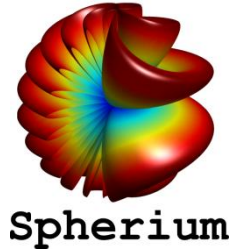




Science in Schools Workshop  
Marseilles March 2011.



# Mathematics and the Machine

The utility and beauty of computer  
visualization



Dr Andrew French



# Summary

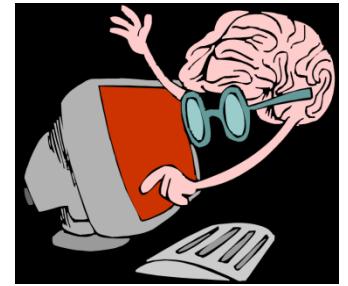
- Introduction
  - Computers and mathematics - a harmonious relationship!
- Software demonstrations
  - Harmonograph
  - Spherium
  - Julia
- Rene Descartes' theory of the rainbow



# Computers and mathematics

- All computer operations are fundamentally the change of a set of binary (0 or 1) switches in *memory* and in a *microprocessor*
- Binary digits e.g. 11110101 are a representation of *numbers*. (In this case 245)
- A 3GHz PC with a 64 bit data bus implies 192 billion switches can potentially be altered every second!
- Despite the appearance of a modern multi-media machine, a computer is essentially a 'number cruncher'

# Computers vs humans



- Computers can perform millions of calculations per second
  - They follow mathematical recipes perfectly as directed
  - They don't get bored
- The human brain is massively more powerful than any computer yet built. We can all perform incredible feats of computation (try programming a computer to recognise a face from any angle...)
- ...Yet most of us cannot add up two double digit numbers in less than a few seconds

# Humans + computers

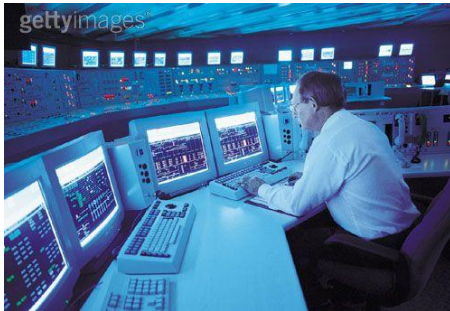


Use a computer to become a superhuman calculator!

# So why do we need all this computational power?



Computers are everywhere!



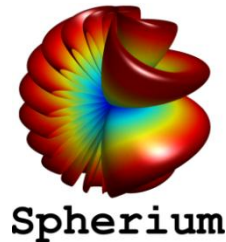
Copyright © Phob International  
All Rights Reserved





# The utility and beauty of computation


- Computers perform fast arithmetic to enable many modern appliances to work
- In this workshop we will make use of this ‘super-power’ to create **art from mathematics**



- We will then look at how maths can be used as a **language to describe the physical world**, and how the use of computers can enable us to relate the maths to what we can readily measure.



# Summary

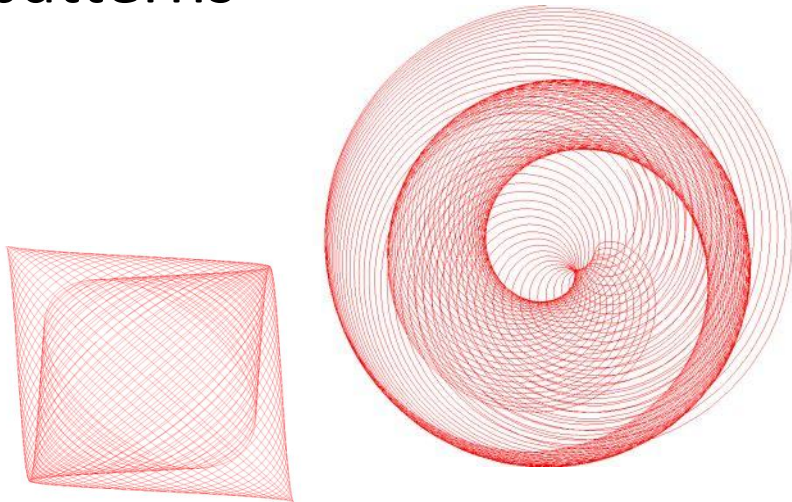
- Introduction 
  - Computers and mathematics - a harmonious relationship!
- **Software demonstrations**
  - **Harmonograph**
  - **Spherium**
  - **Julia**
- Rene Descartes' theory of the rainbow





# Harmonograph

- The Harmonograph was a Victorian curiosity attributed to Professor Blackburn in 1844
- Use two or three pendulums to create strange and beautiful patterns



