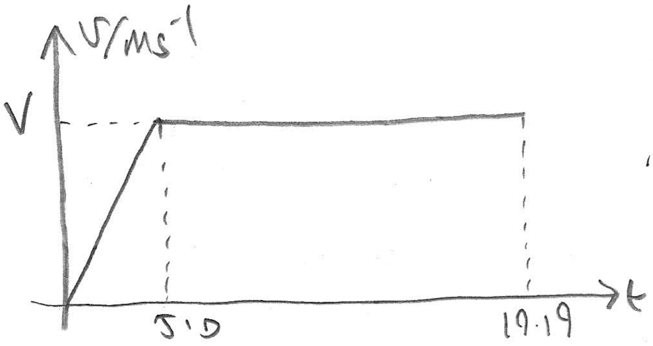
$$\boxed{h = 55 - 1.5t}$$ metres

so after $\sqrt{\frac{2\times 55}{9.81}} = 3.34s$

$$\boxed{h = 50.0\,m}$$

Feather hits the ground after 36.7s, so you have to wait 33.4s after the cannon ball hits the ground for the feather to join it. $(36.7 - 3.34 = 33.4)$.

vi) Usain Bolt's 2009 200m world record in Berlin (2)



Area under graph is 200m.

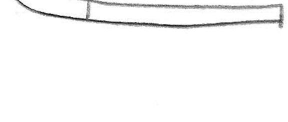
$\therefore \frac{1}{2}(19.19 + 14.19)V = 200$

$\therefore \boxed{V = 11.98\,m/s}$

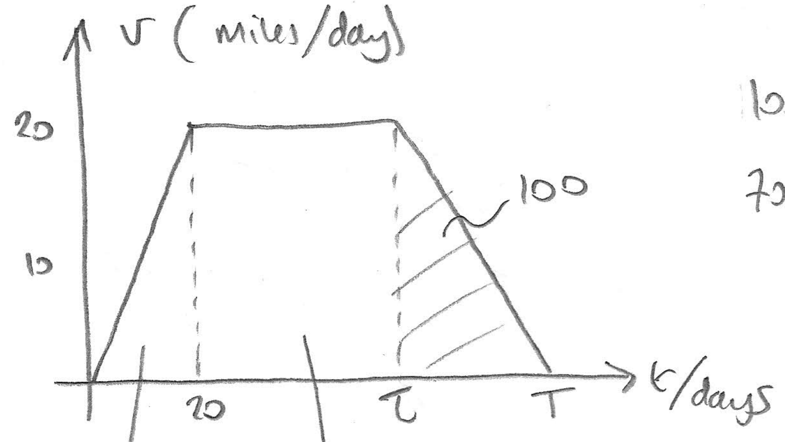
Initial acceleration was $a = \frac{V}{5.0}$ ms^{-2}

$$\boxed{a = 2.40\,m/s^2}$$

[Actually this is just the tangential acceleration. As he was running a bend, he would have to accelerate radially inwards too].



vii/



$\frac{1}{2}(20)(20)$
$= \boxed{200}$

$1000 - 200 - 100$
$= \boxed{700}$

$$100 = \frac{1}{2}(20)(T - \tau)$$

$$700 = 20(\tau - 20)$$

$$\therefore \quad \frac{700}{20} + 20 = \tau$$

$$\boxed{\tau = 55 \text{ days}}$$

$$\therefore \quad \frac{200}{20} + \tau = T$$

$$\therefore \quad T = 10 + 55 = \boxed{65 \text{ days}}$$

(viii)
$$\boxed{\begin{aligned} x_1 &= 100 - 8t + \tfrac{1}{2}0.1t^2 \\ x_2 &= 105 - 8.2t \end{aligned}}$$

(x is distance from finish.
$t = 0$ when runner #1 hits the home straight)

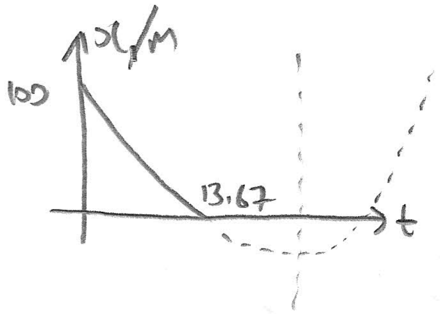
Athlete #1 take t_1 seconds to finish ($x_1 = 0$)

$$\therefore \quad t^2 - 160t + 2000 = 0$$

$$(t - 80)^2 - 6400 + 2000 = 0$$

$$t = 80 \pm \sqrt{4400}$$

$$\boxed{t = 13.67s}$$



(take –ve root)

Athlete #2 take $t_2 = \frac{105}{8.2} = \boxed{12.80s}$ to finish

So $\boxed{\text{Athlete 2 wins the race.}}$ At 12.80s!

