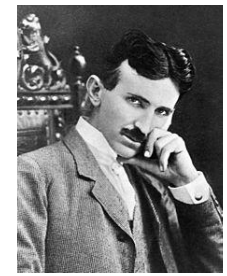
Magnetism is a force which acts on moving charges in the presence of a *magnetic field* $\mathbf{B}$. The latter is similar to a *gravitational field* $\mathbf{g}$ (which acts upon **mass**) and an *electric field* $\mathbf{E}$ (which acts upon **charges**). Like gravitational and electric fields, magnetic fields are also *sourced* by the same type of entity that they act upon i.e. moving charges.

Although magnetic effects have been experienced and recorded since antiquity, the connection to electricity was perhaps first made by Hans Christian Ørsted in 1820. He noticed that compass needles were deflected from magnetic North when placed in the presence of a current carrying wire. If an electric field is essentially 'where a charge will accelerate at a particular location in space', a magnetic field is where a compass needle would align. Iron filings will readily align with a magnetic field, and indeed will cluster where the field is strongest. The strength of a magnetic field is measured in *Tesla*.

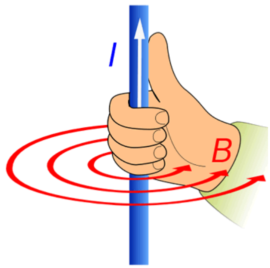
A current carrying wire will result in a magnetic field which forms loops around the conductor. We can use the **right hand grip rule** to predict the direction of the magnetic field.

Hans Christian Ørsted 1777-1851



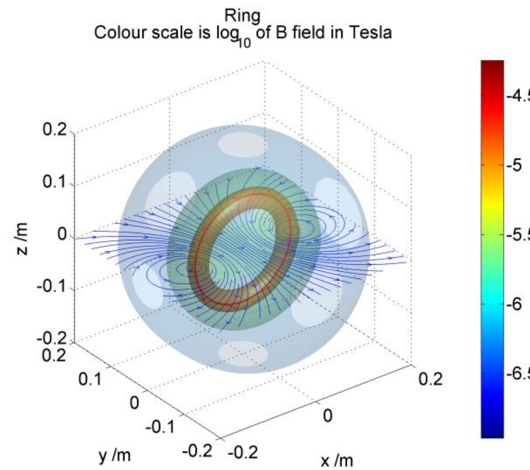
Nikola Tesla 1856-1943

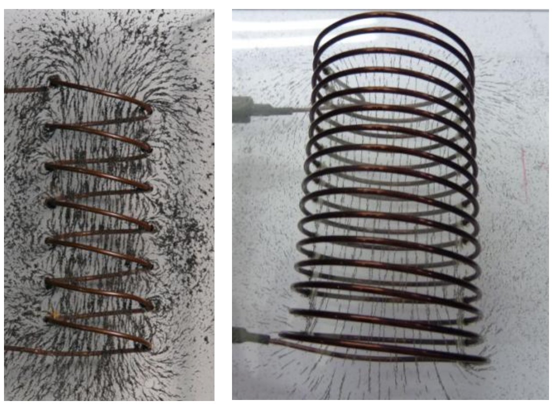
Iron filings aligning with a magnetic field generated by a current carrying wire



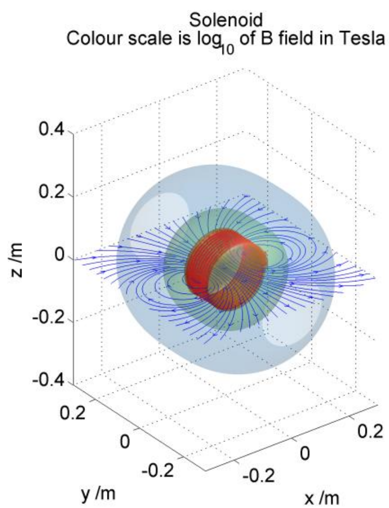
Right hand grip rule

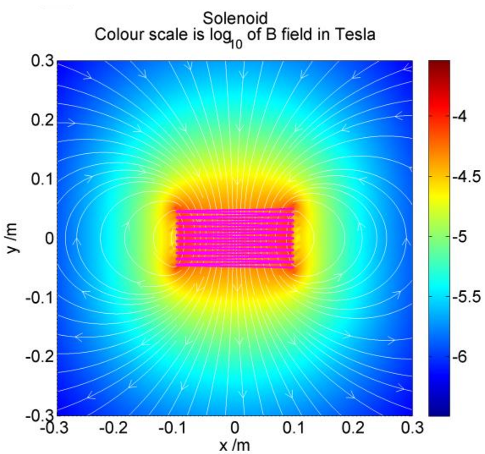
Iron filings aligning with a magnetic field generated by a few loops (a "ring") of current carrying wires

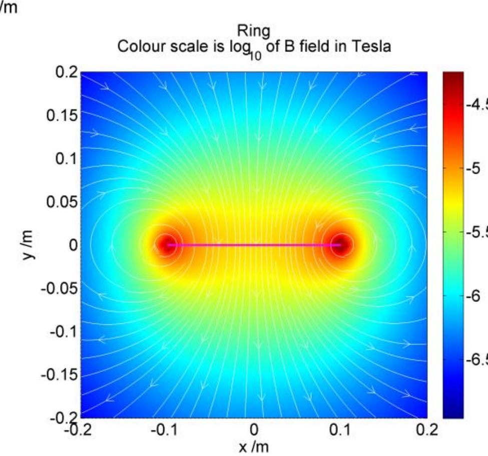




Magnetic field produced by a coil of current carrying wire. This is called a *solenoid*.

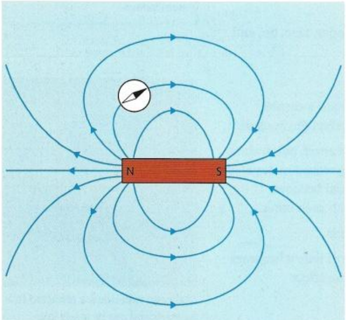


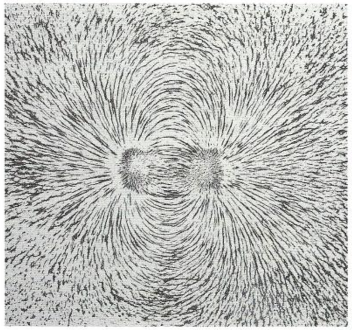


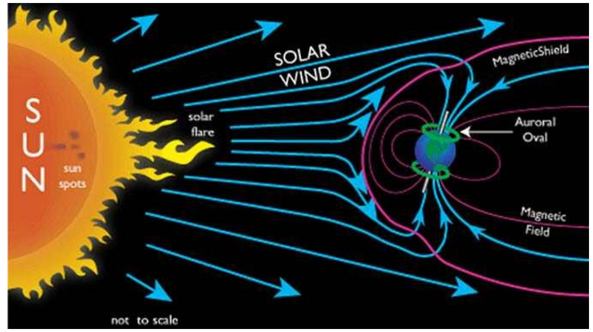


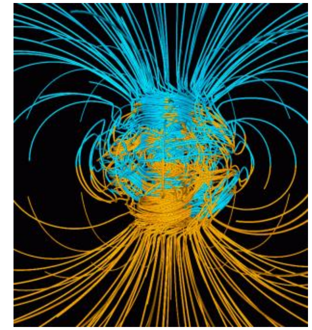
Magnetic field lines are 'where small magnetic compasses' would point in a magnetic field. The shading in the above plots is related to the strength of the magnetic field

