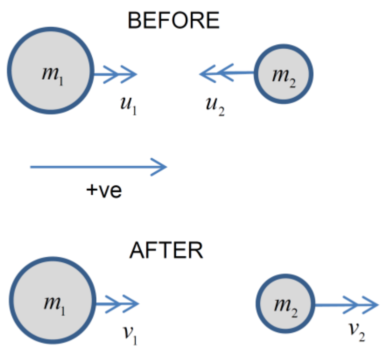
Note we are assuming speeds $v \ll c$, where c is the speed of light $2.998\times10^{8}\,\text{ms}^{-1}$. i.e. the laws of *Classical Mechanics*. ***Momentum*** $\mathbf{p} = m\mathbf{v}$. **Net force is the rate of change of momentum**. If mass is constant: $\mathbf{f} = m\frac{d\mathbf{v}}{dt}$ i.e. Newton's Second Law (*vector sum of force = mass × acceleration*). If mass is not constant (e.g. in a rocket): $m\frac{d\mathbf{v}}{dt} = \mathbf{f} + \mathbf{u}_R\frac{dm}{dt}$ where $\mathbf{u}_R$ is the relative velocity of ejected mass. Rocket thrust (in N) has magnitude $T = u_R\frac{dm}{dt}$.

Impulse is the *change of momentum*, and the *area* under a *force vs time* graph. $\Delta p = \int f(t)dt$.

For **collisions** in a straight line, draw Before and After diagrams to define masses and velocities. Use **conservation of momentum** + **restitution** to determine unknown parameters. The *coefficient of restitution* $C = \dfrac{\text{speed of separation}}{\text{speed of approach}}$.

For an *elastic* collision, $C = 1$. Note this means the *total kinetic energy* of objects is the same after the collision as before.

For a perfectly *inelastic* collision $C = 0$. This means collinear objects will join together after impact, and also a maximum amount of kinetic energy will be lost in the form of heat etc.



An efficient method for analyzing 2D or 3D collisions is to firstly subtract a velocity vector $\mathbf{V}$ to all objects such that the *total system momentum is zero*. A symmetry argument determines the result of the collision: the velocity vectors in this 'Zero Momentum Frame' *reverse*, after being scaled by C.

$$m_1(\mathbf{u}_1 - \mathbf{V}) + m_2(\mathbf{u}_2 - \mathbf{V}) = \mathbf{0} \qquad \therefore \mathbf{V} = \frac{m_1\mathbf{u}_1 + m_2\mathbf{u}_2}{m_1 + m_2}$$

After transforming back to the 'laboratory frame' from the 'Zero Momentum Frame' (ZMF), the resulting velocities $\mathbf{v}_{1,2}$ following collision are:

$$\mathbf{v}_{1,2} = -C(\mathbf{u}_{1,2} - \mathbf{V}) + \mathbf{V}$$

Question 1

(i) A basketball of mass 0.625kg is dropped from a height of 1.23m. It bounces elastically off the ground, and the ground exerts an average force of 500N. Determine the impulse applied by the ground to the ball, and then calculate the time Δt that the ball is in contact with the ground. $g = 9.81$N/kg.

(ii) Show that the kinetic energy of a mass m with momentum p is given by $p^2/2m$.

(iii) A common feat for many superheroes like Spiderman or Superman is to be able to stop a train that has gone off the rails. Assume a locomotive is going at 80mph (35.8m/s) and has a mass of 700 tonnes. It is pulling 6 carriages which are 150 tonnes each. In the movies, a superhero may stop the train in about 5.0s. Calculate the (average) force required to achieve this.

(iv) A mass of $5m$ and velocity $2u$ moving to the right, collides perfectly inelastically with a mass $4m$ moving at velocity u to the left. Determine (a) the resultant velocity of the conjoined masses, and (b) the kinetic energy loss factor k, where the energy loss is $\Delta E = kmu^2$. Show that the impulse each mass transfers to each other has the same magnitude, but opposite direction (and calculate what this impulse is).

(v) A mass of $7m$ and velocity $3u$ moving to the right, collides perfectly elastically with a mass $5m$ moving at velocity $2u$ to the left. Determine the resulting velocities post-collision, and show that there is no change in total kinetic energy.

(vi) A mass of $3m$ and velocity $3u$ moving to the right, collides with a stationary mass m. Determine the resulting velocities post-collision, and calculate the fractional loss in total kinetic energy, if the coefficient of restitution is $C = \frac{2}{3}$.

(vii) A mass of $k \times m$ and velocity $5u$, moving to the right, collides with a mass $5m$, also moving to the right, with a velocity u. The final velocities are $\frac{7}{2}u$ and $\frac{43}{10}u$ respectively. Determine (a) k and (b) the coefficient of restitution C. Calculate the kinetic energy loss factor α such that the energy loss is $\Delta E = \alpha m u^2$.

