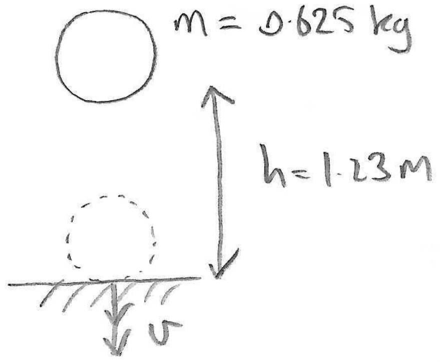
LINEAR MOMENTUM

1/ (i)

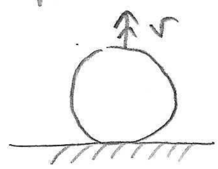


$g = 9.81\ N/kg$

$mgh = \frac{1}{2}mv^2$

$\therefore \boxed{\sqrt{2gh} = v}$

if the ball bounces elastically



ie a momentum change of $\boxed{2mv}$

This is the impulse applied by the ground to the ball. (Δp)

So $\Delta p = 2mv$

$\Delta p = 2m\sqrt{2gh}$

$= 2 \times 0.625\sqrt{2 \times 9.81 \times 1.23}$

$= \boxed{6.14\ Ns}$

Now if f is the average force exerted when the ball is in contact with the ground

$f\Delta t = \Delta p$

$\therefore \Delta t = \frac{\Delta p}{f}$

$\therefore \Delta t = \frac{6.14\ Ns}{500\ N}$

$\boxed{\Delta t = 0.012\ s}$

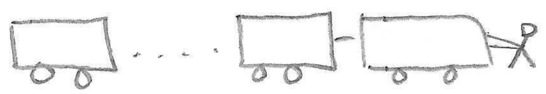
(ii) $p = mv$ $\quad \therefore v = p/m$

$E = \frac{1}{2}mv^2$ $\quad \therefore E = \frac{1}{2}m\left(\frac{p}{m}\right)^2$

$\therefore E = \frac{1}{2} mp^2/m^2$

$\therefore \boxed{E = p^2/2m}$

(iii)



Train momentum is $(700 + 6 \times 150) \times 6^3 \times 35.8$ Ns. let this $= f\Delta t$ where

① $\Delta t = 5.0s$. $\therefore \boxed{f = 1.15 \times 10^7\ N}$ (!) (11.5 Mega N)

(iv)

BEFORE AFTER

INELASTIC COLLISION

Conservation of momentum $\longrightarrow +$

$$(5m)(2u) + (4m)(-u) = 9mv$$

$$10mu - 4mu = 9mv$$

$$6u = 9v$$

$$\frac{2 \times 3u}{3 \times 3} = v$$

$$\therefore \boxed{v = \tfrac{2}{3}u}$$

Energy loss $\Delta E = kmu^2$

$$\therefore \quad \tfrac{1}{2} 9mv^2 + kmu^2 = \tfrac{1}{2} 5m(2u)^2 + \tfrac{1}{2} 4mu^2$$

$$\tfrac{1}{2} 9 \left(\tfrac{2}{3}\right)^2 + k = \tfrac{1}{2} 5 \times 4 + \tfrac{1}{2} 4$$

$$2 + k = 10 + 2$$

$$\boxed{k = 10}$$

For 5m mass : $(5m)(2u) + \Delta p = 5m \times \tfrac{2}{3}u$

$$\therefore \Delta p = \frac{10mu}{3} - 10mu$$

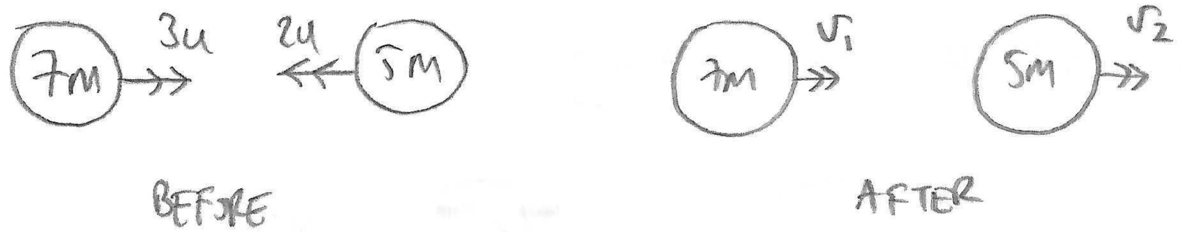
$$\boxed{\Delta p = -\frac{20}{3}mu}$$

For 4m mass : $(4m)(-u) + \Delta p = 4m \times \tfrac{2}{3}u$

$\therefore \Delta p = \tfrac{8}{3}mu + 4mu = \frac{(8 + 12)}{3} mu = \boxed{\frac{20}{3}mu}$

(2) is equal and opposite to impulse on 5m mass.

v)



Conservation of momentum: $\rightarrow +$

$$(7M)(3u) + (5M)(-2u) = (7M)v_1 + (5M)v_2$$

$$11Mu = 7Mv_1 + 5Mv_2$$

$$\boxed{11u = 7v_1 + 5v_2} \qquad ①$$

Restitution: (since elastic collision)

coefficient of restitution

$$1 = \frac{v_2 - v_1}{5u}$$

speed of separation

speed of approach

$$\therefore \boxed{v_2 = 5u + v_1} \qquad ②$$

$\therefore$ Substituting for v_2 in ① from ②

$$11u = 7v_1 + 5(5u + v_1)$$

$$11u = 12v_1 + 25u$$

$$\frac{-14u}{12} = v_1$$

$$\therefore \boxed{v_1 = -\tfrac{7}{6}u}$$

$$\left(-1\tfrac{1}{6}u\right)$$

$$\therefore v_2 = 5u + v_1 \qquad \therefore v_2 = \left(5 - \tfrac{7}{6}\right)u$$

$$v_2 = \frac{30-7}{6}u$$

$$\boxed{v_2 = \tfrac{23}{6}u} \qquad \left(3\tfrac{5}{6}u\right)$$

So AFTER is actually:

Energy before: $\frac{1}{2}7m(3u)^2 + \frac{1}{2}5m(2u)^2$

$$= \frac{63mu^2 + 20mu^2}{2}$$

$$= \boxed{\frac{83}{2}mu^2}$$

Energy after: $\frac{1}{2}7m\left(\frac{7}{6}u\right)^2 + \frac{1}{2}5m\left(\frac{23}{6}u\right)^2$

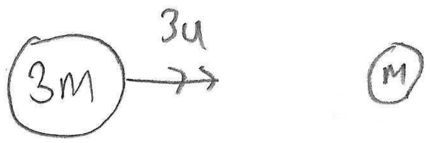
$$= \left(\frac{1}{2} \times \frac{7^3}{36} + \frac{1}{2} \times 5 \times \frac{23^2}{36}\right)mu^2$$

$$= \frac{2988}{2 \times 36}mu^2$$

$$= \boxed{\frac{83}{2}mu^2}$$

So energy (kinetic) conserved – consistent with an elastic collision.

