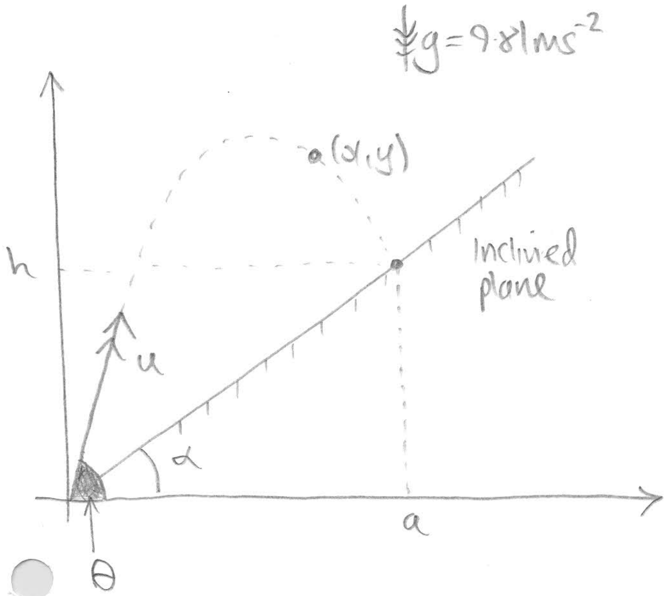
Maximising the vertical distance gained by a projectile that is launched from an inclined plane at position (0,0)



Inclined plane has Cartesian equation

$$y = x \tan\alpha \quad ①$$

Projectile has trajectory defined in terms of time t

$$x = ut\cos\theta \quad ②$$

$$y = ut\sin\theta - \tfrac{1}{2}gt^2 \quad ③$$

$$② \Rightarrow \quad t = \frac{x}{u\cos\theta} \quad ④ \quad \therefore \text{ in } ③: \quad y = \frac{ux\sin\theta}{u\cos\theta} - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$$

$\Rightarrow$ The parabolic trajectory equation $\quad y = x\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)x^2 \quad ⑤$

[using trigonometric identity $\frac{1}{\cos^2\theta} \equiv \sec^2\theta \equiv 1 + \tan^2\theta$]

The projectile trajectory strikes the inclined plane when $x = a$, $y = h$

Equating ① and ⑤

$$a\tan\alpha = a\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)a^2$$

$$\Rightarrow \quad \frac{g}{2u^2}(1+\tan^2\theta)a = \tan\theta - \tan\alpha$$

$$\therefore \quad a = \frac{\tan\theta - \tan\alpha}{1+\tan^2\theta}\left(\frac{2u^2}{g}\right)$$

$\Rightarrow$ (using $y = x\tan\alpha$) $\quad \boxed{h = \frac{(\tan\theta - \tan\alpha)\tan\alpha}{1+\tan^2\theta}\left(\frac{2u^2}{g}\right)}$

To maximize h, given a fixed α we need to find θ s.t. $dh/d\theta = 0$ {which we expect to be a maximal}

$$\therefore \frac{dh}{d\theta} = \frac{2u^2}{g}\left\{\frac{(1+\tan^2\theta)^2\tan\alpha - 2\tan\alpha(\tan\theta-\tan\alpha)\tan\theta(1+\tan^2\theta)}{(1+\tan^2\theta)^2}\right\}$$

Quotient rule

$$\left[\frac{d}{d\theta}\tan\theta = 1+\tan^2\theta \;;\; \frac{d}{d\theta}\tan^2\theta = 2\tan\theta(1+\tan^2\theta)\right]$$

$$\therefore \frac{dh}{d\theta} = \frac{2u^2}{g}\left\{\tan\alpha - \frac{2\tan\alpha(\tan\theta-\tan\alpha)\tan\theta}{1+\tan^2\theta}\right\}$$

$$\therefore \frac{dh}{d\theta} = \frac{2u^2\tan\alpha}{g}\left(1 - \frac{2\tan\theta(\tan\theta-\tan\alpha)}{1+\tan^2\theta}\right)$$