The magnetic field of the solenoid is very similar to that of a permanent bar magnet, or indeed the Earth itself. It appears to be the case that, unlike charges, sources of magnetic fields are always paired with sinks. In other words, a magnetic *dipole* appears to be the basic element of magnetism. A **North pole** is defined to be a magnetic field **source** and a **South pole** a magnetic field **sink**.

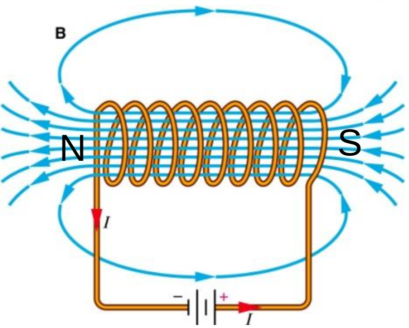








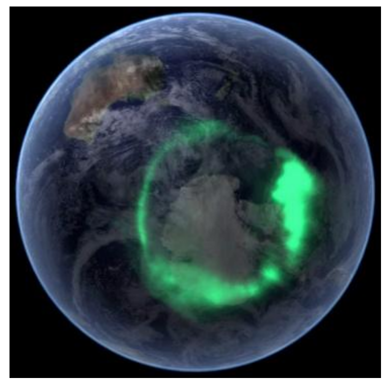
The magnetic field of the Earth results from the movement of charged fluid within its liquid interior. The interior field is very complex, but outside the Earth it resembles a dipole. At present magnetic North is close to the geographic North.



Solenoid magnetic fields are very similar to those of a permanent bar magnet

The magnetic field of the Earth helps to shield it from charged particles which constitute the solar wind.

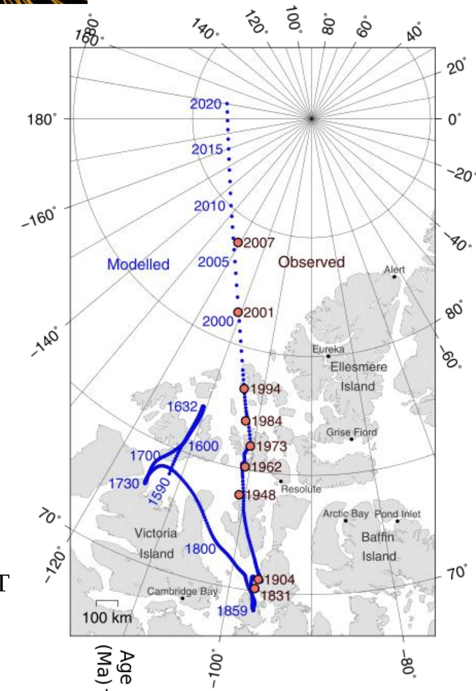
A beautiful result of this acceleration of charged particles are the *Aurora Boralis* and *Aurora Australis*.



The direction of the Earth's magnetic field can be preserved in certain rocks such as cooled lava deposited on the sea floor of a separating tectonic plate boundary like the mid-Atlantic ridge. Polarity reversals are common and appear to be chaotic in regularity. Most modern estimates are between 1000 and 10,000 years between transitions.

Variation in Earth's magnetic field strength is:

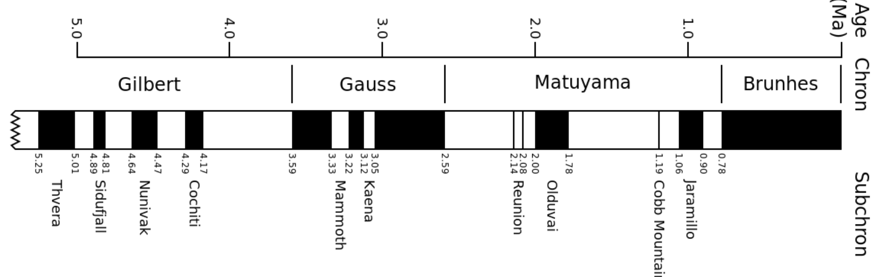
$$25\times10^{-6}\mathrm{T} < B < 65\times10^{-6}\mathrm{T}$$



All materials contain *atoms*, which comprise of negatively charged *electrons* distributed in 'orbitals' about a positively charged nucleus. Although a *Quantum Mechanical* description is required to understand the 'motion' of electrons, atoms will produce a small magnetic field in the fashion of a *dipole*.

If an external magnetic field is applied, the effect upon electron 'orbits' is for a material to be *repelled* by the magnetic field. This is called **diamagnetism** and is universal. However, if there are any *unpaired electrons** then an applied field will cause a alignment of the magnetic fields resulting from these electrons. The net result is an *enhancement* of magnetism and is called **paramagnetism**.

In a few **ferromagnetic** substances such as iron, nickel and their alloys there is a tendency for unpaired electrons to produce magnetic dipoles which align with *each other*, whether an applied field is present or not. This is the origin of permanent magnetism. **Hard magnetic materials** (e.g. steel) retain their magnetism once aligned. **Soft magnetic materials** (e.g. ferrite) will align with an applied field, then return to a more disordered state once a field is switched off. This is why iron cores are useful in electromagnets based upon a solenoid.. They intensify a magnetic field when current flows, but 'switch off' when the current ceases. Above their **Curie Temperature**, all ferromagnetic materials become paramagnetic. Applying an external field will set the alignment, and then cooling and hammering will make the effect permanent.



Change of polarity of the Earth's magnetic field over the past five million years

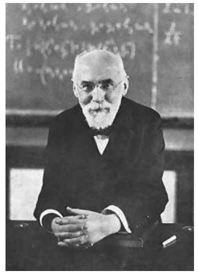
The *Pauli Exclusion* principle states up to *only two* electrons can have the same Quantum state characterized by their *orbital*.
The **spin** of the electrons (a form of intrinsic magnetism which is either 'up' or 'down') is what distinguishes paired electrons.