$x_1 = 100 - 8(12.80) + \frac{1}{2}0.1(12.80)^2 = \boxed{5.176m}$ from the finish.

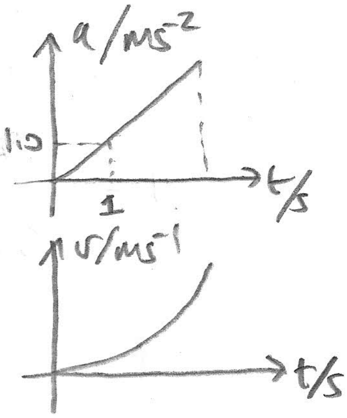
(ix) $\dot{a} = 1\ m/s^3$

$\therefore\ a = t$

$\therefore\ v = \frac{1}{2}t^2$

$\therefore\ x = \frac{1}{6}t^3$

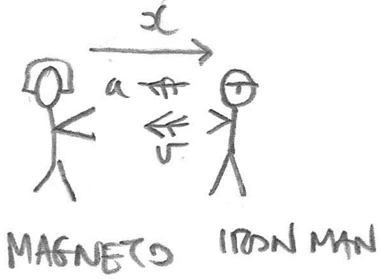
$\left.\begin{matrix} \\ \\ \\ \end{matrix}\right\}$ Since $a(0) = 0$, $v(0) = 0$, $x(0) = 0$



$\therefore$ when $x = 100\ m$, $t = \sqrt[3]{600}$ (s)

$\boxed{t = 8.43 s}$

(x) $a = \dfrac{k}{x^4}$



{ Force between two magnetic dipoles is $\frac{1}{x^4}$ }

$10 = \dfrac{k}{0.25^4}$

$\therefore\ k = 10 \times 0.25^4$

$\boxed{k = 3.91 \times 10^{-2}\ m^5/s^2}$

$\therefore\ \boxed{a(x) = 10\left(\dfrac{0.25}{x}\right)^4}$

Now $a = -\dfrac{v\,dv}{dx}$ (−ve since a is in opposite direction to displacement x)

$\therefore\ \dfrac{v\,dv}{dx} = -10\left(\dfrac{0.25}{x}\right)^4$

$$\int_0^v v'\,dv' = -10 \times 0.25^4 \int_{1.0}^{x} x^{-4}\,dx$$

$$\frac{1}{2}v^2 = -10 \times 0.25^4 \left[-\frac{1}{3}x^{-3}\right]_{1.0}^{x}$$

$$\therefore\ \boxed{v = \sqrt{20 \times 0.25^4\left(-\frac{1}{3} + \frac{1}{3x^3}\right)}^{\ \frac{1}{2}}}$$

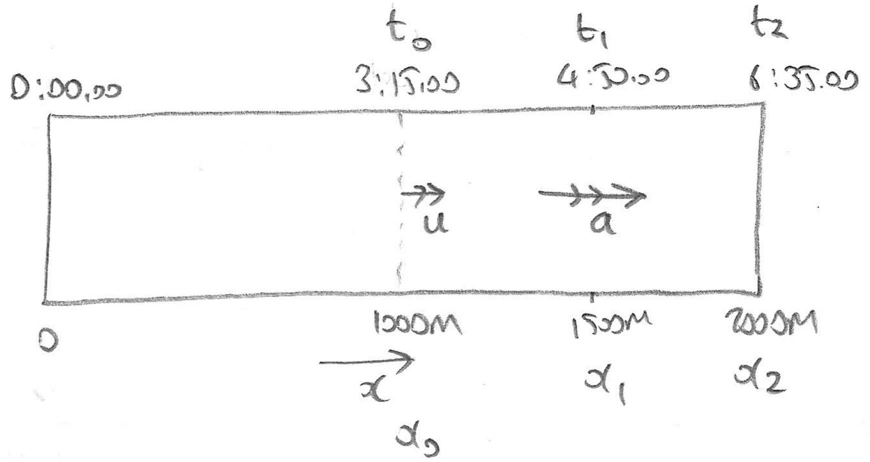
So when $x = 0.1\ m$, $\boxed{v = 5.10\ m/s}$

2/ 2013 USA Women's rowing eight:

$$v = \frac{2000\,m}{(5\times 60 + 54.16)\,s} = \boxed{5.65\ m/s}$$

↑ Average speed

Camford ladies:



Assume constant acceleration motion for the final 1000m.

