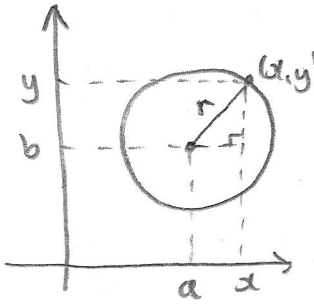
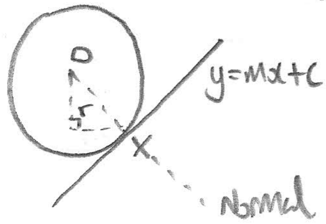
MP extra topics



Cartesian Equation of a circle

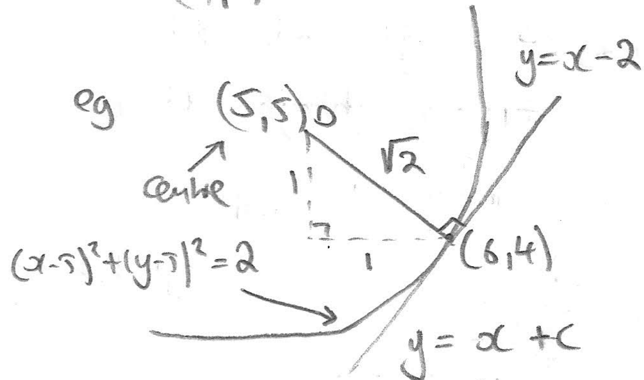
Pythagoras: $(x-a)^2+(y-b)^2=r^2$

Tangent to a circle through X



- Work out gradient of normal through OX
- Tangent gradient is $-\dfrac{1}{\text{normal gradient}} = m$
- If X has coordinates (α,β) find c using $\beta = m\alpha + c$

eg

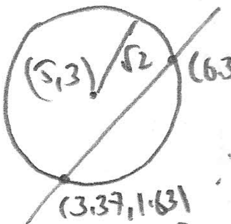


$y = x + c$

$4 = 6 + c$

$\boxed{-2=c}$

Intersection of line with a circle



$(x-5)^2+(y-3)^2=2$

$y=x-3$ substitute

$\therefore (x-5)^2+(x-6)^2=2$

$x^2-10x+25+x^2-12x+36=2$

$2x^2-22x+59=0$

$$x=\frac{22\pm\sqrt{22^2-4(2)(59)}}{2(2)}$$

$$\therefore \boxed{\begin{aligned} x &= 6.37,\ 4.63 \\ y &= 3.37,\ 1.63 \end{aligned}}$$

Surds

$\times$ by one $\dfrac{a-\sqrt{b}}{a-\sqrt{b}}$

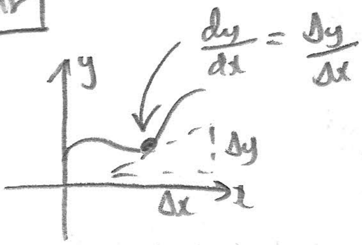
$$\frac{1}{a+\sqrt{b}} = \frac{a-\sqrt{b}}{(a+\sqrt{b})(a-\sqrt{b})} = \frac{a-\sqrt{b}}{a^2-b}$$

eg $$\frac{1}{2-\sqrt{3}} = \frac{2+\sqrt{3}}{2^2-3} = \boxed{2+\sqrt{3}}$$

Differential Calculus

$$\boxed{\begin{aligned} y &= ax^n \\ \frac{dy}{dx} &= nax^{n-1} \end{aligned}}$$

$\dfrac{dy}{dx}$ is the $\boxed{\text{gradient}}$ of y vs x.