A moving charge in a magnetic field will feel a force in proportion to the strength of the magnetic field, and also the velocity of the charge.

$$\boxed{\mathbf{F} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}}$$

Electric force

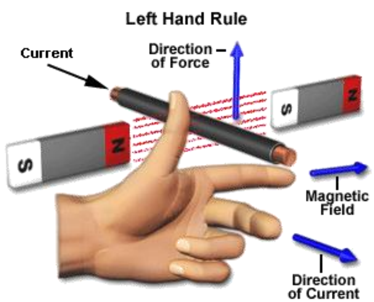
Lorentz magnetic force



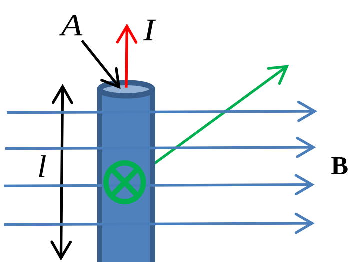
Hendrick Antoon Lorentz
1853-1928

The magnetic force acts in a *mutually perpendicular* direction to both the charge velocity and the magnetic field.

Since *direct current* is the flow of *positive charge*, we can easily understand the geometric relationship of these quantities using **Fleming's Left Hand Rule.**



St John Ambrose Fleming
1849-1945



⊗ Force on current carrying wire *into* page

If electric current I flows at velocity v through a cylinder of cross section A and length l, and the charge density is n coulombs per unit volume

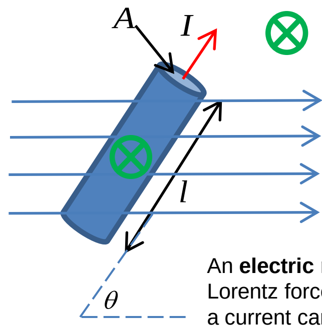
$$q = nAl$$

$$I = nAv \quad \therefore v = \frac{I}{nA}$$

$\therefore F = qvB$ if **v** and **B** perpendicular

$$\therefore F = nAl \times \frac{I}{na}B$$

$$\boxed{F = BIl}$$

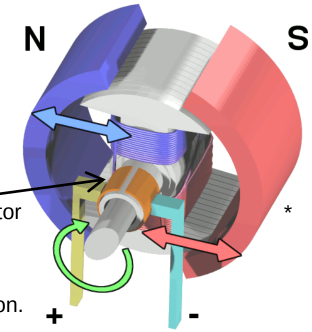


⊗ $\boxed{F = BIl\sin\theta}$

If the conductor is *not* perpendicular to the magnetic field then the net force will vary as sine of the angle i.e. the degree of perpendicular projection.

We can generalize the force on a current carrying conductor element d**l**

$$\boxed{d\mathbf{F} = I d\mathbf{l}\times\mathbf{B}}$$



An **electric motor** is a practical application of the Lorentz force, which results in a *turning moment* on a current carrying coil which is free to rotate within a magnetic field. A **split ring commutator** swaps the direction of current once every half revolution to maintain torque, and hence rotation, continuously in the same anticlockwise or clockwise direction. The maximum force is when current and field are perpendicular. To optimise, a practical motor will have curved magnets, and coils in different orientations

Application of the Lorentz force – the Hall Efffect

A semiconductor of width w and height h is placed in a magnetic field B. Current I passes through the semiconductor as shown.The Lorentz force on charges will cause a charge separation, which in turn will result in an electric field E perpendicular to both the magnetic field and the current direction.

Equilibrium is reached when the electric force and Lorentz magnetic forces balance.

Example calculation:

$$n = 7\times10^{21}\,\mathrm{m}^{-3}$$

$$q = e = 1.6\times10^{-19}\,\mathrm{C}$$

$$h = 0.1\mathrm{mm}$$

$$\therefore qnh = \boxed{0.112}$$

$\therefore \frac{V_H}{I}$ can be a ratio of near-unity quantities, which are readily measureable i.e. B fields not too many orders of magnitude less than 1.0T can be easily measured.

$$qE = qvB$$

$$\therefore E = vB$$

$$E = \frac{V_H}{w}$$

The electric potential resulting from the separated charges is called the **Hall Voltage**

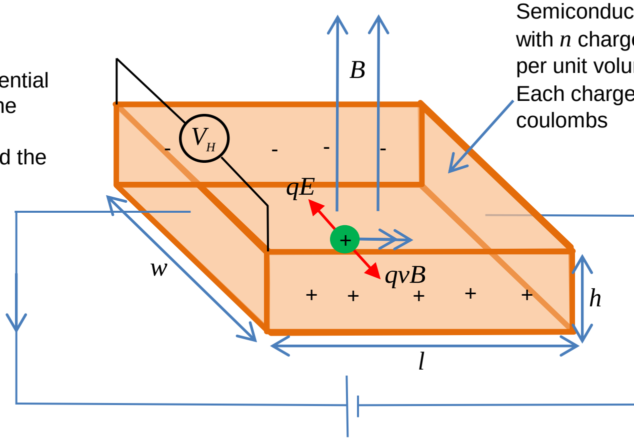
$$I = qnwhv$$

$$\therefore v = \frac{I}{qnwh}$$

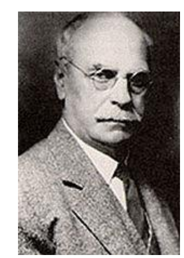
$$\therefore \frac{V_H}{w} = \frac{IB}{qnwh}$$

$$V_H = \frac{IB}{qnh}$$

$$\boxed{B = \frac{qnhV_H}{I}}$$



Semiconductor with n charges per unit volume
Each charge has q coulombs



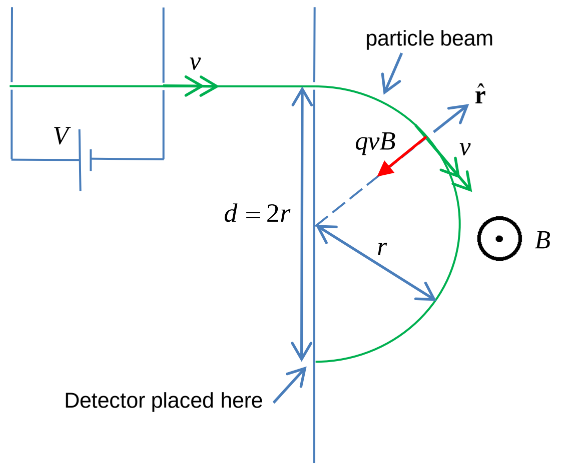
Edwin Hall
1855-1938

It is possible to measure the Hall effect in a small semiconductor, so the effect can be used to determine the how a non uniform magnetic field varies in time and space.

Application of the Lorentz force – the mass spectrometer

The charge q to mass m ratio of the molecular constituents of a material can be determined using a mass-spectrometer. The material is incinerated and then a beam of the resulting particulates is accelerated through an electric potential V.

