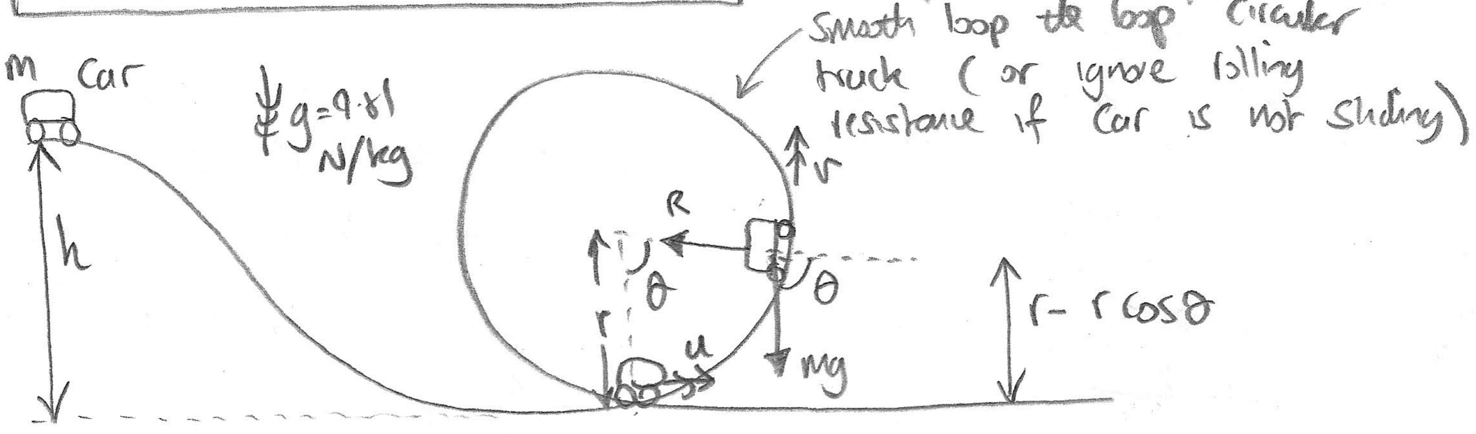
LOOP THE LOOP PROBLEM(S)



Radially inward Newton II $\quad \frac{mv^2}{r} = R - mg\cos\theta \quad ①$

conservation of energy $\quad mgh = \frac{1}{2}mu^2 \quad ②$

$$\frac{1}{2}mu^2 = mgr(1-\cos\theta) + \frac{1}{2}mv^2 \quad ③$$

So from ③, ②: $\quad mgh - mgr(1-\cos\theta) = \frac{1}{2}mv^2$

$$\frac{2mgh}{r} - 2mg(1-\cos\theta) = \frac{mv^2}{r}$$

$\therefore$ from ①: $\quad R = mg\cos\theta + \frac{2mgh}{r} - 2mg(1-\cos\theta)$

$$\boxed{R = 3mg\cos\theta - 2mg + 2mgh/r}$$

Car is in contact with loop-the-loop if $R > 0$

$$\therefore \quad 3mg\cos\theta + 2mgh/r > 2mg$$

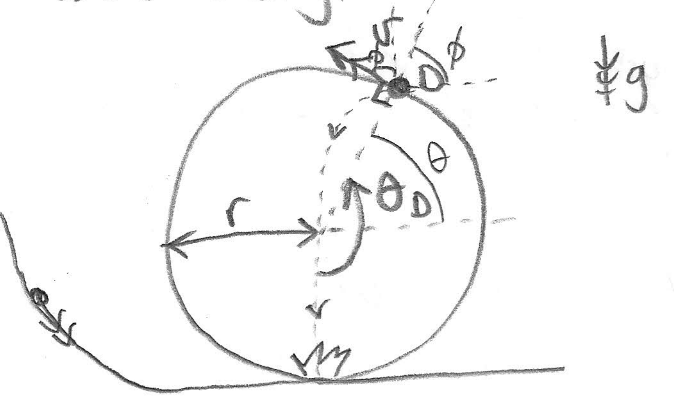
$$3\cos\theta + 2h/r > 2$$

$$\cos\theta > \frac{2}{3}(1 - h/r)$$

Now $\cos\theta \geq -1$ $\quad$ So $R > 0$ $\forall\theta$ if $\frac{2}{3}(1-h/r) < -1$