eg $y = x^3$

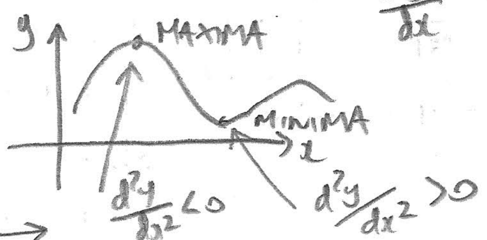
$\dfrac{dy}{dx} = 3x^2$

$y = \dfrac{1}{\sqrt{x}} = x^{-\frac{1}{2}}$

$\dfrac{dy}{dx} = -\frac{1}{2}x^{-3/2}$

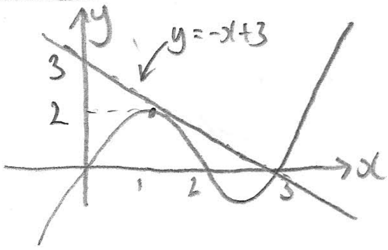
Stationary point

when $\dfrac{dy}{dx} = 0$



Second derivative

Tangents and normals to curves using Calculus



$y = x(x-2)(x-3)$

$y = x(x^2-5x+6)$

$y = x^3 - 5x^2 + 6x$

$$\frac{dy}{dx} = 3x^2 - 10x + 6$$

$$= 3\{x^2 - \tfrac{10}{3}x\} + 6$$

Complete the square

$$= 3\{(x-\tfrac{5}{3})^2 - \tfrac{25}{9}\} + 6$$

$$= 3(x-\tfrac{5}{3})^2 - \tfrac{25}{3} + 6$$

$$= \boxed{3(x-\tfrac{5}{3})^2 - 2\tfrac{1}{3}}$$

so when $x = 1$, $\frac{dy}{dx} = 3(1-\tfrac{5}{3})^2 - 2\tfrac{1}{3}$

$= 3 \times \tfrac{4}{9} - 2\tfrac{1}{3}$

$= -1$

$\therefore$ Since $y = 1(1-2)(1-3) = 1(-1)(-2) = 2$ when $x = 1$

tangent through $(1,2)$ is $y_T = -x + c$. Now $2 = -1 + c \therefore c = 3$ $\boxed{y_T = -x+3}$

Normal is $y_N = x + d$ $\therefore 2 = 1 + d \therefore \boxed{d = 1}$ $\therefore \boxed{y_N = x+1}$

gradient $= \frac{-1}{(-1)}$

$\boxed{\text{Integral calculus}}$ $\int f(x)\,dx$

Use definite integrals to find areas

$$\boxed{\int x^n\,dx = \frac{x^{n+1}}{n+1} + c}$$

x^2

Anti differential

Differentiate

"indefinite integral"

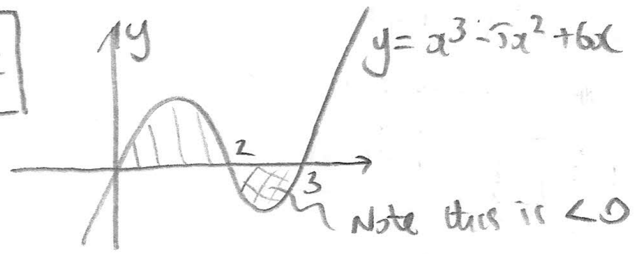
$\frac{1}{3}x^3 + c = \int x^2\,dx$

Constant of integration

$\frac{d}{dx}\left(\frac{1}{3}x^3 + 1\right) = x^2$

$\frac{d}{dx}\left(\frac{1}{3}x^3 - 10\right) = x^2$

$\vdots$



$\int_0^2 (x^3 - 5x^2 + 6x)\,dx = A$ is "Sum $f(x)$ dx"