(viii) The force (in N) vs time (in s) applied to a mass of 42kg varies with time t as $f(t) = 6t(10-t)$. Calculate the total impulse transferred to the mass between 0 and 10s, and hence determine the velocity change.

(ix) A rocket of mass 450kg blasts off from rest, and ejects 300kg of fuel at a rate of $\mu =$ 10kg/s. Fuel is ejected at $u =$ 200m/s relative to the rocket. If gravity and air resistance can be ignored (e.g. the rocket is in space), determine the final velocity of the rocket, once all the fuel has been ejected.

(x) A spacecraft fires a rocket booster, which ejects gas at 1234m/s and provides a constant thrust of 2.2kN. After a 1000s burn, the spacecraft increases speed from 10,000m/s to 11,000m/s. Calculate the initial mass M_0 of the rocket.

Question 2 A mass of $5m$ is moving at velocity $(1, \frac{1}{2})u$. It collides elastically with a mass m, initially moving with velocity $(-2,-1)u$. Determine the velocities of both masses post-collision using the *Zero Momentum Frame* idea.

Question 3 A mass of $2m$ is moving at velocity $\mathbf{u}_1 = (1,2)u$. It collides with a mass $3m$, initially moving with velocity $\mathbf{u}_2 = (-2,0)u$. If $C = 0.5$, show that the resulting velocities are: $\mathbf{v}_1 = (-1.7, 0.2)u$ and $\mathbf{v}_2 = (-0.2, 1.2)u$

Question 4 A *point mass* of $3m$ is moving at velocity $(3,1)u$. It collides perfectly inelastically with a point mass $7m$, initially moving with velocity $(1,-1)u$. (a) Determine the velocities of both masses post-collision, and show that the kinetic energy loss is $8.4mu^2$. (b) If the collision occurs at $(0,0)$, what is the position of both masses after 5.0s if $u = 2.0\text{m/s}$? Note a *point mass* is a 'particle' i.e. something that has mass concentrated at one location, with no extension.

Question 5 *After* a collision, a mass of m is moving at velocity $(\frac{1}{3}, -3)u$. It previously collided with a mass $8m$, which was *originally* moving with velocity $(1,-2)u$. The coefficient of restitution for the collision was 0.5. Determine the pre-collision velocity $\mathbf{u}_1$ of the mass m and the post-collision velocity $\mathbf{v}_2$ of mass $8m$. Also show that $\frac{13}{3}mu^2$ of energy is lost. HINT: Express ZMF frame velocity $\mathbf{V}$ in terms of $\mathbf{u}_1$, the solve for $\mathbf{u}_1$ using $\mathbf{v}_1 = -C(\mathbf{u}_1 - \mathbf{V}) + \mathbf{V}$.

Question 6 A ball of mass M with a smaller mass m balanced on top, are dropped together from height h onto a hard surface. Assuming all collisions are perfectly elastic, and by drawing a sequence of Before and After diagrams, show that the maximum height gain post-collision by the smaller mass (from its lowest position) is: $H = \left(\frac{3 - m/M}{1 + m/M}\right)^2 h$.

Sketch H/h vs the mass ratio m/M.

Question 7 A ball of mass m is dropped onto a hard floor from height h. It collides with the floor with coefficient of restitution C and continues to bounce, but reaching a lower height each time. (i) Show that the total *distance* travelled by the ball is $D_\infty = 2h\left(\frac{1}{1-C^2} - \frac{1}{2}\right)$, and the time taken is $t_\infty = \sqrt{\frac{8h}{g}}\left(\frac{1}{1-C} - \frac{1}{2}\right)$.

(ii) If $h = 1.20\text{m}$ and $g = 9.81\text{ms}^{-2}$, calculate C if the ball takes 4.2s to stop bouncing.

(iii) Make careful sketches of (a) displacement vs time and (b) velocity vs time for the ball.

Question 8 A rocket of empty mass M kg is filled with m_0 kg of fuel. This is ejected at relative speed u and at mass ejection rate μ. Assuming u, μ are constant, and the rocket is launched vertically from rest, use a spreadsheet/computer program (e.g. the *Verlet* method) to determine the acceleration, velocity and displacement vs time for the rocket until it runs out of fuel. Use $M = 0.3\text{kg}$, $m_0 = 0.3\text{kg}$, $u = 220\text{ms}^{-1}$, $\mu = 0.11\text{kgs}^{-1}$ and a time step of $\Delta t = 10^{-4}\text{s}$. Ignore air resistance (although incorporating it would be a nice extension), and take $g = 9.81\text{ms}^{-2}$. As a further extension, model the dynamics of the rocket until it falls back to the ground. Ignore any horizontal motion due to windage etc.