The accelerated beam is then bent into a circle via the Lorentz force, resulting from a uniform magnetic field of strength B. The charge to mass ratio can then be determined from the diameter of the circle.



By Newton's second law, and noting acceleration is *centripetal* since the motion of the particle beam is uniform and circular:

$$\frac{mv^2}{r} = qvB \quad \therefore r = \frac{mv}{qB}$$

$$\tfrac{1}{2}mv^2 = qV \quad \therefore v = \sqrt{\frac{2qV}{m}}$$ Conservation of energy

$$\therefore r = \frac{m}{qB}\sqrt{\frac{2qV}{m}} = \sqrt{\frac{m^2}{q^2B^2}}\sqrt{\frac{2qV}{m}}$$

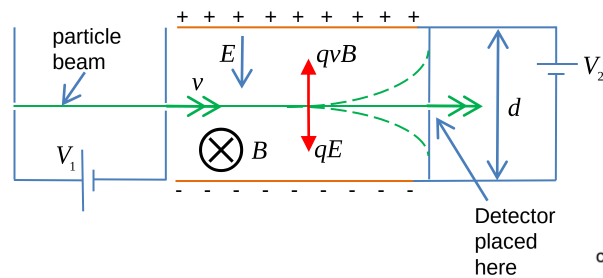
$$\therefore \quad d = \sqrt{\frac{8V}{B^2}}\left(\frac{q}{m}\right)^{-\frac{1}{2}} \quad \therefore \boxed{\frac{q}{m} = \frac{8V}{B^2d^2}}$$

Note relativistic effects ignored here!

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}$$

Application of the Lorentz force – velocity selector

An alternative system for determining the charge q to mass m ratio of the molecular constituents of a material is the velocity selector. In this case a second electric potential is varied until the Lorentz force balances the electric field between two plates, which are placed in a uniform magnetic field. When these forces balance, a particle beam will not be deflected and can therefore be detected.



Force balance

$$qE = qvB \therefore E = vB$$

$$E = \frac{V_2}{d}$$

$$\tfrac{1}{2}mv^2 = qV_1 \therefore v = \sqrt{\frac{2qV_1}{m}}$$

$$\therefore \frac{V_2}{d} = B\sqrt{\frac{2qV_1}{m}}$$

$$\frac{V_2^2}{d^2} = B^2\frac{2qV_1}{m}$$

$$\boxed{\frac{q}{m} = \frac{V_2^2}{2V_1B^2d^2}}$$

Application of the Lorentz force – Cyclotron

A Cyclotron is a space efficient mechanism for accelerating charges to high speeds. Much like a mass-spectrometer, a beam of charges are bent into a circle. However, in this case the charges receive a boost twice per cycle from a *alternating* potential between two halves ('Dees') of the cyclotron.

For a given arc of radius r, beam enters gap between the Dees with velocity u

$$\frac{mv^2}{r} = qvB \quad \therefore \boxed{r = \frac{mv}{qB}}$$

$$\tfrac{1}{2}mv^2 = \tfrac{1}{2}mu^2 + qV \quad \therefore \boxed{v = \sqrt{u^2 + \frac{2qV}{m}}}$$

Now since motion is uniform circular

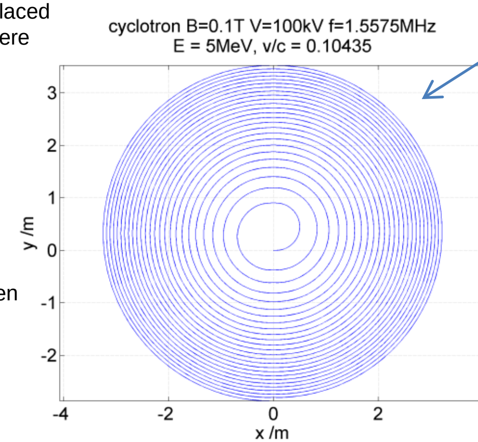
$$v = r\omega \qquad v = \frac{rqB}{m}$$

$$\therefore \omega = \frac{qB}{m} \qquad \therefore \boxed{f_{cyc} = \frac{qB}{2\pi m}}$$

Since the *cyclotron frequency* is *independent of radius*, an alternating potential of

$$\boxed{V(t) = V_0 \cos\left(\frac{qB}{2\pi m}t\right)}$$

will always provide a maximal boost in the right direction when the particles exit the dees.

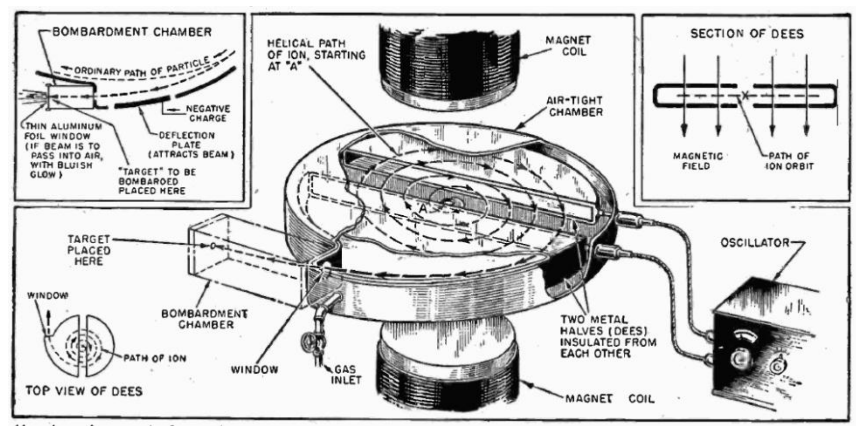


Paths of a proton beam cyclotron. 'Dee' separation is 0.1m

$m = 1.637 \times 10^{-27}\,\text{kg}$

$q = 1.602 \times 10^{-19}\,\text{C}$

Note many modern particle accelerators (e.g. the LHC in CERN) are **Synchrotrons**. *Time varying* magnetic fields maintain a *fixed circular beam* as charges are accelerated. As of 2015, energies of 6.5TeV per beam at the LHC.

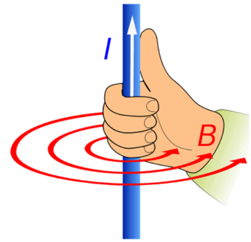


Note as particles approach the speed of light, *relativistic* effects become significant*

*Multiplying B by the *Lorentz factor* γ, or alternatively dividing the cyclotron frequency by γ, can optimize power transfer as v approaches c. Note most modern high energy sources are typically *Synchrotrons*. However, these are much less compact!