So:
$$x_1 = u(t_1-t_0) + \tfrac{1}{2}a(t_1-t_0)^2 + x_0$$
$$x_2 = x_0 + u(t_2-t_0) + \tfrac{1}{2}a(t_2-t_0)^2$$

① $x_1(t_2-t_0) = x_0(t_2-t_0) + u(t_1-t_0)(t_2-t_0) + \tfrac{1}{2}a(t_1-t_0)^2(t_2-t_0)$

② $x_2(t_1-t_0) = x_0(t_1-t_0) + u(t_2-t_0)(t_1-t_0) + \tfrac{1}{2}a(t_2-t_0)^2(t_1-t_0)$

① - ②:
$$x_1(t_2-t_0) - x_2(t_1-t_0) = x_0(t_2-t_0) - x_0(t_1-t_0) + \tfrac{1}{2}a\left((t_1-t_0)^2(t_2-t_0) - (t_2-t_0)^2(t_1-t_0)\right)$$

$$\therefore \boxed{a = \frac{2\left[(x_1-x_0)(t_2-t_0) - (x_2-x_0)(t_1-t_0)\right]}{(t_1-t_0)^2(t_2-t_0) - (t_2-t_0)^2(t_1-t_0)}}$$

$$\therefore \quad \boxed{\frac{x_1 - x_0 - \frac{1}{2}a(t_1-t_0)^2}{t_1 - t_0} = u}$$

$x_0 = 1000M$ $\quad t_0 = 195s$

$x_1 = 1500m$ $\quad t_1 = 290s$

$x_2 = 2000M$ $\quad t_2 = 395s$

$$\therefore \quad a = 2\left[\frac{(1500-1000)(395-195) - (2000-1000)(290-195)}{(290-195)^2(395-195) - (395-195)^2(290-195)}\right]$$

$$a = \frac{2 \times 5000}{-1995000}$$

$$\boxed{a = -5.01 \times 10^{-3} \ m/s^2} \rightarrow \text{calc memory.}$$

$$\therefore \quad u = \frac{1500 - 1000 - \frac{1}{2}a(290-195)^2}{290-195}$$

$$\boxed{u = 5.50 \ m/s}$$

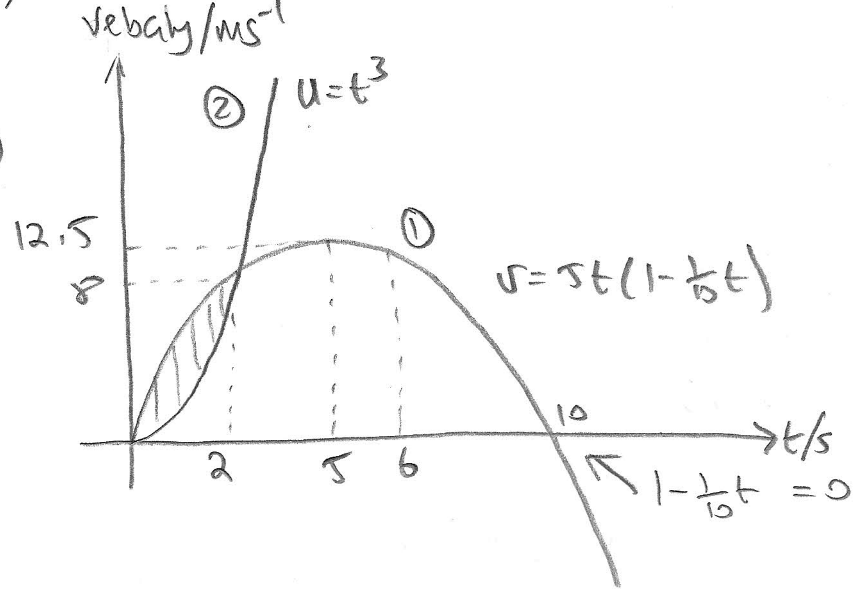
The Camford ladies cross the finishing line at

$$v = \underbrace{5.50}_{u} - \underbrace{5.01 \times 10^{-3}}_{a} \times \underbrace{(395-195)}_{t_2 - t_0} \quad m/s.$$

$$\boxed{v = 4.50 \ m/s}$$

3/ ① $v = 5t(1-\frac{1}{10}t)$ ② $u = t^3$

a)



$1-\frac{1}{10}t = 0 \Rightarrow t = 10\text{s}$

b) when $t = 2$

$u = 8$ ✓

$v = 5(2)(1-0.1\times 2)$

$= 8$ ✓

i.e. $u = v$ when $t = 2$.