(vi)



BEFORE

AFTER

Conservation of momentum: $(3M)(3u) = 3Mv_1 + Mv_2$

$$\boxed{9u = 3v_1 + v_2} \quad (1)$$

Restitution $\quad \frac{2}{3} = \frac{v_2 - v_1}{3u}$

$$\therefore \quad \boxed{v_2 = v_1 + 2u} \quad (2)$$

$$\therefore \quad 9u = 3v_1 + (v_1 + 2u) \qquad ((2) \text{ in } (1)).$$

$$9u = 4v_1 + 2u$$

$$\boxed{\frac{7u}{4} = v_1} \qquad \therefore \text{ in } (2): \quad v_2 = \left(\frac{7}{4} + \frac{8}{4}\right)u$$

$$\boxed{v_2 = \frac{15}{4}u}$$

$(v_1 = 1\tfrac{3}{4}u)$ $\qquad (v_2 = 3\tfrac{3}{4}u)$

$$\Delta E = \tfrac{1}{2}(3m)(3u)^2 - \tfrac{1}{2}(3m)\left(\tfrac{7u}{4}\right)^2 - \tfrac{1}{2}m\left(\tfrac{15}{4}u\right)^2$$

$$= \tfrac{1}{2}mu^2\left(3^3 - 3 \times \tfrac{49}{16} - \tfrac{15^2}{16}\right)$$

$$= \tfrac{1}{2} \times 3\tfrac{3}{4}\, mu^2$$

$$= 1\tfrac{7}{8}\, mu^2$$

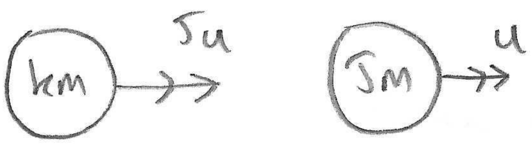
or $\boxed{\Delta E = \tfrac{15}{8} mu^2}$

(Fractional loss is:

$$\frac{15/8}{\frac{1}{2} \times 3^3} = 5/36$$

$\approx \boxed{13.9\%}$

(vii)



BEFORE

AFTER

Conservation of Momentum:

$$(km)(5u) + (5m)u = (km)(\tfrac{7}{2}u) + (5m)(\tfrac{43}{10}u)$$

$$k(5 - \tfrac{7}{2}) = \frac{5 \times 43}{10} - 5$$

$$k\tfrac{3}{2} = \frac{33}{2}$$

$$\boxed{k = 11}$$

Restitution:

$$\frac{\frac{43}{10} - \frac{7}{2}}{4} = e$$

$$\therefore \quad e = \frac{43 - 35}{40}$$

$$\boxed{e = \tfrac{1}{5}} \qquad (\text{i.e. } e = 0.2)$$

$$\Delta E = \tfrac{1}{2}(11m)(5u)^2 + \tfrac{1}{2}(5m)u^2$$
$$- \tfrac{1}{2}(11m)(\tfrac{7}{2}u)^2 - \tfrac{1}{2}(5m)(\tfrac{43}{10}u)^2$$

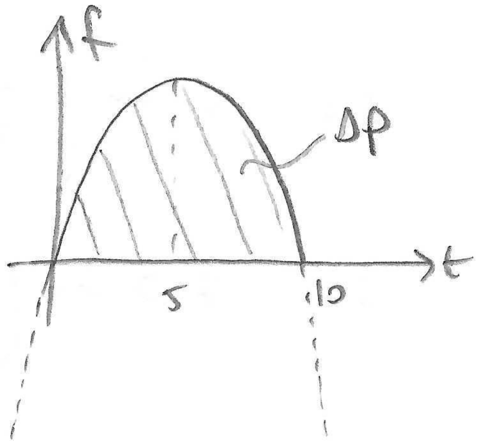
$$= \tfrac{1}{2}mu^2\left(280 - 134\tfrac{3}{4} - 92\tfrac{9}{20}\right)$$

$$= 26\tfrac{2}{5}mu^2$$

$$= \boxed{\frac{132}{5}mu^2} \qquad \text{So if } \Delta E = \alpha mu^2$$

$$\boxed{\alpha = 26.4}$$

(viii) $\boxed{f(t) = 6t(10-t)}$



$$\Delta p = \int_0^{10} f(t)\,dt$$

$$\Delta p = \int_0^{10} (60t - 6t^2)\,dt$$

$$\Delta p = \left[\frac{60t^2}{2} - \frac{6t^3}{3} \right]_0^{10}$$

$$\Delta p = (30)(100) - 2(1000)$$

$$\Delta p = \boxed{1000 \text{ Ns}}$$

Now $1N = 1\,kg\,ms^{-2}$

("mass × acceleration = vector sum of force" NII)

Now $\Delta p = m \Delta v$

So $\Delta v = \dfrac{1000 \text{ kgms}^{-1}}{42\text{kg}}$

$$\boxed{\Delta v = 23.8 \text{ m/s}}$$

(ix) $\overset{u}{\leftarrow}$ ▷[m ⟩ $\overset{v}{\rightarrow}$ $u = 200$ m/s. Fuel ejection rate is $\mu = 10$ kg/s

Newton II $\boxed{m\dfrac{dv}{dt} = u\dfrac{dm}{dt}}$ $\boxed{m = m_0 - \mu t}$

$\dfrac{dm}{dt} = \mu$ so $\boxed{(m_0 - \mu t)\dfrac{dv}{dt} = u\mu}$

$m_0 = 450$ kg. if $\mu t = 300$ kg (ie fuel ejected)

$t = \dfrac{300 \text{ kg}}{10 \text{kg/s}} = \boxed{30\text{s burn time}}$

$$\int_0^v dv' = u\mu \int_0^{30} \frac{dt}{m_0 - \mu t} = \frac{u\mu}{-\mu}\int_0^{30} \frac{-\mu}{m_0 - \mu t}\,dt$$

$$v = -u\left[\ln(m_0 - \mu t)\right]_0^{30}$$

$$v = -u\left(\ln\left(\frac{m_0 - 30\mu}{m_0}\right)\right)$$

$$v = u \ln\left(\frac{M_0}{M_0 - 30\mu}\right) \qquad \left\{ \text{in general } v = u \ln\left(\frac{M_0}{M_0 - \mu t}\right) \right\}$$