Calculating magnetic field strength

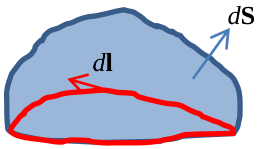
A current will generate a magnetic field, as indicated by the **Right hand grip rule.** But what is the field strength B a radial distance r away from a current carrying wire?



Right hand grip rule

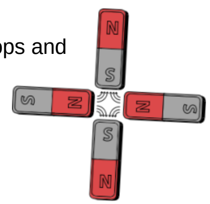
Ampère's Theorem, combined with **Maxwell's Displacement Current** when electrical fields vary with time, describes this relationship

$$\boxed{\oint_{loop} \mathbf{B}\cdot d\mathbf{l} = \mu_0 I + \mu_0\varepsilon_0 \int_{surface} \frac{\partial \mathbf{E}}{\partial t}\cdot d\mathbf{S}}$$



i.e. an open surface bounded by a loop corresponding to the integral on the left hand side

Magnetic quadropole



Carl Friedrich Gauss
1777-1855

If there are no magnetic monopoles (i.e. magnetic fields are closed loops and all magnets come in dipole, quadropole etc sets of North,South pairs)

$$\boxed{\oint_{closed\,surface} \mathbf{B}\cdot d\mathbf{S} = 0}$$

Note this is different from electric fields, which are sourced by charges

$$\boxed{\oint_{closed\,surface} \mathbf{E}\cdot d\mathbf{S} = \frac{Q}{\varepsilon_0}}$$

Gauss's Law

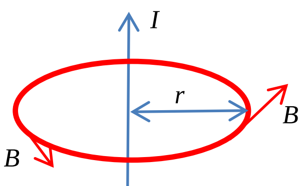
André-Marie Ampère
1775-1836

If nearby electric fields are static so we can ignore the Displacement current

$$\boxed{\oint_{loop} \mathbf{B}\cdot d\mathbf{l} = \mu_0 I}$$

Ampère's Theorem

For a current carrying wire, the magnetic field strength at distance r from the wire is therefore given by



$$B \times 2\pi r = \mu_0 I$$

$$\therefore \boxed{B(r) = \frac{\mu_0 I}{2\pi r}}$$

A magnetic field of *exactly* 2 x 10^{-7} Tesla is therefore produced at a distance of 1m from a wire carrying current of 1A

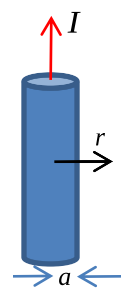
A similar result follows for a uniform cylinder of radius a, carrying current I, when $r < a$.

When $r < a$, the current closed by a loop of radius r is

$$I' = I\frac{r^2}{a^2} \quad \therefore B \times 2\pi r = \mu_o I'$$

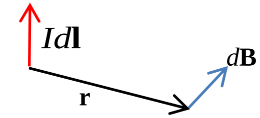
$$\therefore B = \frac{Ir^2}{2\pi r a^2} \Rightarrow B = \frac{Ir}{2\pi a^2}$$

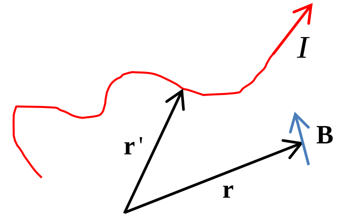
$$\boxed{B = \begin{cases} \dfrac{\mu_0 I r}{2\pi a^2} & r < a \\ \dfrac{\mu_0 I}{2\pi r} & r \geq a \end{cases}}$$



Another useful expression for calculating magnetic fields from current elements is the **Biot-Savart law**

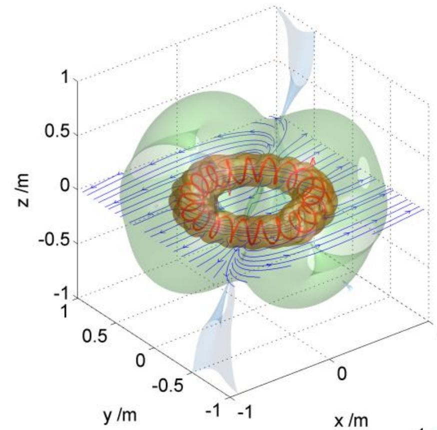
$$\boxed{d\mathbf{B} = \frac{\mu_0 I}{4\pi}\frac{d\mathbf{l}\times\mathbf{r}}{r^3} \quad r = |\mathbf{r}|}$$





For a arbitrary current carrying line

$$\boxed{\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_{line}\frac{d\mathbf{l}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}}$$

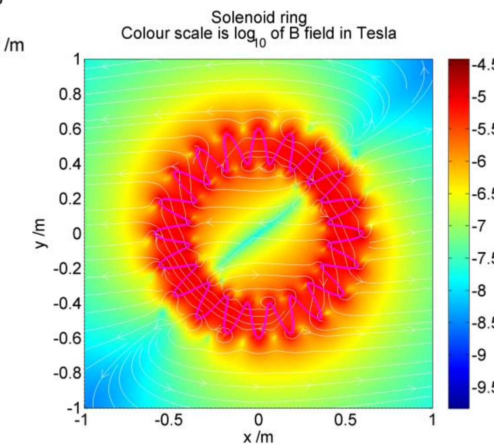


We can use the Biot Savart law to compute magnetic fields for more complex geometries, such as a coil of current carrying wire bent into a torus.

Note we can find the field inside the torus easily with Ampère's theorem – see page 8

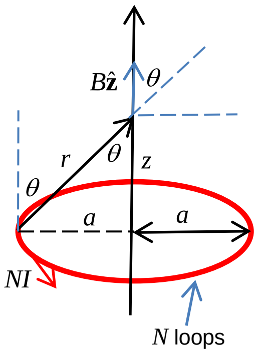
$$\boxed{B = \frac{\mu_0 \mu N I}{2\pi r}}$$

James Clerk Maxwell
1831-1879

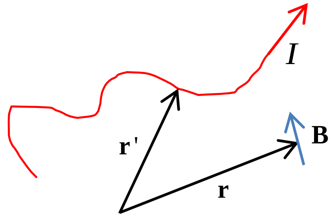


μ_0 is the permeability of free space. It is defined to be $\mu_0 = 4\pi \times 10^{-7}\,\mathrm{NA}^{-2}$ N being Newtons and A being Amps

Magnetic field on axis from a current loop
of N turns and radius a, carrying current I



On the z axis, the only magnetic field direction can be in the z direction since the circular symmetry of the situation means magnetic field in the x and y directions must cancel



Biot-Savart Law

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0 I}{4\pi}\int_{\text{line}} \frac{d\mathbf{l}\times(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}$$