$\boxed{\text{Accelerations:}}$ $\frac{dv}{dt} = 5 - t$ $\therefore \left.\frac{dv}{dt}\right|_{t=2} = \boxed{3}$ (m/s^2)

$[\ v = 5t - \frac{1}{2}t^2\ ]$

$\frac{du}{dt} = 3t^2$

$\therefore \left.\frac{du}{dt}\right|_{t=2} = \boxed{12}$ (m/s^2)

c) //// area is the extra distance travelled by ① than ② after 2s.

This distance is: $\int_0^2 (5t - \frac{1}{2}t^2 - t^3)\,dt$

$= \left[\frac{5}{2}t^2 - \frac{1}{6}t^3 - \frac{1}{4}t^4\right]_0^2$

$= \frac{5}{2}(4) - \frac{1}{6}(8) - \frac{1}{4}(16)$

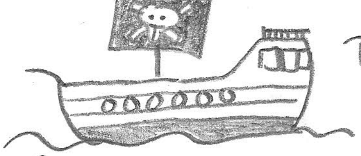
$= 4\frac{2}{3}$ so $\boxed{4.67\text{ m}}$

d) v is largest when $t = 5$s, ie when $\frac{dv}{dt} = 0$.

③ $\therefore$ Extra distance travelled by ② after 5s is $-\left[\frac{5}{2}(5)^2 - \frac{1}{6}(5)^3 - \frac{1}{4}(5)^4\right]$

$= 114\frac{7}{12} = \boxed{114.58\text{ m}}$

4/



The Black Pig

cannon ball $M = 5.44\,kg$, $\rho = 7874\,kg/m^3$

x

$\uparrow \frac{4}{3}\pi r^3 \rho_w g + \frac{1}{2} C_D \rho_w \pi r^2 v^2$

$\downarrow Mg$

$\downarrow v \quad \downarrow a = v\frac{dv}{dx}$

Drag coefficient $C_D = 0.47$

$\rho_w = 1029\,kg/m^3$.

(Seawater density).

Challenger Deep

(10,929m)

a) $M = \frac{4}{3}\pi r^3 \rho$

$\therefore \sqrt[3]{\frac{3M}{4\pi\rho}} = r$

$\therefore r = \sqrt[3]{\frac{3 \times 5.44}{4\pi \times 7874}}$

$\boxed{r = 0.0548\,M}$ (5.48cm)

b) Newton II: $Ma = Mg - \underbrace{\frac{4}{3}\pi r^3 \rho_w g}_{\text{upthrust}} - \underbrace{\frac{1}{2} C_D \rho_w \pi r^2 v^2}_{\text{drag}}$

Now $\frac{4}{3}\pi r^3 = M/\rho \quad \therefore \quad \pi r^2 = \frac{3M}{4\rho r}$

$\therefore \quad Ma = Mg - \frac{M\rho_w}{\rho} g - \frac{1}{2} C_D \rho_w \frac{3M}{4\rho r} v^2$

$\therefore \quad \boxed{\frac{v\,dv}{dx} = g\left(1 - \frac{\rho_w}{\rho}\right) - \frac{3}{8} C_D \frac{\rho_w}{\rho} \frac{v^2}{r}}$

c) $$\int_0^v \frac{v\,dv\left(-\frac{3}{4} C_D \frac{\rho_w}{\rho r}\right)}{g\left(1 - \frac{\rho_w}{\rho}\right) - \frac{3}{8} C_D \frac{\rho_w}{\rho} \frac{v^2}{r}} \times \frac{1}{-\frac{3}{4} C_D \frac{\rho_w}{\rho r}} = x$$

$$\Rightarrow \quad \frac{-4\rho r}{3 C_D \rho_w} \left[\ln\left(g\left(1 - \frac{\rho_w}{\rho}\right) - \frac{3}{8} C_D \frac{\rho_w}{\rho} \frac{v^2}{r} \right) \right]_0^v = x$$

$$\therefore \quad \ln\left(\frac{g(1-\rho_w/\rho)}{g(1-\rho_w/\rho) - \frac{3}{8}C_D \frac{\rho_w}{\rho}\frac{v^2}{r}} \right) = \frac{3C_D\rho_w}{4\rho r}x$$