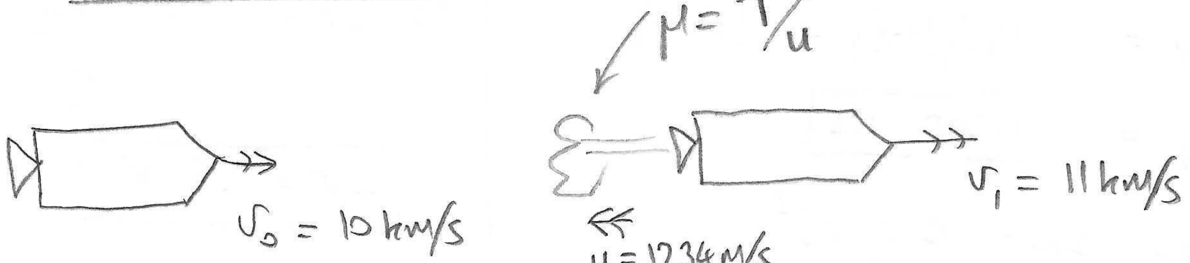
for $\mu t \leq M_{fuel}$ and $M_0 > M_{fuel}$.

$$v = 200 \ln\left(\frac{450}{450 - 300}\right)$$

$$v = 200 \ln 3$$

$$\boxed{v = 219.7 \text{ m/s}}$$

(x)



$t = 0$ $\qquad$ $t = 600$s

Thrust $\boxed{T = \mu u}$ $\qquad$ $T = 2200$ N

Rocket equation $\quad (M_0 - \mu t)\dfrac{dv}{dt} = \mu u$

$$\int_{v_0}^{v_1} dv = \mu u \int_0^t \frac{dt'}{M_0 - \mu t'}$$

$$v_1 - v_0 = \frac{-u\mu}{\mu}\int_0^t \frac{-\mu \, dt'}{M_0 - \mu t'}$$

$$v_1 - v_0 = -u\Big[\ln(M_0 - \mu t)\Big]_0^t$$

$$v_1 - v_0 = u \ln\left(\frac{M_0}{M_0 - \mu t}\right)$$

$$e^{\frac{v_1 - v_0}{u}} = \frac{1}{1 - \mu t/M_0}$$

$$1 - \frac{\mu t}{M_0} = e^{-(v_1 - v_0)/u}$$

$$1 - e^{-(v_1 - v_0)/u} = \frac{\mu t}{M_0}$$

$$\therefore \quad M_0 = \frac{\mu t}{1 - e^{-(v_1 - v_0)/u}}$$

Now $\mu = T/u$

$$\therefore \quad \boxed{M_0 = \frac{\frac{T}{u} t}{1 - e^{-(v_1 - v_0)/u}}}$$

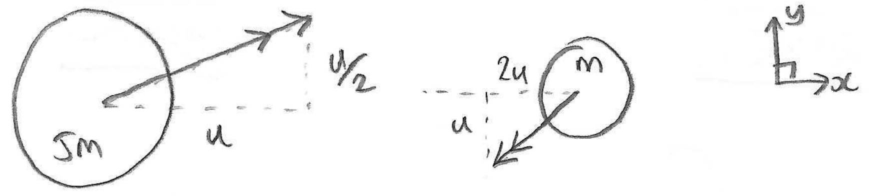
So
$$M_0 = \frac{\frac{2200}{1234} \times 1000}{1 - e^{-(1000)/1234}}$$

$$\boxed{M_0 = 3211 \text{ kg}}$$

[Note $\mu = \frac{2200}{1234} = 1.78$ kg/s of fuel ejection.
So in 1000s, 1783 kg of fuel is ejected
which means a remaining mass of $\boxed{1428 \text{ kg}}$].

Perhaps this could be a probe rather than a manned spacecraft?

2/ BEFORE:



$\underline{u}_1 = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} u \qquad \underline{u}_2 = \begin{pmatrix} -2 \\ -1 \end{pmatrix} u$

velocity of zero-momentum frame: $\underline{V} = \dfrac{5m\begin{pmatrix} u \\ \frac{u}{2} \end{pmatrix} + m\begin{pmatrix} -2u \\ -u \end{pmatrix}}{5m + m}$

$= \frac{1}{6}\begin{pmatrix} 3u \\ \frac{3}{2}u \end{pmatrix} = \begin{pmatrix} u/2 \\ u/4 \end{pmatrix}$

$\therefore$ using post-collision velocity formula: (and $e = 1$)
$\uparrow$ elastic collision

$\boxed{\underline{v}_1 = -(\underline{u}_1 - \underline{V}) + \underline{V}}$

$\underline{v}_1 / u = -\left(\begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}\right) + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}$

$= \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{4} \end{pmatrix} + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}$

$= \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ ie $5m$ mass is stationary

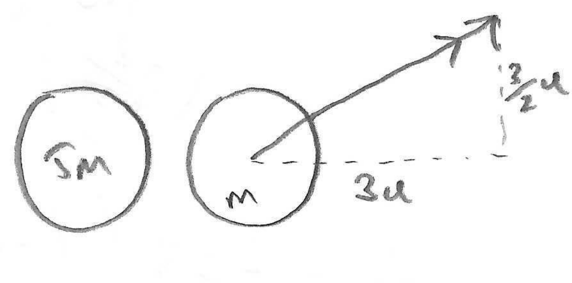
$\underline{v}_2 = -(\underline{u}_2 - \underline{V}) + \underline{V}$

$\underline{v}_2 / u = -\left(\begin{pmatrix} -2 \\ -1 \end{pmatrix} - \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}\right) + \begin{pmatrix} \frac{1}{2} \\ \frac{1}{4} \end{pmatrix}$

$= \begin{pmatrix} 2.5 \\ 1.25 \end{pmatrix} + \begin{pmatrix} 0.5 \\ 0.25 \end{pmatrix}$

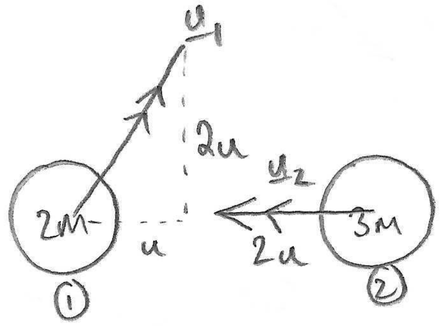
$= \begin{pmatrix} 3.0 \\ 1.5 \end{pmatrix}$

AFTER:

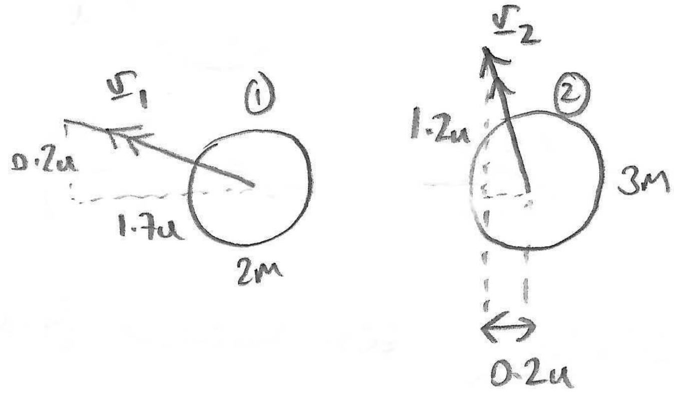


3/

BEFORE:



AFTER



$$\underline{v}_1 = -c(\underline{u}_1 - \underline{V}) + \underline{V}$$

$$\underline{v}_2 = -c(\underline{u}_2 - \underline{V}) + \underline{V}$$

Coefficient of restitution $\boxed{c = 0.5}$

ZMF velocity : $$\underline{V} = \frac{2mu\begin{pmatrix}1\\2\end{pmatrix} + 3mu\begin{pmatrix}-2\\0\end{pmatrix}}{2m+3m}$$

$$= \frac{1}{5}\begin{pmatrix}-4u\\4u\end{pmatrix} = \begin{pmatrix}-\frac{4}{5}u\\ \frac{4}{5}u\end{pmatrix}$$

$\therefore$ Post collision velocities are:

$$\underline{v}_1 = -0.5\left(\begin{pmatrix}u\\2u\end{pmatrix} - \begin{pmatrix}-\frac{4}{5}u\\ \frac{4}{5}u\end{pmatrix}\right) + \begin{pmatrix}-\frac{4}{5}u\\ \frac{4}{5}u\end{pmatrix}$$

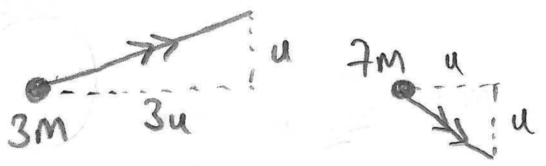
$$= \begin{pmatrix}-\frac{9}{10}\\ -\frac{3}{5}\end{pmatrix}u + \begin{pmatrix}-\frac{4}{5}u\\ \frac{4}{5}u\end{pmatrix} = \boxed{\begin{pmatrix}-1.7\\0.2\end{pmatrix}u}$$

$$\underline{v}_2 = -0.5\left(\begin{pmatrix}-2u\\0\end{pmatrix} - \begin{pmatrix}-\frac{4}{5}u\\ \frac{4}{5}u\end{pmatrix}\right) + \begin{pmatrix}-\frac{4}{5}u\\ \frac{4}{5}u\end{pmatrix}$$

$$= \begin{pmatrix}-\frac{u}{5}\\ \frac{6u}{5}\end{pmatrix} = \boxed{\begin{pmatrix}-0.2u\\1.2u\end{pmatrix}}$$

4/ BEFORE

AFTER



ZMF velocity $\quad \underline{V} = \dfrac{3m\begin{pmatrix}3u\\u\end{pmatrix} + 7m\begin{pmatrix}u\\-u\end{pmatrix}}{3m+7m}$

$$= \frac{1}{10}\begin{pmatrix}16u\\-4u\end{pmatrix} = \begin{pmatrix}1.6u\\-0.4u\end{pmatrix}$$

$\therefore$ Post collision velocities are: (if $C=0$)

INELASTIC COLLISION

$$\left.\begin{aligned}\underline{v}_1 &= -C(\underline{u}_1-\underline{V}) + \underline{V}\\ \underline{v}_2 &= -C(\underline{u}_2-\underline{V}) + \underline{V}\end{aligned}\right\} \text{ so } \underline{v}_{1,2} = \underline{V} \text{ if } C=0$$

$$\therefore \underline{v}_{1,2} = \begin{pmatrix}1.6u\\-0.4u\end{pmatrix}$$

KE lost is: $\quad \Delta E = \frac{1}{2}(3m)(3^2+1^2)u^2 + \frac{1}{2}(7m)(1^2+1^2)u^2$
$\quad - \frac{1}{2}(10m)(1.6^2+0.4^2)u^2$

$$= \frac{1}{2}mu^2\left\{3\times 10 + 7\times 2 - 136/5\right\}$$

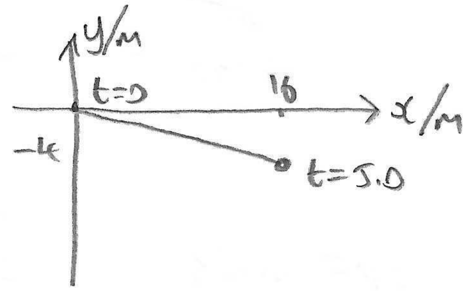
$$= \frac{42}{5}mu^2 = \boxed{8.4mu^2}$$

So after 5.0s, position of both masses is $\underline{r} = \underline{V}t$ ← as they move together

$$\underline{r} = \begin{pmatrix} 1.6 \times 2.0 \\ -0.4 \times 2.0 \end{pmatrix} \times 5.0 \quad (m)$$

if $u = 2.0$ m/s

$$\therefore \quad \underline{x} = \begin{pmatrix} 16 \\ -4 \end{pmatrix}$$



5/ $\boxed{\text{BEFORE:}}$ mass #1: $\underline{u}_1 = ?$ $m_1 = m$

mass #2: $\underline{u}_2 = \begin{pmatrix} u \\ -2u \end{pmatrix}$ $m_2 = 8m$

$\boxed{\text{AFTER:}}$ $\underline{v}_1 = \begin{pmatrix} \frac{1}{3}u \\ -3u \end{pmatrix}$ coefficient of restitution $C = 0.5$

$\underline{v}_2 = ?$

ZMF velocity $\underline{V} = \dfrac{m_1 \underline{u}_1 + m_2 \underline{u}_2}{m_1 + m_2}$

$$\underline{V} = \frac{m\underline{u}_1 + 8m\begin{pmatrix} u \\ -2u \end{pmatrix}}{9m}$$

$$\underline{V} = \frac{1}{9}\underline{u}_1 + \frac{8}{9}\begin{pmatrix} u \\ -2u \end{pmatrix}$$

$$\boxed{\underline{v}_1 = -C(\underline{u}_1 - \underline{V}) + \underline{V}}$$

$$\therefore \quad \underline{v}_1 = -C\left(\underline{u}_1 - \tfrac{1}{9}\underline{u}_1 - \tfrac{8}{9}\begin{pmatrix} u \\ -2u \end{pmatrix} \right) + \frac{\underline{u}_1}{9} + \tfrac{8}{9}\begin{pmatrix} u \\ -2u \end{pmatrix}$$

$$= \underline{u}_1\left(-C + \tfrac{C}{9} + \tfrac{1}{9} \right) + \left[\tfrac{8C}{9} + \tfrac{8}{9} \right] \begin{pmatrix} u \\ -2u \end{pmatrix}$$