This $= 0$ when $1 = \dfrac{2\tan\theta(\tan\theta-\tan\alpha)}{1+\tan^2\theta}$

$$\Rightarrow \quad 1+\tan^2\theta = 2\tan^2\theta - 2\tan\theta\tan\alpha$$

$$\Rightarrow \quad \tan^2\theta - 2\tan\theta\tan\alpha - 1 = 0 \qquad \text{[Quadratic in } \tan\theta]$$

$$\therefore \tan\theta = \frac{2\tan\alpha \pm \sqrt{4\tan^2\alpha + 4}}{2} = \tan\alpha \pm \sqrt{\tan^2\alpha+1}$$

obviously ignore $-$ve solution, since $0 \leq \theta \leq \pi/2$ (or $90°$)

$$\therefore \tan\theta = \tan\alpha + \sqrt{\tan^2\alpha+1} = \tan\alpha + \frac{1}{\cos\alpha} = \frac{\sin\alpha+1}{\cos\alpha}$$

Hence h is maximized when $\boxed{\theta = \tan^{-1}\left(\dfrac{\sin\alpha+1}{\cos\alpha}\right)}$

Note this is independent of u and g.

Does this make sense? when $\alpha = 0 \Rightarrow \theta = \tan^{-1}(1)$
$= 45°$ or $\pi/4$. Yes!

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Now $\tan\theta = \dfrac{\sin\alpha + 1}{\cos\alpha}$ can be simplified

by noting the following identities.

$$\sin\alpha = \frac{2t}{1+t^2} \qquad t = \tan\alpha/2$$

$$\cos\alpha = \frac{1-t^2}{1+t^2}$$

$$\tan\left(\tfrac{\pi}{4} + \tfrac{\alpha}{2}\right) = \frac{\tan\frac{\pi}{4} + \tan\frac{\alpha}{2}}{1 - \tan\frac{\pi}{4}\tan\frac{\alpha}{2}} = \boxed{\frac{1+t}{1-t}}$$

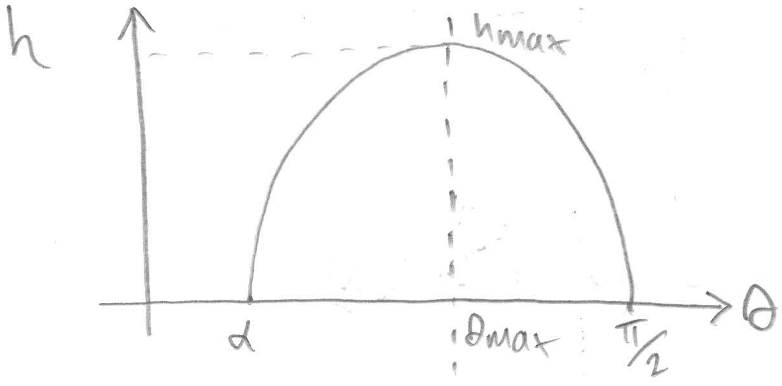
Since $\tan\frac{\pi}{4} = 1$ $\quad (\pi/4 = 45°)$

$$\therefore \frac{\sin\alpha + 1}{\cos\alpha} = \frac{\left(\frac{2t}{1+t^2} + 1\right)(1+t^2)}{1-t^2} = \frac{2t + 1 + t^2}{1-t^2}$$

$$= \frac{(1+t)^2}{(1-t)(1+t)}$$

$$= \boxed{\frac{1+t}{1-t}}$$

Hence $\tan\theta = \tan\left(\frac{\pi}{4} + \frac{\alpha}{2}\right) \Rightarrow \boxed{\theta = \frac{\pi}{4} + \frac{\alpha}{2}}$



$$\theta_{max} = \alpha + \frac{\left(\frac{\pi}{2} - \alpha\right)}{2}$$

$$= \frac{\pi}{4} + \frac{\alpha}{2}$$