$$g\left(1-\frac{\rho_w}{\rho}\right)e^{-\frac{3C_D\rho_w}{4\rho r}x} = g\left(1-\frac{\rho_w}{\rho}\right) - \frac{3}{8}C_D\frac{\rho_w}{\rho}\frac{v^2}{r}$$

$$\frac{3}{8}C_D\frac{\rho_w}{\rho}\frac{v^2}{r} = g\left(1-\frac{\rho_w}{\rho}\right)\left(1-e^{-\frac{3C_D\rho_w}{4\rho r}x}\right)$$

$$v^2 = \frac{8\rho r}{3C_D\rho_w}g\left(1-\frac{\rho_w}{\rho}\right)\left(1-e^{-\frac{3C_D\rho_w}{4\rho r}x}\right)$$

$$\therefore \boxed{v = \sqrt{\frac{8\rho r}{3C_D\rho_w}g\left(1-\frac{\rho_w}{\rho}\right)\left(1-e^{-\frac{3C_D\rho_w}{4\rho r}x}\right)}}$$

Let $\quad l = \dfrac{4\rho r}{3C_D\rho_w}$

$$\therefore \boxed{v = \sqrt{2lg\left(1-\frac{\rho_w}{\rho}\right)\left(1-e^{-x/l}\right)}}$$

$$l = \frac{4 \times 7874 \times 0.0548}{3 \times 0.47 \times 1029} = \boxed{1.19\,\mathrm{m}}$$

This is a characteristic distance when terminal velocity manifests.

Terminal velocity is when $a=0$

$$\therefore \frac{3}{8} C_D \frac{\rho_w}{\rho} \frac{v^2}{r} = g\left(1 - \frac{\rho_w}{\rho}\right)$$

$$\therefore \boxed{v = \sqrt{\frac{8\rho r}{3C_D\rho_w} g\left(1 - \frac{\rho_w}{\rho}\right)}}$$

ie $\boxed{v_T = \sqrt{2lg(1-\rho_w/\rho)}}$

In our case: $v_T = \sqrt{2 \times 1.19 \times 9.81\left(1 - \frac{1029}{7874}\right)}$

$$\simeq \boxed{4.50 \text{ m/s}}$$

Note $\boxed{v(x) = v_T\sqrt{1 - e^{-x/l}}}$

So

v/ms^{-1}	x/m
3.40	1.0
4.50	10
4.50	100
4.50	1000
4.50	10 929

d)

$\therefore$ Ignore acceleration over first few metres and assume ball falls at 4.50 m/s.

$\therefore$ It takes

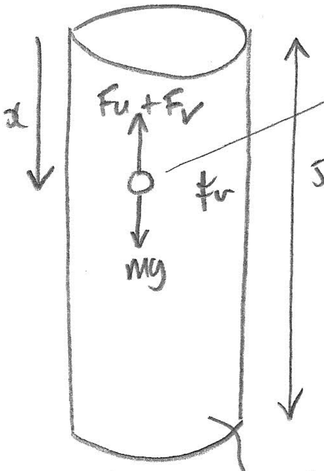
$$t = \frac{10929 \text{ m}}{4.50 \text{ m/s}}$$

$$= 2426s$$

$$\simeq 40 \text{ mins}$$

(40 mins, 26.3s) to hit the bottom.

5/



Everlasting Gobstopper mass M, density ρ and radius r

Molten chocolate test column
$\rho_c = 1325\ kg/m^3$
Viscosity μ.

(1/r)

$\rho = 1500\ kg/m^3$
$M = 1.00 \times 10^{-4}\ kg$

a) $\frac{4}{3}\pi r^3 \rho = M \quad \therefore \quad r = \sqrt[3]{\frac{3M}{4\pi\rho}}$

$\therefore$ radius of miniature gobstopper is:

$$r = \sqrt[3]{\frac{1.00 \times 10^{-4} \times 3}{4\pi \times 1500}} \quad (m)$$

$$= \boxed{2.52\ mm}$$

b) Newton II: $$M\frac{dv}{dt} = \underbrace{mg}_{\text{weight}} - \underbrace{\frac{4}{3}\pi r^3 \rho_c g}_{\text{upthrust}} - \underbrace{6\pi\mu r v}_{\text{viscous drag}}$$