$A = \left[\frac{1}{4}x^4 - \frac{5}{3}x^3 + \frac{6}{2}x^2\right]_0^2$

$A = \left(\frac{1}{4}2^4 - \frac{5}{3}2^3 + 3 \times 2^2\right) - (0)$

$A = 4 - 13\tfrac{1}{3} + 12$

$A = \boxed{2\tfrac{2}{3}}$

$\boxed{\text{Arithmetic sequence}}$

$U_n = a + (n-1)d$

$S_n = a + a+d + a+2d + \dots a + (n-1)d$

$S_n = a+(n-1)d + \dots + a$

$2S_n = n(2a + (n-1)d)$

$S_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}(U_1 + U_n)$

a is the first term

d is the common difference

Geometric Sequences

$u_n = ar^{n-1}$ ($n \geq 1$, integer)

$$S_n = a + ar + ar^2 + \dots + ar^{n-1}$$
$$rS_n = ar + ar^2 + \dots + ar^{n-1} + ar^n$$

$\therefore S_n(1-r) = a(1-r^n) \Rightarrow \boxed{S_n = \frac{a(1-r^n)}{1-r}}$

Note $\boxed{\text{if } |r| < 1}$

As $n \to \infty$

$|r^n| \to 0 \quad \therefore$

$\boxed{S_\infty = \frac{a}{1-r}}$

if $|r| < 1$

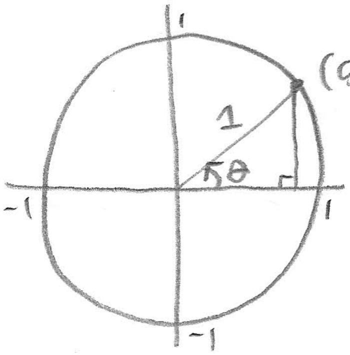
eg $1 + \frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \dots \quad a = 1, \; r = \frac{1}{10}$

$$= \frac{1}{1-\frac{1}{10}} = \frac{1}{\frac{9}{10}} = \frac{10}{9} = \boxed{1\tfrac{1}{9}}$$

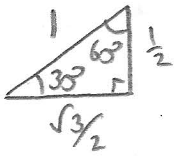
Unit circle trigonometry

$\cos\theta$ is the x coordinate of the unit circle

$\sin\theta$ " " y " " " "



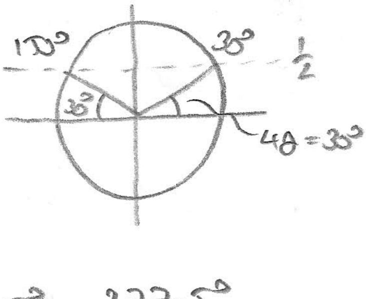
Solve $\sin 4\theta = \frac{1}{2}$



So $4\theta = 30° + 360°N$

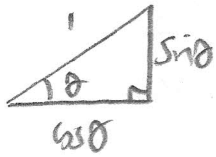
$\boxed{\theta = 7.5° + 90°N}$

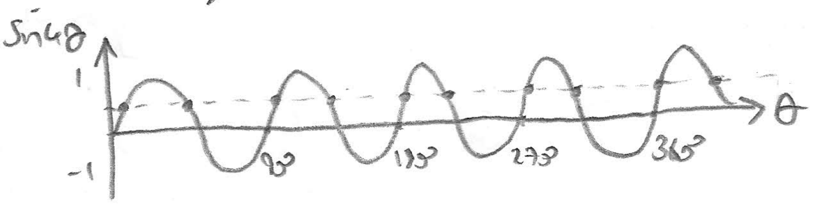
ie $7.5°, 97.5°, 187.5°, 277.5° \dots$



Also $4\theta = 150° + 360°M \quad \therefore \boxed{\theta = 37.5° + 90°M}$

ie $37.5°, 127.5°, 217.5°, 307.5°$





$\sin^2\theta + \cos^2\theta = 1$

$\tan\theta = \frac{\sin\theta}{\cos\theta}$

FACTOR THEOREM

Define $\boxed{f(x) = (x-a)Q(x) + R}$ eg $f(x) = (x-1)(x^2+3x+1) + 10 = x^3 + 2x^2 - 2x + 9$

If $x-a$ is a factor of $f(x)$ then $f(a) = R = 0$

eg $f(1) = 1 + 2 - 2 + 9 = 10$, which means $(x-1)$ is not a factor of $f(x)$.