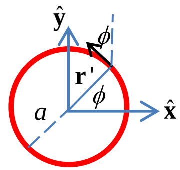
$$\mathbf{r} = z\hat{\mathbf{z}}, \quad \mathbf{r}' = a(\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}})$$

$$|\mathbf{r}-\mathbf{r}'| = r$$

$$d\mathbf{l} = a\,d\phi(-\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}})$$

$$d\mathbf{l}\times\mathbf{r}' = -a^2 d\phi\hat{\mathbf{z}}$$

line element and loop vector are perpendicular



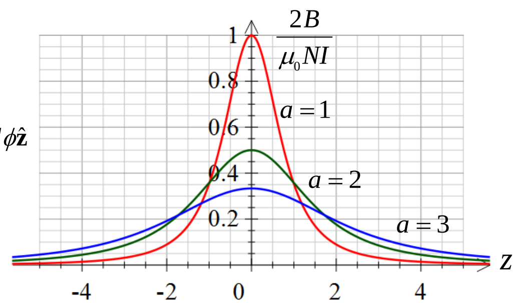
$$d\mathbf{l}\times\mathbf{r} = a\,d\phi(-\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}})\times z\hat{\mathbf{z}}$$

$$= azd\phi(\sin\phi\hat{\mathbf{y}} + \cos\phi\hat{\mathbf{x}})$$

$$d\mathbf{l}\times(\mathbf{r}-\mathbf{r}') = azd\phi(\sin\phi\hat{\mathbf{y}} + \cos\phi\hat{\mathbf{x}}) + a^2 d\phi\hat{\mathbf{z}}$$

$$\therefore \int_{\phi=0}^{2\pi} d\mathbf{l}\times(\mathbf{r}-\mathbf{r}') = 2\pi a^2\hat{\mathbf{z}}$$

Integral of cos and sin terms over full period is zero



Biot-Savart becomes:

$$\mathbf{B} = B\hat{\mathbf{z}} = \frac{\mu_0 NI}{4\pi r^3}\int_{\phi=0}^{2\pi} d\mathbf{l}\times(\mathbf{r}-\mathbf{r}')$$

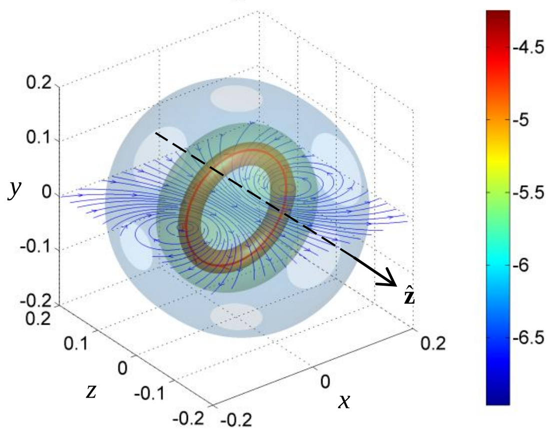
$$\therefore B = \frac{\mu_0 NI}{4\pi r^3}\times 2\pi a^2$$

$$r = \sqrt{a^2+z^2}$$

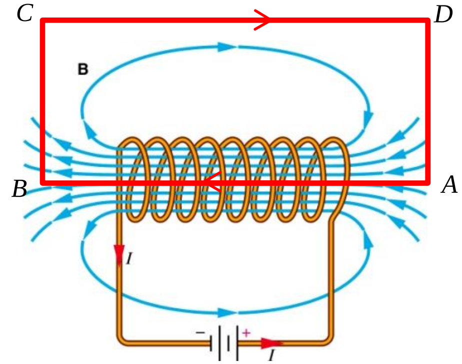
$$\boxed{\therefore B = \tfrac{1}{2}\mu_0 NI\frac{a^2}{(a^2+z^2)^{\frac{3}{2}}}}$$

$$\therefore B(0) = \tfrac{1}{2}\mu_0\frac{NI}{a}$$

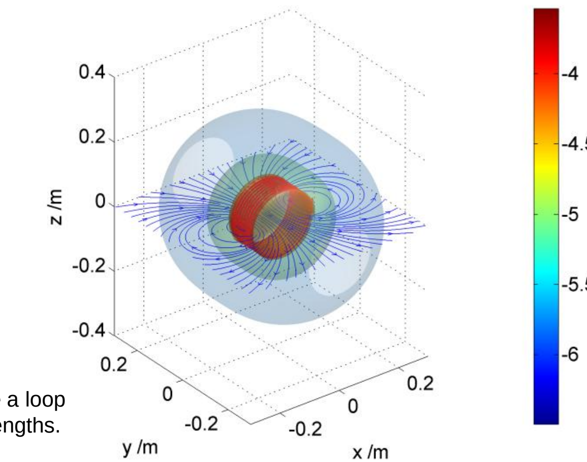
Ring
Colour scale is $\log_{10}$ of B field in Tesla



Magnetic field inside a solenoid of n turns per unit length and radius a carrying current I. Unlike the finite solenoid in the simulation plot below, let us consider an infinitely long. We might anticipate this to be a good approximation to a finite solenoid in terms of the field within the coils.



Solenoid
Colour scale is $\log_{10}$ of B field in Tesla



Let us define a loop ABCD which passes through the solenoid. Let us assume the magnetic field is uniform within the solenoid tending to zero outside. We will take a loop where CB and DA distances tend to infinite lengths.

$$\oint_{ABCD} \mathbf{B}\cdot d\mathbf{l} = \mu_0 I\times nl \qquad \textbf{Ampère's Theorem}$$

$$\int_B^C \mathbf{B}\cdot d\mathbf{l} = -\int_D^A \mathbf{B}\cdot d\mathbf{l}, \quad \int_C^D \mathbf{B}\cdot d\mathbf{l}\to 0$$

$$\int_A^B \mathbf{B}\cdot d\mathbf{l} \approx Bl \quad \therefore \oint_{ABCD} \mathbf{B}\cdot d\mathbf{l}\approx Bl$$

$$\therefore Bl = \mu_0 I\times nl$$

$$\boxed{B = \begin{cases}\mu_0 nI & r<a \\ 0 & r\gg a\end{cases}}$$

Typical magnetic field strengths in Tesla

Interstellar space 10^{-10}

Earth's magnetic field 10^{-5}

Small bar magnet 10^{-2}

Within a sunspot 0.15

Small Neodymium magnet 0.2

Large electromagnet 1.5

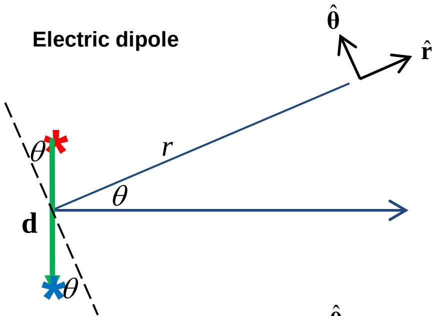
Strong laboratory magnet 20

Surface of a Neutron Star 10^{8}

Magnetar 10^{11}

The field of a **Magnetic dipole** is mathematically very similar to that if an electric dipole (see *Electric dipole* notes).