Now $\frac{4}{3}\pi r^3 = M/\rho$ $\quad \therefore \quad \boxed{\frac{dv}{dt} = g\left(1 - \frac{\rho_c}{\rho}\right) - \frac{v}{\tau}}$

where $\boxed{\tau = \frac{M}{6\pi\mu r}}$

$\Rightarrow$ Terminal velocity: v_T is when $\frac{dv}{dt} = 0$

$$\Rightarrow \quad v_T = g\tau\left(1 - \rho_c/\rho\right)$$

$$v_T = \frac{g\frac{4}{3}\pi r^3 \rho}{6\pi\mu r}\left(1 - \rho_c/\rho\right)$$

$$\boxed{v_T = \frac{\frac{2}{9}gr^2}{\mu}(\rho - \rho_c)}$$

(c)

Hence:

$$-\tau\int_0^v \frac{dv\,(-1/\tau)}{g(1-\rho_c/\rho) - v/\tau} = t$$

[Note: $\int \frac{f'}{f}dx = \ln|f| + c$]

$$\therefore \quad -\tau\left[\ln\left(g(1-\rho_c/\rho)-\frac{v}{\tau}\right)\right]_0^v = t$$

$$\therefore \quad \ln\left(\frac{g(1-\rho_c/\rho)-v/\tau}{g(1-\rho_c/\rho)}\right) = -t/\tau$$

$$\therefore \quad g(1-\rho_c/\rho)-\frac{v}{\tau} = g(1-\rho_c/\rho)e^{-t/\tau}$$

$$\therefore \quad \boxed{v = g\tau(1-\rho_c/\rho)(1-e^{-t/\tau})} = \boxed{v_T(1-e^{-t/\tau})}$$

since $\quad x = \int_0^t v\,dt \quad \Rightarrow \quad x = g\tau(1-\rho_c/\rho)\left[t+\tau e^{-t/\tau}\right]_0^t$

$$\therefore \quad x = g\tau(1-\rho_c/\rho)(t+\tau e^{-t/\tau}-\tau)$$

$$\boxed{x = g\tau^2(1-\rho_c/\rho)(t/\tau + e^{-t/\tau}-1)}$$

d) $\quad \tau = \dfrac{m}{6\pi\mu r} = \dfrac{1.00\times10^{-4}}{6\pi\times60\times10^{-3}\times2.52\times10^{-3}} = \boxed{0.0352\,s}$

(which indicates terminal speed will be established on this timescale).

when $t = 1.28\,s$

$$x = 9.81\tau^2\left(1-\frac{1325}{1500}\right)\left(\frac{1.28}{\tau}+e^{-\frac{1.28}{\tau}}-1\right)$$

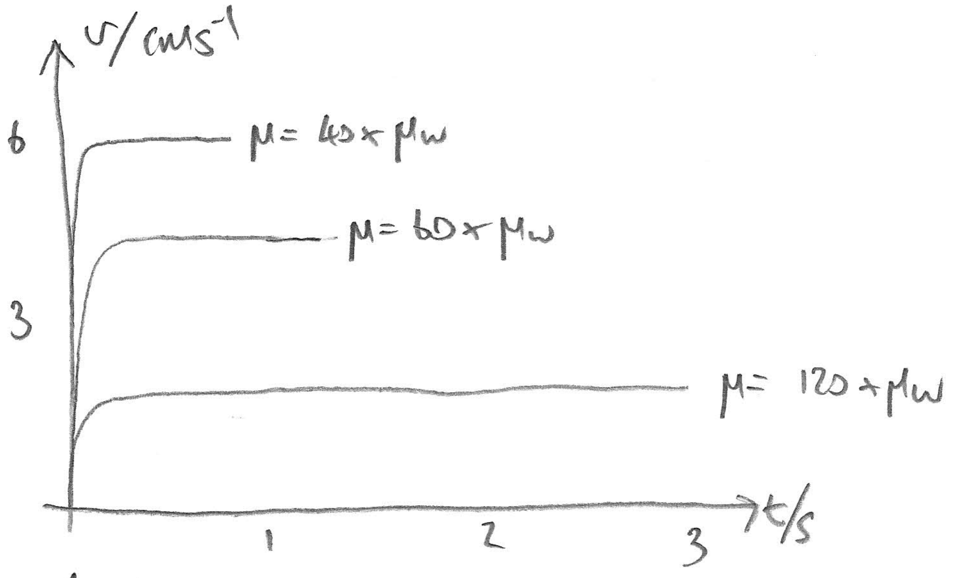
$= 0.0501\ m \quad$ ie $\boxed{5.10\ cm}$ as required.