$\therefore$ using $\underline{v}_1 = \begin{pmatrix} u/3 \\ -3u \end{pmatrix}$ and $C = 1/2$

$$\underline{u}_1 = \frac{\begin{pmatrix} u/3 \\ -3u \end{pmatrix} - \tfrac{4}{3}\begin{pmatrix} u \\ -2u \end{pmatrix}}{-\tfrac{1}{2} + \tfrac{1}{18} + \tfrac{1}{9}} = \frac{\begin{pmatrix} -1 \\ -\tfrac{1}{3} \end{pmatrix} u}{-\tfrac{1}{3}}$$

$$\left(\tfrac{3}{2} \times \tfrac{8}{9} = \tfrac{4}{3} \right)$$

$$\therefore \quad \boxed{\underline{u}_1 = \begin{pmatrix} 3u \\ u \end{pmatrix}}$$

So $$\underline{V} = \tfrac{1}{9}\underline{u}_1 + \tfrac{8}{9}\begin{pmatrix} u \\ -2u \end{pmatrix} = \begin{pmatrix} \tfrac{u}{3} \\ \tfrac{u}{9} \end{pmatrix} + \begin{pmatrix} \tfrac{8u}{9} \\ -\tfrac{16u}{9} \end{pmatrix}$$

$$= \begin{pmatrix} \tfrac{11}{9}u \\ -\tfrac{5}{3}u \end{pmatrix} \simeq \begin{pmatrix} 1.22u \\ -1.67u \end{pmatrix}$$

$$\therefore \quad \underline{v}_2 = -0.5\left(\begin{pmatrix} u \\ -2u \end{pmatrix} - \begin{pmatrix} \tfrac{11}{9}u \\ -\tfrac{5}{3}u \end{pmatrix} \right) + \begin{pmatrix} \tfrac{11}{9}u \\ -\tfrac{5}{3}u \end{pmatrix}$$

$$= \begin{pmatrix} \tfrac{4}{3} \\ -\tfrac{3}{2} \end{pmatrix} u$$

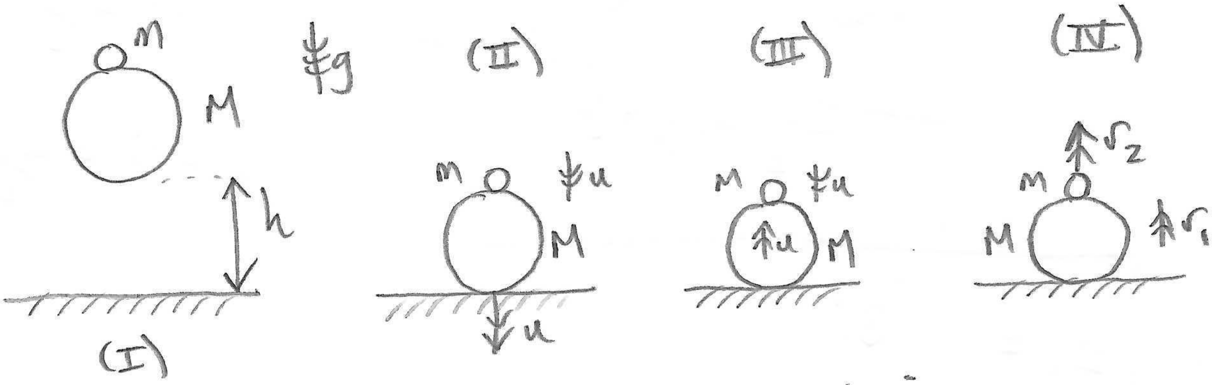
$$\Delta E = \tfrac{1}{2}m|\underline{u}_1|^2 + \tfrac{1}{2}8M|\underline{u}_2|^2 - \tfrac{1}{2}M|\underline{v}_1|^2 - \tfrac{1}{2}(8M)|\underline{v}_2|^2$$

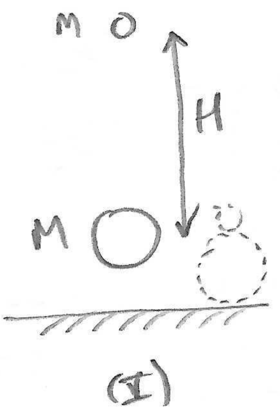
$$= \tfrac{1}{2}Mu^2\left\{3^2+1 + 8\times(1+2^2) - \left(\tfrac{1}{3^2}+3^2\right) - 8\left(\tfrac{16}{9}+\tfrac{9}{4}\right)\right\}$$

$$= \tfrac{1}{2}Mu^2\left\{10 + 40 - \tfrac{82}{9} - \tfrac{290}{9}\right\}$$

$$= \boxed{\tfrac{13}{3}Mu^2}$$

6/





If all collisions are elastic, by conservation of momentum between (III) and (IV) $\uparrow +$

$$Mu - mu = Mv_1 + mv_2 \qquad (1)$$

Restitution (ie elastic collisions) $\quad 1 = \dfrac{v_2 - v_1}{2u}$

$$\therefore \quad v_1 = v_2 - 2u \qquad (2)$$

In (1):

$$(M-m)u = M(v_2 - 2u) + mv_2$$

$$(3M-m)u = v_2(M+m)$$

$$\therefore \quad \boxed{v_2 = \frac{3 - m/M}{1 + m/M}}$$

Now by conservation of energy

$$\frac{1}{2} M u^2 = M g h \quad \Rightarrow \quad \boxed{u = \sqrt{2gh}}$$

Also:

$$\frac{1}{2} m v_2^2 = m g H \quad \Rightarrow \quad \boxed{v_2 = \sqrt{2gH}}$$

$\therefore$ Since

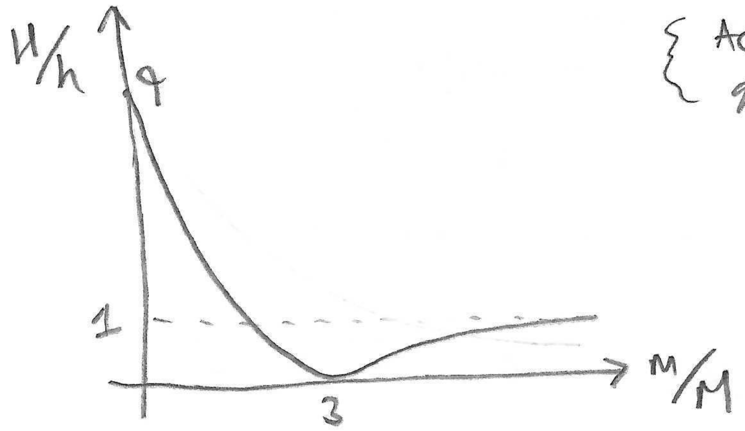
$$v_2^2 = \left(\frac{3 - m/M}{1 + m/M} \right)^2 u^2$$

$$\boxed{H = \left(\frac{3 - m/M}{1 + m/M} \right)^2 h}$$

as required.

