Kinematics is the quantitative study of *motion*, and underpins much of the fundamental Physics discipline of *Mechanics*. **Displacement** $\mathbf{x}$ is the position *vector* of an object at a particular time t. **Velocity** $\mathbf{v}$ is the *rate of change* of displacement: $\mathbf{v} = \frac{d\mathbf{x}}{dt} = \dot{\mathbf{x}}$. **Acceleration** $\mathbf{a}$ is the rate of change of velocity: $\mathbf{a} = \frac{d\mathbf{v}}{dt} = \ddot{\mathbf{x}}$.
There are higher *derivatives* too, but these don't often feature in physical laws. '*Jerk'* is the rate of change of acceleration, and '*Snap'*, '*Crackle'* and '*Pop'* are the subsequent rates of change of each other. (!)

In one direction (1D): $v = dx/dt = \dot{x}$ and $a = dv/dt = d^2x/dt^2 = \ddot{x}$. Note also $dv/dt = v dv/dx$.

x, v, a, t can also be related *graphically*. Displacement (change) Δx is the *area* bounded by the horizontal axis of a (t, v) graph, and velocity (change) Δv is the area bounded by the horizontal axis of a (t, a) graph. In Calculus notation: $\Delta x = \int_{t_1}^{t_2} v dt$ and $\Delta x = \int_{t_1}^{t_2} a dt$. Note this means 'areas' below the horizontal axis *count negatively*. Similarly, velocity is the instantaneous *gradient* of a (t, x) graph, and acceleration is the instantaneous gradient of a (t, v) graph.

If **acceleration is constant,** the (t, v) graph is a *straight line* $v = u + at$. The area under this is a *trapezium*, hence: $x = \frac{1}{2}(u+v)t$ which means $x = ut + \frac{1}{2}at^2$. This assumes $x = 0$ when $t = 0$. If not, add an x_0 to x.

$v = u + at \Rightarrow t = \frac{v-u}{a}$. Hence by substituting into $x = \frac{1}{2}(u+v)t$ and rearranging: $v^2 = u^2 + 2ax$.

Useful conversions: 1.000 miles = 1,609m ; 2.24mph = 1.00m/s; 3.6km/h = 1.00m/s

Question 1

(i) Sybil the cat rushes into a room at 5m/s and decelerates (i.e. negatively accelerates) at 20m/s^2. Work out how long it takes her to stop moving, and sketch her velocity vs time graph. How far has she travelled since she entered the room?

(ii) Alice enters the Cliff Diving World Series competition at the *Blue Lagoon* at Abereiddi in Pembrokeshire, Wales. She leaps off a 27m diving board above the (25m deep) water-filled quarry. If she drops from rest, and air resistance can be neglected, determine (a) the speed (in /s) she strikes the water and (b) the time falling (in s). Assume the acceleration due to gravity is $g = 9.81\text{ms}^{-2}$.

(iii) A sports car accelerates from 30mph to 60mph in 40m. What is the acceleration in m/s^2?

(iv) A runner hits the home straight and completes the last 100m in 12.0s, accelerating at 0.1m/s^2. How fast did he start this sprint finish?

(v) Galileo and his apprentice drop a cannon ball and a feather at the same time on a windless day from the top of the Leaning Tower of Pisa, a height of 55m. The feather reaches a terminal velocity of 1.5m/s almost instantaneously, whereas the effect or air resistance on the cannonball is negligible and it accelerates towards the ground at 9.81m/s^2. (a) Calculate the impact speed of the cannon ball. (b) Overlay velocity vs time graphs for the cannon ball and the feather, and also height vs time graphs. (c) How high is the feather when the cannon ball hits the ground, and how long do you have to wait before the feather hits the ground too?

(vi) Usain Bolt set the 200m world record of 19.19s in Berlin in 2009. Assume he accelerated from rest to maximum speed in 5.0s, and then maintained this top speed until he completed the race. What was the maximum speed, and what was his initial acceleration?

(vii) Charlie and Craig decide to walk 500 miles, and then 500 more. Excluding any stops, they build up (at a constant rate) from zero to 20 miles per day over 20 days. They carry on until they have 100 miles left. At this point they slow at a constant rate until they reach their thousand mile target. Sketch a speed (miles per day) vs time (days) graph, and hence work out how long it takes them.