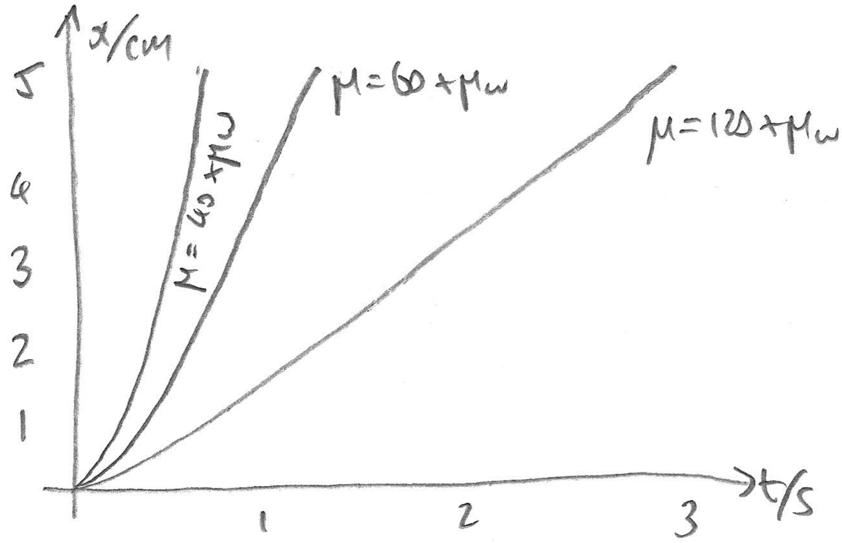
$v_T = g\tau(1-\rho_c/\rho) = \boxed{4.02\ cm/s} \qquad \boxed{v = v_T(1-e^{-t/\tau})}$

So $v = 4.02\,cm/s \times (1-e^{-1.28/\tau})$
$= \boxed{4.02\ cm/s}$ ie terminal speed attained.

(13)

See MATLAB code and graphs.



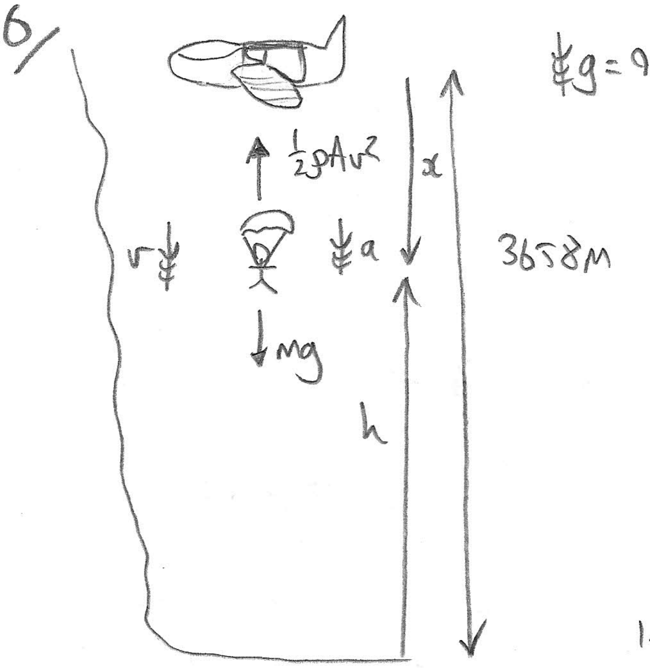


So as $\mu \uparrow$, $\tau \downarrow$ and it takes less time for terminal velocity to be established, and indeed for the gobstopper to fall 5.0cm as $v_T \downarrow$.

Note for viscous drag: $\boxed{v = v_T(1-e^{-t/\tau})}$

compared to $v = v_T\sqrt{1-e^{-x/\ell}}$ for hydrodynamic "v^2" drag

6/



$g = 9.81$ N/kg

a) Terminal speed is when $mg = \frac{1}{2}\rho A v_T^2$

$$\therefore \boxed{v_T = \sqrt{\frac{2mg}{\rho A}}}$$

ie v_T for James Bond is:

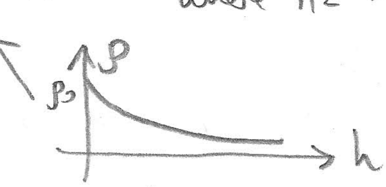
$$v_T = \sqrt{\frac{2 \times 80 \times 9.81}{1.225 \times 4.0}}$$

$$\boxed{v_T = 17.9 \text{ m/s}}$$

which is $\simeq$ 40 mph. Alas Mr Bond might not walk away from this one... However, since he is initially moving towards the mountain at aircraft speed, there may be a possibility of uplift by using the cape as an aerofoil... But then he might just hit the mountain!

b) See MATLAB code.

$$\rho \simeq \rho_0 e^{-h/H} \quad \text{where } H = 10{,}400 \text{ m}$$



Approximate model of atmospheric density.