Now when m/M is small $(M \gg m)$, $\boxed{H \to 9h}$

when $m/M \gg 1$, $H \to h$. Also, when $\frac{m}{M} = 3$, $H = 0$.



{ Actually, in the question m < M.... }

However, which parts of the curve have physical meaning?

$v_2 > 0$, otherwise 'balls move through the floor' (!)

So

$$\frac{3 - m/M}{1 + m/M} u > 0 \quad \Rightarrow \quad 3 - \frac{m}{M} > 0$$

$$\Rightarrow \quad \boxed{m/M < 3}$$

Also $v_2 > v_1$, otherwise there would be an additional collision

$$v_1 = v_2 - 2u$$ so this is assured

However, if $v_1 < 0$ it will result in another elastic bounce, which means M will then rise with $-v_1$.

So for M to not catch m, $|v_1| < |v_2|$

$$v_1 = \frac{3 - m/M}{1 + m/M}u - 2u$$

$$v_1 = \frac{3 - m/M - 2 - 2m/M}{1 + m/M}u$$

$$\boxed{v_1 = \frac{1 - 3m/M}{1 + m/M}u}$$

So $|v_1| < |v_2| \Rightarrow v_1^2 < v_2^2$

$$\left(1 - 3m/M\right)^2 < \left(3 - m/M\right)^2$$

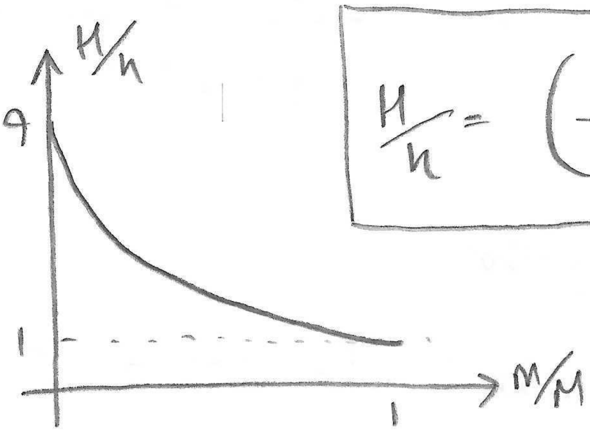
$$1 - \cancel{\frac{6m}{M}} + \frac{9m^2}{M^2} < 9 - \cancel{\frac{6m}{M}} + \frac{m^2}{M^2}$$

$$\frac{8m^2}{M^2} < 8$$

$$\frac{m^2}{M^2} < 1 \Rightarrow \boxed{m < M}$$

(17)

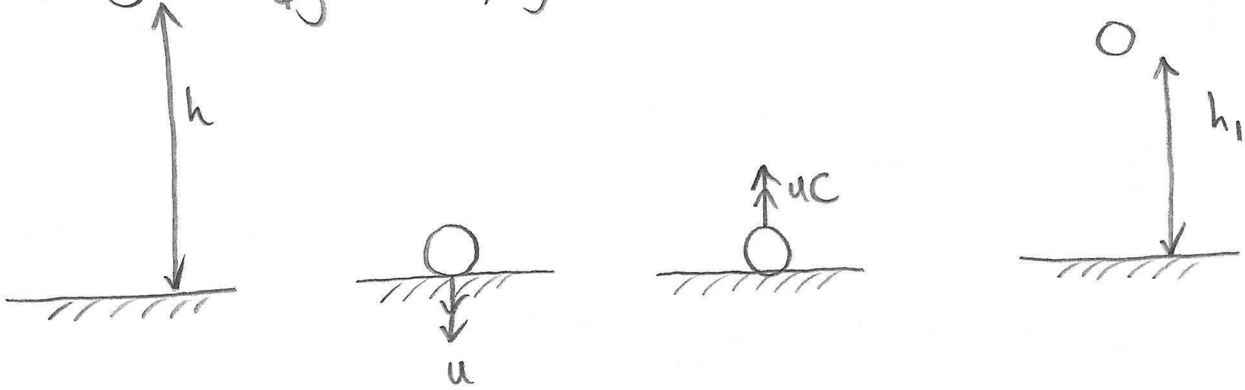
So $\boxed{m < M}$ and $\therefore$ overall graph is:



$$\boxed{\frac{H}{h} = \left(\frac{3 - m/M}{1 + m/M}\right)^2}$$

$$\left[\frac{H}{h} = \left(\frac{3-1}{1+1}\right)^2 = 1 \quad \text{ie when } \frac{m}{M} = 1\right]$$

7/ $g = 9.81$ N/kg



(i) $h = \frac{1}{2}gt_1^2$ $\quad t_1$ is time to first bounce.

$\frac{1}{2}u^2 = gh \Rightarrow u = \sqrt{2gh}$

Conservation of energy
$\frac{1}{2}mu^2 = mgh$

Between 1st and second bounce, ball rises to height h_1

$\frac{1}{2}u^2c^2 = gh_1 \Rightarrow h_1 = \frac{1}{2}u^2c^2/g$

ie $\boxed{h_1 = c^2 h}$

Clearly generalizes so $\boxed{h_n = c^2 h_{n-1}}$

$\therefore$ distance travelled by ball is (after n bounces)

$$D_n = h + 2c^2h + 2c^2(c^2h) + 2c^2(c^2c^2h) + \dots 2(c^2)^{n-1}$$

is a geometric progression:

$$D_n = h + 2h\left(c^2 + (c^2)^2 + (c^2)^3 + \dots (c^2)^{n-1}\right)$$

Now $a + ar + ar^2 + \dots + ar^{n-1}$

$$= \frac{a(1-r^n)}{1-r} \qquad \text{let } a=1,\ r=c^2$$

$$\therefore\ c^2 + (c^2)^2 + (c^2)^3 + \dots (c^2)^{n-1} = \frac{1-c^{2n}}{1-c^2} - 1$$

So $$D_n = h + 2h\left(\frac{1-c^{2n}}{1-c^2} - 1\right) \qquad [h = 2h(\tfrac{1}{2})]$$

$$D_n = 2h\left(\frac{1-c^{2n}}{1-c^2} - \frac{1}{2}\right)$$

So $$\boxed{D_\infty = 2h\left(\frac{1}{1-c^2} - \frac{1}{2}\right)}$$ assuming $c<1$ (true since "ball reaches a lower height each time").