Polynomial Division

$$\begin{array}{r} x^2 + 3x + 1 \\ x-1 \,\big)\, x^3 + 2x^2 - 2x + 9 \\ -x^2(x-1) = -x^3 + x^2 \\ \hline 3x^2 - 2x + 9 \\ -3x(x-1) = -3x^2 + 3x \end{array} \quad \begin{array}{l} \\ \\ \text{\} Add} \\ \\ \text{\} Add} \end{array}$$

$$\begin{array}{r} x + 9 \\ -1(x-1) = -x + 1 \\ \hline 10 \end{array} \quad \begin{array}{l} \text{\} Add} \\ \\ \uparrow \text{ Remainder} \end{array}$$

$\boxed{\text{So } x^3 + 2x^2 - 2x + 9 = (x-1)(x^2+3x+1) + 10}$

Combinatorics

n letters, r repeats of letter A, q repeats of letter B etc

ways of arranging ("permutations") is $\frac{n!}{r!\,q!\ldots}$

eg ZOOROPA

$$\frac{7!}{3!} = 7\times6\times5\times4 = \boxed{840}$$

Note $\frac{6!}{4!} = \frac{6\times5\times4!}{4!} = \boxed{30}$

If you have n letters, r repeats of letter A and <u>the rest</u> letter B

permutations is $\boxed{\frac{n!}{(n-r)!\,r!}} = {}^nC_r$ ← Binomial probability distribution

or $\binom{n}{r}$ →

" If win chance is 0.4, $P(3 \text{ out of } 5) = \binom{5}{3}0.4^3\,0.6^2 = \frac{5!}{2!\,3!}0.4^3\,0.6^2$

$= 0.2304$

Upper and lower bounds

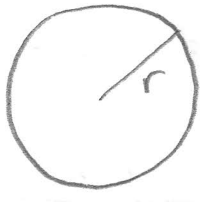
" I travel ten miles to the nearest mile in six hours to the nearest minute"

$$\frac{9.5 \ (\leftarrow \text{smallest})}{6+\frac{30}{3600} \ (\leftarrow \text{largest})} < \text{speed}/\text{mph} < \frac{10.5 \ (\leftarrow \text{largest})}{6-\frac{30}{3600} \ (\leftarrow \text{smallest})}$$

[3600s in 1 hour]

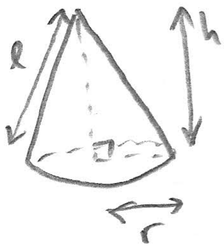
$\boxed{1.58\,\text{mph} < \text{speed} < 1.75\,\text{mph}}$

Volumes and areas



Sphere volume = $\boxed{\frac{4}{3}\pi r^3}$

Sphere area = $\boxed{4\pi r^2}$

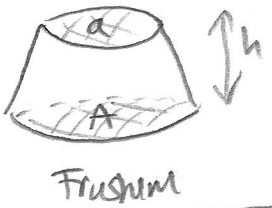


Cone volume = $\boxed{\frac{1}{3}\pi r^2 h}$

Cone surface area = $\boxed{\underbrace{\pi r^2}_{\text{base}} + \pi r l}$

Note $\boxed{l^2 = r^2 + h^2}$

Slant height l



Frustum

Volume = $\boxed{\frac{1}{3}h\left(a + A + \sqrt{aA}\right)}$

Direct and inverse proportion

$y \propto x \Rightarrow y = kx$

y inversely proportional to x^2

$\Rightarrow y = \frac{k}{x^2}$

Trick in these problems is to find k (the constant of proportionality) first.

eg $z = 3x^2/\sqrt{y}$

z is proportional to x^2 and inversely proportional to $\sqrt{y}$