$\rho_0 = 1.225$ kg/m³

NII: $\quad ma = mg - \frac{1}{2}\rho A v^2$

$$a = g - \frac{1}{2}\rho A/m \, v^2$$

$$a = g - \frac{\rho A}{2mg} g v^2$$

$$\boxed{a = g\left(1 - \left(\frac{v}{v_T}\right)^2\right)}$$

But $v_T = \sqrt{\frac{2mg}{A\rho_0}} \, e^{\frac{h}{2H}}$

so [use v_T in (a). $\rho \to \rho_0$]

So $$\boxed{a = g\left(1 - \left|\frac{v}{v_T}\right|^2 e^{-\frac{(x_{max}-x)}{H}}\right)}$$

$x_{max} = 3658$ m.

VDAV iterative method:

$h_0 = 3658,\quad t_0 = 0,\quad v_0 = 0$ (m, s, ms^{-1}) $\Delta t = 0.01s$

$$\alpha = g\left(1 - \left(\frac{v_n}{v_T}\right)^2 e^{-h_n/H}\right)$$

$$t_{n+1} = t_n + \Delta t \qquad h_{n+1} = h_n - v_n \Delta t - \tfrac{1}{2} A \Delta t^2$$

$$\beta = g\left(1 - \left(\frac{v_n}{v_T}\right)^2 e^{-h_{n+1}/H}\right) \qquad v_{n+1} = v_n + \tfrac{1}{2}(\alpha + \beta)\Delta t$$

Run until $h_{n+1} = 0$ $\quad v_T = \sqrt{\frac{2mg}{A\rho_0}}$

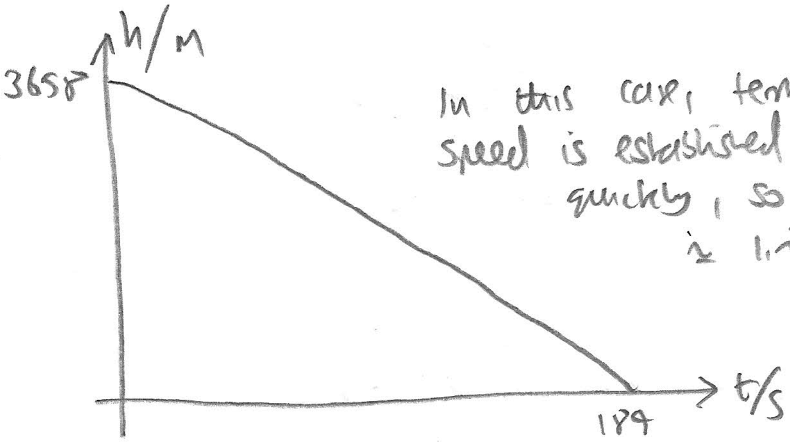
$H = 10{,}400m$

$\rho_0 = 1.225\ kg/m^3$

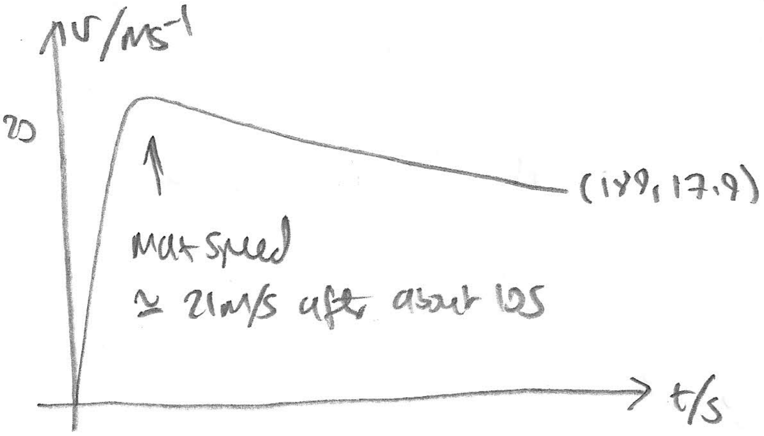
$A = 4.0m^2$

$m = 80\ kg$

$g = 9.81\ N/kg$



In this case, terminal speed is established fairly quickly, so h vs t ≈ linear.



Max speed ≈ 21m/s after about 10s

See MATLAB code and output graphs "Q6_Skyfall.m" for details.