(viii) Two athletes are running the last leg of a 4x400m relay. The first athlete hits the home straight 5.0m clear of the second. The first athlete leaves the bend at 8.0m/s but is starting to tire, resulting in an acceleration of -0.1m/s^2. The second athlete maintains a speed of 8.2m/s. Who wins the race, and by what margin?

(ix) A particle is subjected to a force which results in a constant 'jerk' of 1m/s^3. At time $t=0$, acceleration, velocity and displacement are zero. Calculate how long it takes the particle to travel 100m.

(x) Iron Man and Magneto decide unwisely to participate in the same film franchise. Iron Man moves from rest towards Magneto with acceleration inversely proportional to the fourth power of their separation x in meters. At 0.25m the acceleration is 10m/s^2. If the initial separation is 1.0m, calculate the velocity when the separation is 0.1m. *Hint*: use the nice calculus trick $dv/dt = vdv/dx$.

Question 2 In 2013, a the USA Women's rowing eight completed 2000m in 5:54.16. What was their average speed in m/s? Camford Ladies race the same distance at Eton Dorney. At 1000m their split time is 3:15.00, at 1500m their split time is 4:50.00 and they finish the race in 6:35.00. Assume constant acceleration motion for the final 1000m. What speed did they pass the 1000m mark, and what was the acceleration in the final 1000m? What speed did they cross the finish line? (*Hint*: work out u, a algebraically in terms of distances x_1, x_2 and t_1, t_2, *then* substitute in the numbers. You now have a general method for solving this type of problem!)

Question 3 The velocity of a particle (in m/s) varies with time t (in s) according to the equation $v = 5t(1-\frac{1}{10}t)$. Another particle is initially at the same displacement as the first, but has a velocity vs time curve $u = t^3$.

(a) Sketch the velocity vs time curves for both particles on the same axis for $0 \le t \le 12$.
(b) Show that $u = v$ when $t = 2$ and calculate the acceleration of both particles at $t = 2$.
(c) How much further has the first particle travelled than the second, at the time when their speeds equal?
(d) How much further has the second particle travelled when v is largest?

Question 4 Captain Pugwash takes a morning stroll on the decks of *The Black Pig* and accidentally trips on a cannon ball. The ball is 12 lbs (about 5.44kg) and is made from iron of density $\rho = 7874\text{kg/m}^3$. The ball rolls over the side of the ship and splashes into the sea. The Black Pig is currently sailing over *Challenger Deep*, near Guam, which has a depth of x = 10,929m. This is the deepest part of the *Marianas Trench*, the deepest section of ocean on Earth.

(a) Determine the radius r of the cannon ball in metres.

The drag force on the ball of cross-sectional area $A = \pi r^2$ is given by $F_D = \frac{1}{2} c_D \rho_w A v^2$ where v is the velocity and the density of seawater is $\rho_w = 1029\text{kg/m}^3$. c_D is the drag coefficient, and is about 0.47 for a sphere. The upthrust on the ball is the weight of fluid displaced, i.e. $F_U = \frac{4}{3}\pi r^3 \rho_w g$.

(b) Hence show from Newton II that: $\dfrac{vdv}{dx} = g\left(1-\frac{\rho_w}{\rho}\right) - \frac{3}{8}\frac{\rho_w}{\rho} c_D \dfrac{v^2}{r}$ and show that the terminal velocity is $v_T = \sqrt{2gl(1-\frac{\rho_w}{\rho})}$ where $l = 4\rho r/3c_D\rho_w$. Calculate l and v_T.

(c) Use the standard integral: $\displaystyle\int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$ (where $f'(x) = df/dx$) to show that: $v = v_T\sqrt{1-e^{-\frac{x}{l}}}$ and hence determine the velocity at 1m, 10m, 100m, 1000m and 10,929m.

(d) Hence without doing any further calculus, determine the time it takes the cannon ball to hit the bottom of Challenger Deep.