Now $h_1 = \frac{1}{2}gt_2^2 \Rightarrow t_n = \sqrt{\frac{2h_{n-1}}{g}}$

This is the time to fall (or rise to) height h_{n-1}

$\therefore$ Time of bouncing is $\sqrt{\frac{2h}{g}} + 2\sqrt{\frac{2h_1}{g}} + 2\sqrt{\frac{2h_2}{g}} + \dots$

$\dots + 2\sqrt{\frac{2h_{n-1}}{g}}$ (for n bounces).

i.e $$t_n = 2\sqrt{\frac{2h}{g}}\left(\frac{1}{2} + C + C^2 + C^3 + \dots C^{n-1}\right)$$

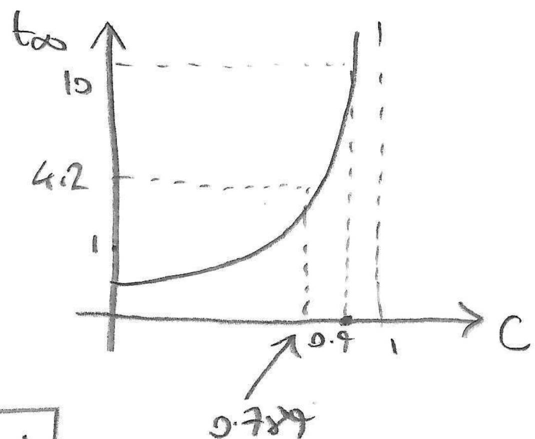
$$\left[\quad h_n = C^2 h_{n-1} \quad \text{so} \quad \sqrt{h_n} = C\sqrt{h_{n-1}}\quad\right]$$

$$\therefore\ t_n = \sqrt{\frac{8h}{g}}\left(\frac{1}{2} + \frac{1-C^n}{1-C} - 1\right)$$

$$\therefore\ t_n = \sqrt{\frac{8h}{g}}\left(\frac{1-C^n}{1-C} - \frac{1}{2}\right)$$

So $$\boxed{t_\infty = \sqrt{\frac{8h}{g}}\left(\frac{1}{1-C} - \frac{1}{2}\right)}$$ as required.

(ii) So if $t_\infty = 4.2\,s$, $h = 1.20\,m$ and $g = 9.81\,m/s^2$



$$t_\infty\sqrt{\frac{g}{8h}} + \frac{1}{2} = \frac{1}{1-C}$$

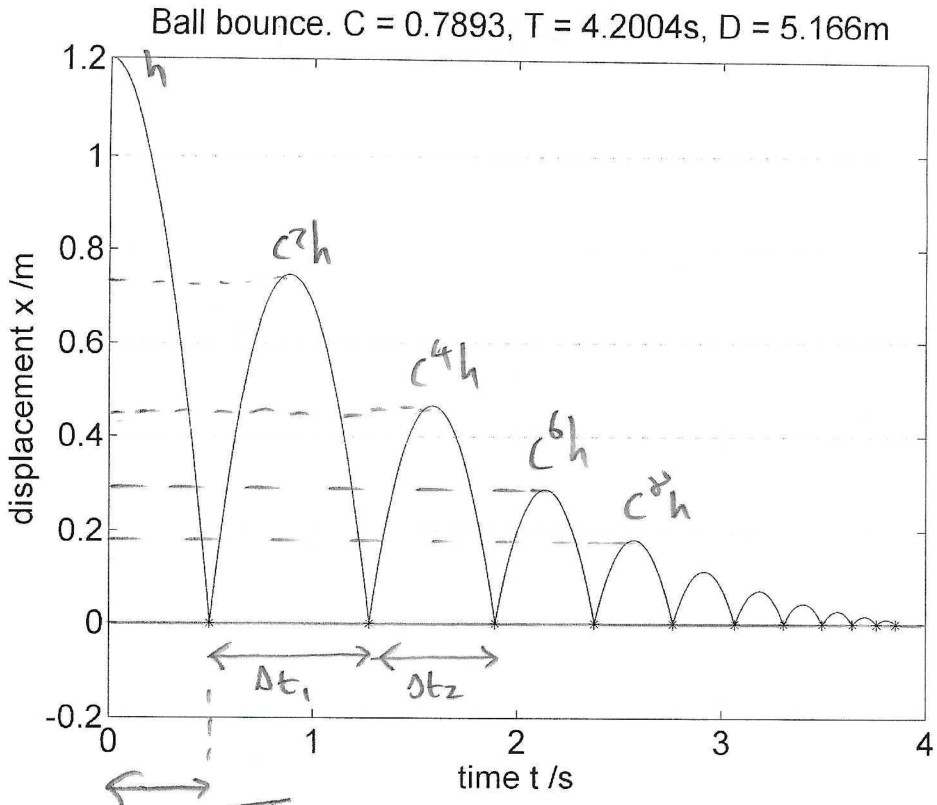
$$1 - C = \frac{1}{\frac{1}{2} + t_\infty\sqrt{g/8h}}$$

$$\therefore\ \boxed{C = 1 - \left(\frac{1}{2} + t_\infty\sqrt{\frac{g}{8h}}\right)^{-1}}$$

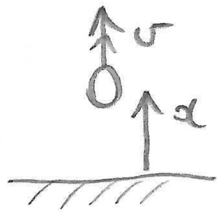
So $$C = 1 - \left(\frac{1}{2} + 4.2\sqrt{\frac{9.81}{8\times1.20}}\right)^{-1}$$

$$\boxed{C = 0.789}$$

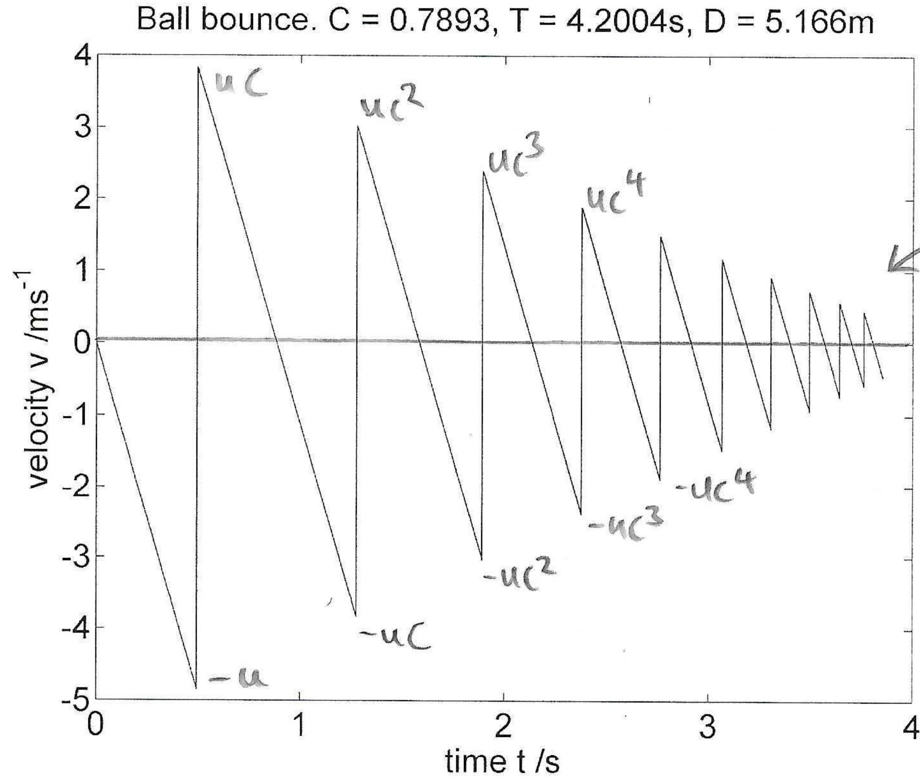
(iii)



$$\boxed{h_n = c^2 h_{n-1}}$$



$t_1 = \sqrt{2h/g}$



gradient is always $-g$.