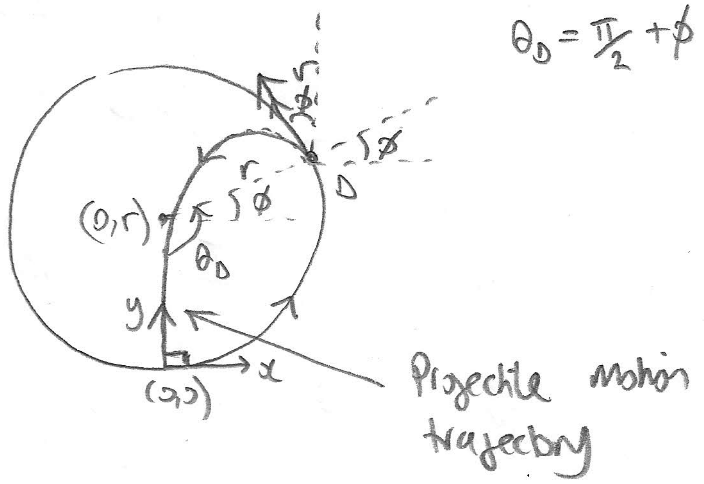
$\therefore \quad 1 - h/r < -\frac{3}{2} \quad \Rightarrow \quad 1 + \frac{3}{2} < \frac{h}{r} \quad \therefore \quad \boxed{h > \frac{5}{2}r}$

Supplementary problem, what h gives a projectile motion where landing is at the base of the loop?



Cart departs loop the loop at D (angle θ_D) and becomes a projectile. At this point speed is v.

$$\theta_D = \frac{\pi}{2} + \phi$$



From base of loop the loop, coordinates of D are

$x_D = r\cos\phi$

$y_D = r + r\sin\phi$

Projectile motion trajectory is:

$$\boxed{\begin{aligned} x &= r\cos\phi - v\sin\phi\, t \\ y &= r + r\sin\phi + v\cos\phi\, t - \tfrac{1}{2}gt^2 \end{aligned}}$$

Now $(0,0)$ must be on the trajectory

$\therefore \boxed{t = \dfrac{r\cos\phi}{v\sin\phi}}$ $\qquad \therefore\ 0 = r(1+\sin\phi) + \dfrac{v\cos^2\phi\, r}{v\sin\phi} - \dfrac{1}{2}g\dfrac{r^2\cos^2\phi}{v^2\sin^2\phi}$

$$\therefore \frac{\frac{1}{2}gr^2\cos^2\phi}{v^2\sin^2\phi} = r(1+\sin\phi) + \frac{\cos^2\phi\, r}{\sin\phi} \qquad (*)$$

Now $R=0$ at $\theta = \frac{\pi}{2}+\phi$. From above: $\boxed{\cos\theta = \frac{2}{3}(1 - h/r)}$

$\cos\left(\frac{\pi}{2}+\phi\right) = \cos\frac{\pi}{2}\cos\phi - \sin\frac{\pi}{2}\sin\phi = -\sin\phi$ $\qquad$ So $\sin\phi = \frac{2}{3}(h/r - 1)$

$\therefore \sin\phi = \frac{2}{3}\left(\frac{h-r}{r}\right)$

$\sin^2\phi = \frac{4}{9}\left(\frac{h-r}{r}\right)^2$

$\cos^2\phi = 1 - \sin^2\phi$

$= 1 - \frac{4}{9}\left(\frac{h}{r} - 1\right)^2$

$\cos^2\phi = 1 - \frac{4}{9}\left(\frac{h^2}{r^2} - \frac{2h}{r} + 1\right)$

$\cos^2\phi = \frac{5}{9} - \frac{4h}{9r}\left(\frac{h}{r} - 2\right)$

Now from above: $v^2 = 2gh - 2gr(1-\cos\theta)$

So $v^2 = 2gh - 2gr\left(1 - \frac{2}{3} + \frac{2}{3}\frac{h}{r}\right)$