Question 5 It is very important that the molten chocolate in Charlie and Mr Wonka's factory is of the correct *viscosity*, which Mr Wonka thinks should be about 60 times that of water. The viscosity μ of water is: $\mu = 1.0\times10^{-3}\,\text{kgm}^{-1}\text{s}^{-1}$. To determine whether the viscosity is correct, a 5.00cm high sample column of molten chocolate is extracted, and a miniature Everlasting Gobstopper of mass $1.00\times10^{-4}\,\text{kg}$ is dropped from rest at the top of the column. If it takes exactly $t =$ 1.28s to fall to the bottom of the test column (as measured using a force-plate datalogger - you are unlikely to see the gobstopper in the chocolate!), the viscosity is correct. The drag force due the chocolate is modeled by $F_D = 6\pi\mu r v$ where r is the radius of the ball and v is the velocity. The other forces[1] acting on the ball are weight and up-thrust (see Q4). Take $g = 9.81\text{Nkg}^{-1}$.

(a) If the density of the Everlasting Gobstopper is $\rho =$ 1500kg/m^3, calculate its radius r in mm.

(b) Show from Newton II : $\dfrac{dv}{dt} = g\left(1-\frac{\rho_c}{\rho}\right) - \dfrac{v}{\tau}$ where $\tau = m/6\pi\mu r$ and the density of chocolate $\rho_c = 1325\text{kgm}^{-3}$.

(c) Hence show that $v(t) = g\tau\left(1-\frac{\rho_c}{\rho}\right)\left(1-e^{-\frac{t}{\tau}}\right)$ and $x(t) = g\tau^2\left(1-\frac{\rho_c}{\rho}\right)\left(\frac{t}{\tau}+e^{-\frac{t}{\tau}}-1\right)$.

(d) Confirm that $x \approx 5.0\text{cm}$ when $t = 1.28\text{s}$, and work out the velocity that the ball strikes the bottom of the test column.

(e) Use a spreadsheet or computer programming environment like MATLAB or Python to plot (t,x) and (t,v) graphs. Investigate the effect of having chocolate of 40 times the viscosity of water, and 120 times the viscosity of water. [*Programming tip:* Use a fixed time-step, say 0.01s, and compute x and v in a `while` loop until $x = 5.00\text{cm}$].

Note Mr Wonka would normally tend to use a longer test column (say 20cm), but in order to better illustrate the fluid dynamics of viscous flow (and the approach to terminal velocity) to his young apprentice, he opts for the smaller version.

Question 6 Q and James Bond become involved in a deadly fight with the agents of *Spectre* inside a small plane at 3658m (12,000ft). The plane is about to crash into a mountainside and the Q and James decide to jump. Obviously Q is carrying some form of compact parachute, but unfortunately James only brought his wits with him on the flight. However, he does manage to wrest a huge cloak from one of the Spectre agents before he jumps. The cloak has a surface area of $A =$ 4.0m^2, and James has a mass of $m = 80$kg. $g = 9.81\text{Nkg}^{-1}$.

Assume drag force (in N) due to the air is of the form $F = \frac{1}{2}\rho A v^2$, where ρ is the density of air.

(a) If James jumps from rest, and $\rho = 1.225\text{kgm}^{-3}$ determine an equation for his velocity v with altitude h, and sketch this variation. What is his terminal velocity?

(b) The density of air actually drops quite significantly with altitude. Ignoring changes in air temperature, an approximate variation is $\rho \approx \rho_0 e^{-h/H}$ where $\rho_0 = 1.225\text{kgm}^{-3}$ and $H = 10{,}400\text{m}$. Construct a spreadsheet (or better, write a computer program) which determines the acceleration a, velocity v and altitude h of James Bond vs time. Use a fixed small timestep of $\Delta t = 0.01\text{s}$ and the *Verlet* method of 'constant acceleration within a timestep' as described below. Note it is assumed positive velocity and acceleration act downwards.

Velocity Dependent Acceleration Verlet (VDAV) method:

$$\alpha = a\left(h_n, v_n\right); \quad t_{n+1} = t_n + \Delta t; \quad h_{n+1} = h_n - v_n\Delta t - \tfrac{1}{2}A\Delta t^2$$
$$\beta = a\left(h_{n+1}, v_n\right); \quad v_{n+1} = v_n + \tfrac{1}{2}\left(\alpha+\beta\right)\Delta t$$

[1] For viscous forces to dominate over hydrodynamic ones, the Reynolds number for the flow must be less than about 25. This means a Gobstopper of radius less than about 4mm, for the viscosities and densities in Q5.