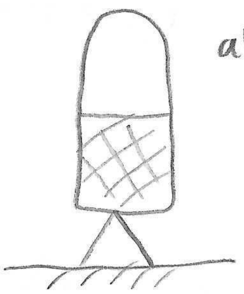
(ie $-9.81\,ms^{-2}$ in our case).

$$\Delta t_1 = 2\sqrt{\frac{2c^2h}{g}} = 2c\sqrt{\frac{2h}{g}}$$

$$\Delta t_2 = 2\sqrt{\frac{2c^4h}{g}} = 2c^2\sqrt{\frac{2h}{g}}$$

$$\boxed{\Delta t_n = 2c^n\sqrt{\frac{2h}{g}}}$$

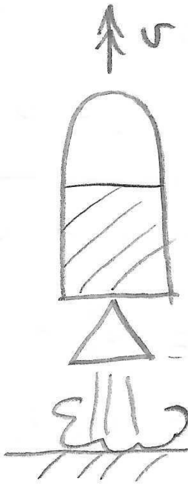
8/ $g = 9.81\ \mathrm{m/s^2}$



$m(0) = M + m_0$ at $t=0$

$M = 0.3\,\mathrm{kg}$

$m_0 = 0.3\,\mathrm{kg}$



$m(t) = M + m_0 - \mu t$

$u = 220\ \mathrm{m/s}$ $\uparrow x$

mass ejection rate $\mu = 0.1\,\mathrm{kg/s}$

$\boxed{\text{Thrust phase:}}$ $m_0 - \mu t > 0 \Rightarrow \boxed{t < m_0/\mu}$

Newton II: $(M + m_0 - \mu t)\dfrac{dv}{dt} = -(M + m_0 - \mu t)g + \mu C$

$$- \underbrace{\tfrac{1}{2}\rho C_D \pi r^2 v|v|}_{\text{Drag model}}$$

$\rho = 1.225\ \mathrm{kg\,m^{-3}}$ air density

$C_D = 0.5$ drag coefficient

$r = 5\times10^{-2}\ \mathrm{m}$ bottle radius

So $$\boxed{a(v,t) = -g + \frac{\mu C - \frac{1}{2}\rho C_D \pi r^2 v|v|}{M + m_0 - \mu t}}$$

$\boxed{\text{Projectile phase.}}$ $t \geq m_0/\mu$, $x \geq 0$.

$$\boxed{a(v,t) = -g - \frac{\frac{1}{2}\rho C_D \pi r^2 v|v|}{M}}$$

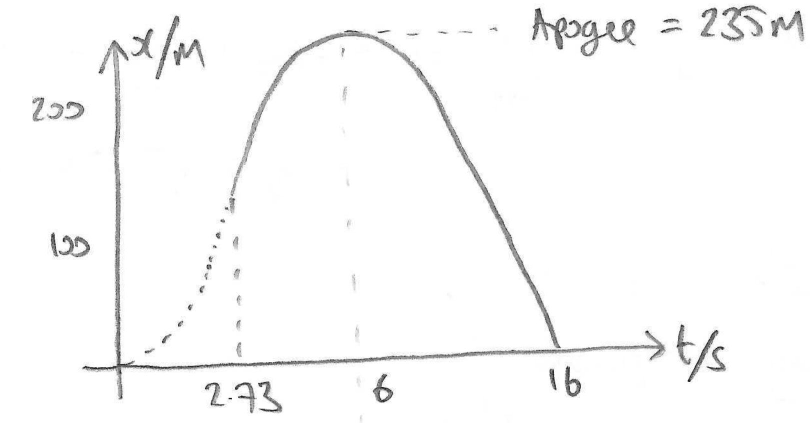
Use $\boxed{\text{Verlet}}$ method to determine dynamics, with timestep Δt.
($\Delta t = 0.001$ s). $x_0 = 0$, $v_0 = 0$, $t_0 = 0$.

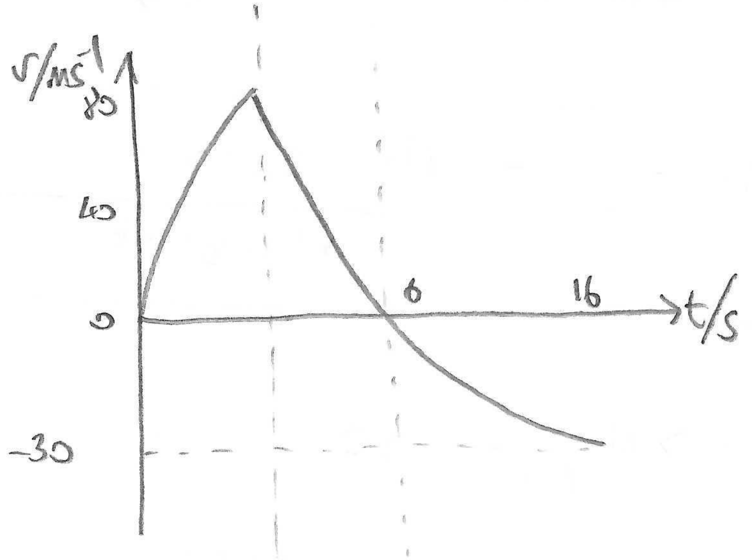
"VDAV"

(Velocity dependent acceleration Verlet)

$$\boxed{\begin{aligned} x_{i+1} &= x_i + v_i \Delta t + \tfrac{1}{2} a_i \Delta t^2 \qquad t_{i+1} = t_i + \Delta t \\ V &= v_i + a(v_i, t_i)\,\Delta t \\ d_V &= a(V, t_{i+1}) \\ v_{i+1} &= v_i + \tfrac{1}{2}\big(a(v_i, t_i) + a(V, t_{i+1}) \big)\Delta t \end{aligned}}$$

See MATLAB model for x, v, a vs t.





$M = 0.3$ kg $\quad M_0 = 0.3$ kg
$\dot{M} = 0.11$ kg/s $\quad u = 220$ m/s
$\rho = 1.225$ kg/m^3 $\quad C_D = 0.5$
$r = 5 \times 10^{-2}$ m $\quad g = 9.81$ m/s^2
$\Delta t = 10^{-4}$ s.