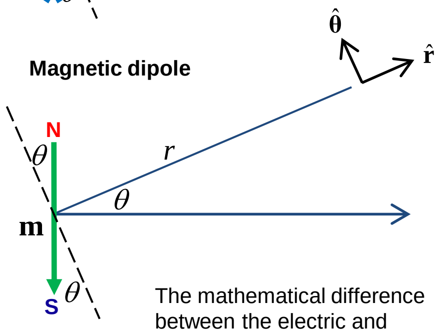
Electric dipole



$$\boxed{\mathbf{E} = \frac{qd}{4\pi\varepsilon_0 r^3}\left(2\hat{\mathbf{r}}\sin\theta - \hat{\boldsymbol{\theta}}\cos\theta\right)}$$

In both cases assume r is much greater than the dimensions associated with the dipole

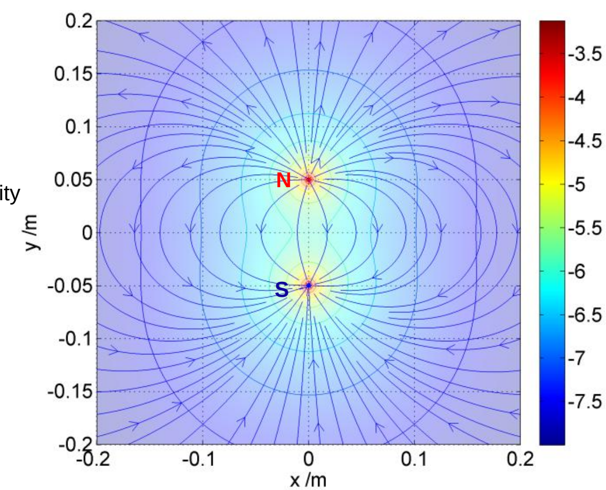
Magnetic dipole



$$\boxed{\mathbf{B} = \frac{\mu_0 m}{4\pi r^3}\left(2\hat{\mathbf{r}}\sin\theta - \hat{\boldsymbol{\theta}}\cos\theta\right)}$$

The mathematical difference between the electric and magnetic dipoles is the quantity

$$\frac{qd}{\varepsilon_0} \to \mu_0 m$$



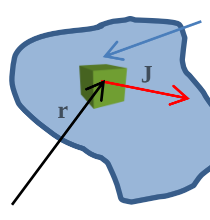
m is the **magnetic dipole moment**

For a magnetic dipole formed from a small current loop (or indeed solenoid) of radius a

$$\boxed{m = I\pi a^2}$$

In general for a volume with current density $\mathbf{J}$ (current per unit area) in a particular direction

$$\boxed{\mathbf{m} = \tfrac{1}{2}\int_{volume} \mathbf{r}\times\mathbf{J}\,dxdydz}$$



The **torque** on an electric dipole in a magnetic field is:

$$\boxed{\boldsymbol{\tau} = \mathbf{m}\times\mathbf{B}}$$

For an atom, the *diamagnetic* moment (i.e. the effect of a magnetic field on an atom, and not inter-atom effects that give rise to *ferromagnetism* in iron, nickel etc) is:

$$m_e = 9.109\times10^{-31}\,\text{kg}$$
$$e = 1.602\times10^{-19}\,\text{C}$$

Electron mass and charge

$$\boxed{\mathbf{m} = -\frac{e^2}{6m_e}Z\left\langle r^2\right\rangle\mathbf{B}}$$

Atomic number

mean squared orbital radius

For a charge q of mass M orbiting around a circle of radius r

$$J = Mrv \;\therefore\; v = \frac{J}{Mr}$$ angular momentum J

$$T = \frac{2\pi r}{v}$$ orbit time

$$I = \frac{q}{T} = \frac{qv}{2\pi r} = \frac{qJ}{2\pi Mr^2}$$

$$m = I\pi r^2 \Rightarrow \boxed{m = \frac{qJ}{2M}}$$

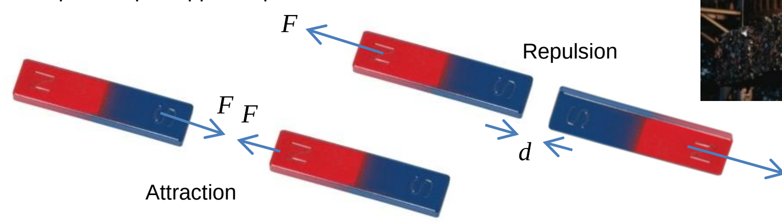
Many magnetic effects can be understood in terms of the **magnetic moment of an electron**. Note this is really a quantum mechanical attribute and therefore it is perhaps unwise to imagine an electron consisting of a tiny current loop!

$$\boxed{\begin{aligned}&\mathbf{m}_e \approx \frac{e}{2m_e}g\mathbf{J}\\ &g = 2 \quad \text{"spin"}\\ &g = 1 \quad \text{orbital angular momentum}\end{aligned}}$$

Total angular momentum vector Note this has the same symbol as current density. Take care to avoid confusion!

Forces between magnetic dipoles
Probably the most basic property of magnetism is that 'Like poles repel, opposite poles attract', but with what force?

e.g. electromagnets



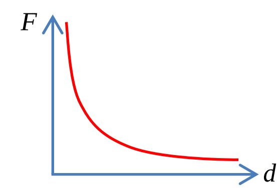
For a magnetic dipole $\mathbf{m}$ in a magnetic field $\mathbf{B}$, the force depends on the local *gradient* of the magnetic field. $\mathbf{B}$ *can't* be locally uniform because there would be an equal and opposite force on either poles of the dipole. The force is given by:

$$\boxed{\begin{aligned}&\mathbf{F} = (\mathbf{m}\cdot\nabla)\mathbf{B}\\ &\nabla = \hat{\mathbf{x}}\frac{\partial}{\partial x} + \hat{\mathbf{y}}\frac{\partial}{\partial y} + \hat{\mathbf{z}}\frac{\partial}{\partial z}\end{aligned}}$$

gradient vector operator

If m and B are parallel:

$$\boxed{\begin{aligned}&\mathbf{m} = m\hat{\mathbf{x}}, \quad \mathbf{B} = B\hat{\mathbf{x}}\\ &F = m\frac{dB}{dx}\end{aligned}}$$



For a small bar magnet near a *wire*:

$$B = \frac{\mu_0 I}{2\pi r} \;\therefore\; F \propto \frac{dB}{dr} \;\therefore \boxed{F = \frac{k}{r^2}}$$

The force between two magnetic dipoles (e.g. *two bar magnets*)

$$B \propto \frac{1}{r^3} \;\therefore\; F \propto \frac{dB}{dr}$$
$$\therefore \boxed{F = \frac{k}{r^4}}$$

Effects of magnetisation

All atoms will exhibit **diamagnetism** i.e. the effect of an applied magnetic field on electron orbits is to produce a magnetic field which opposes the applied field.

However, many materials will form magnetic dipoles that will align with an applied field, strengthening it. This is caused **paramagnetism**. For a few **ferromagnetic** materials (such as steel) the alignment persists even when the field is removed, creating a permanent magnet.

To account for magnetisation effects, the net magnetic field $\mathbf{B}$ is defined as the sum of a magnetisation field $\mathbf{M}$ and an applied field $\mathbf{H}$

$$\boxed{\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})}$$

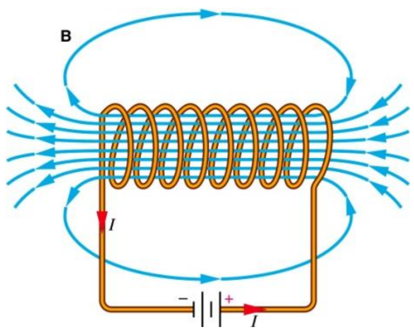
If a material is *isotropic* in a magnetic sense, we might sensibly assume magnetisation effects are parallel to the applied field. hence:

$$\boxed{\mathbf{B} = \mu\mu_0 \mathbf{H}}$$

Relative permeability

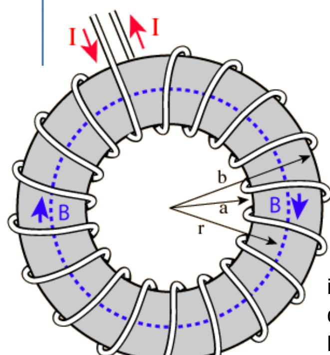
Material	**Relative permeability μ**
Air, wood, water, copper	1
Nickel, carbon steel	100
Ferrite (Fe_2O_3, usually combined with Nickel, Zinc...)	> 640
Pure iron	> 5,000

Modification to magnetic field equations to include an isotropic cores of relative permeability μ



$$\boxed{B = \begin{cases} \mu_0 \mu n I & r < a \\ 0 & r \gg a \end{cases}}$$

'Infinite' solenoid of n turns per unit length



*

i.e. within the dotted B field loop there are N wires exiting, each carrying current I

Ampère's Theorem

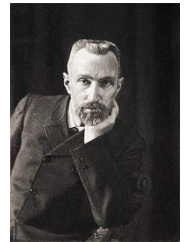
If there are magnetisation effects then we must use $\mathbf{H}$ not $\mathbf{B}$

$$\oint_{loop} \mathbf{H} \cdot d\mathbf{l} = I$$

Solenoid bent into a torus with N turns

$$H \times 2\pi r = NI$$

$$\therefore B = \mu_0 \mu H = \boxed{\frac{\mu_0 \mu N I}{2\pi r}}$$



Pierre Curie
1859-1906

Paul Langevin
1872-1946

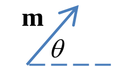
Curie's Law and Paramagnetism

i.e. how magnetism varies with *temperature*

Consider an isotropic magnetic medium with magnetic dipoles aligned in random orientations. Let a magnetic field be applied of strength B.

The **magnetic energy** of a dipole is given by

$$\boxed{E = -\mathbf{m} \cdot \mathbf{B} = -mB\cos\theta}$$



The average magnetisation aligned with the field is therefore:

$$\langle m_{//} \rangle = \int_0^\pi m\cos\theta \times p(\theta)\,d\theta$$

$$p(\theta)\,d\theta = A\frac{2\pi r\sin\theta r d\theta}{4\pi r^2} \times e^{\frac{-mB\cos\theta}{k_B T}}$$

Boltzmann factor

fraction of the surface area of a sphere which corresponds to polar angles of θ

$$1 = \int_0^\pi p(\theta)\,d\theta$$

Normalization of probability density function

$$p(\theta) = \tfrac{1}{2} A \sin\theta e^{\frac{mB\cos\theta}{k_B T}}$$

$$\mu = \cos\theta, \quad x = \frac{mB}{k_B T} \qquad \therefore d\mu = -\sin\theta\,d\theta$$

$$\therefore p(\theta)\,d\theta = \tfrac{1}{2} A(-d\mu)e^{x\mu}$$

$$\therefore \int_0^\pi p(\theta)\,d\theta = -\tfrac{1}{2} A\int_1^{-1} e^{x\mu}\,d\mu = \tfrac{1}{2}\frac{A}{x}\left[e^{x\mu}\right]_{-1}^{1} = \frac{A}{x}\sinh x = 1$$

$$\therefore \boxed{A = \frac{x}{\sinh x}}$$

$$\langle m_{//} \rangle = \int_0^\pi m\cos\theta \times p(\theta)\,d\theta = \frac{mx}{\sinh x}\int_{-1}^{1} \mu \tfrac{1}{2} e^{x\mu}\,d\mu$$

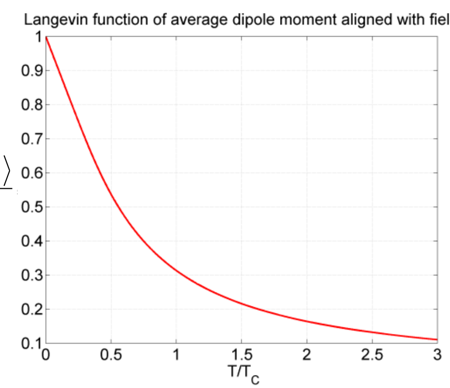
$$\langle m_{//} \rangle = \tfrac{1}{2}\frac{mx}{\sinh x}\left(\left[\frac{\mu}{x}e^{x\mu}\right]_{-1}^{1} - \int_{-1}^{1} (1)\frac{e^{x\mu}}{x}\,d\mu \right)$$

$$\langle m_{//} \rangle = \tfrac{1}{2}\frac{mx}{\sinh x}\left(\frac{2\cosh x}{x} - \left[\frac{e^{x\mu}}{x^2}\right]_{-1}^{1} \right)$$

$$\langle m_{//} \rangle = \tfrac{1}{2}\frac{mx}{\sinh x}\left(\frac{2\cosh x}{x} - \frac{2\sinh x}{x^2} \right)$$

$$\boxed{\langle m_{//} \rangle = m\left(\coth x - \frac{1}{x}\right)}$$

Langevin function



So when T is large, *no magnetisation*, when T is small, the material will align with the applied field.

*http://hyperphysics.phy-astr.gsu.edu/hbase/magnetic/toroid.html#c1