Example of a *lateral* harmonograph

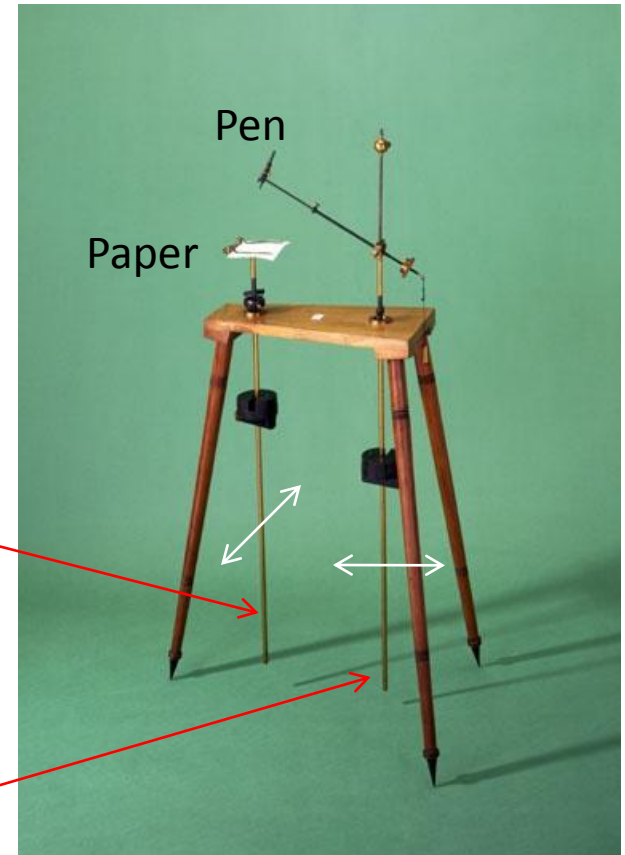
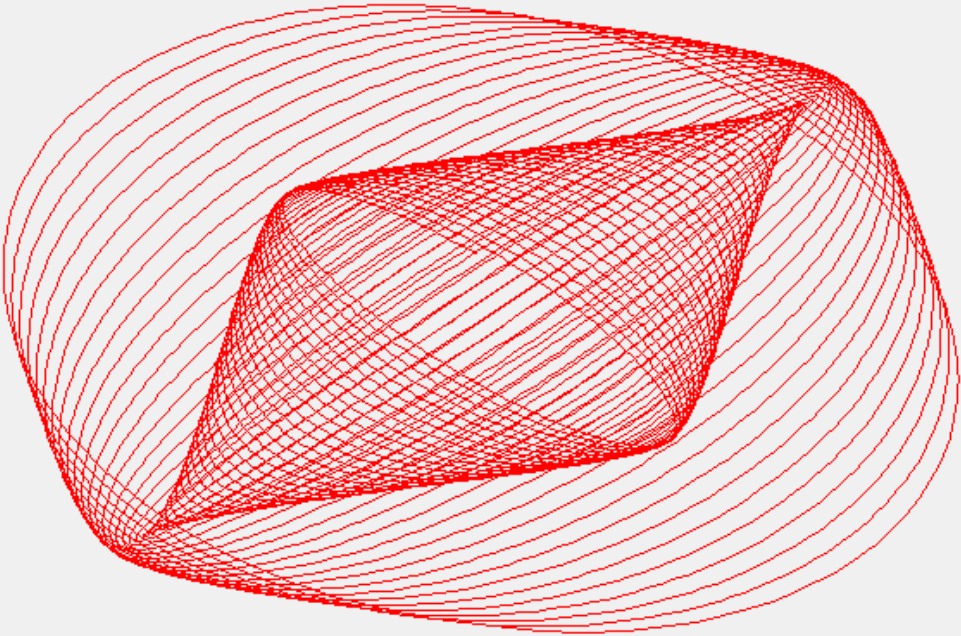


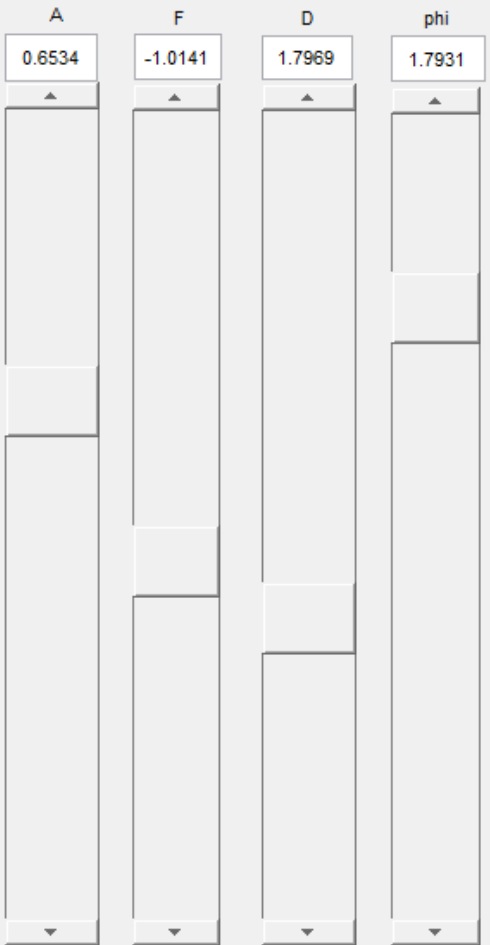
Photo from The Science Museum

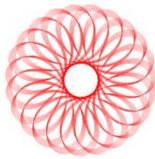
harmonograph

Lateral  
 $N=50$ ,  $A=0.6534$ ,  $F=-1.0141$ ,  $\phi=102.7371^\circ$ ,  $D=1.7969$




A	F	D	phi
0.6534	-1.0141	1.7969	1.7931





# Harmonograph



Written by  
Andy French. March 2011

Play tones
Default
Load settings
Save settings

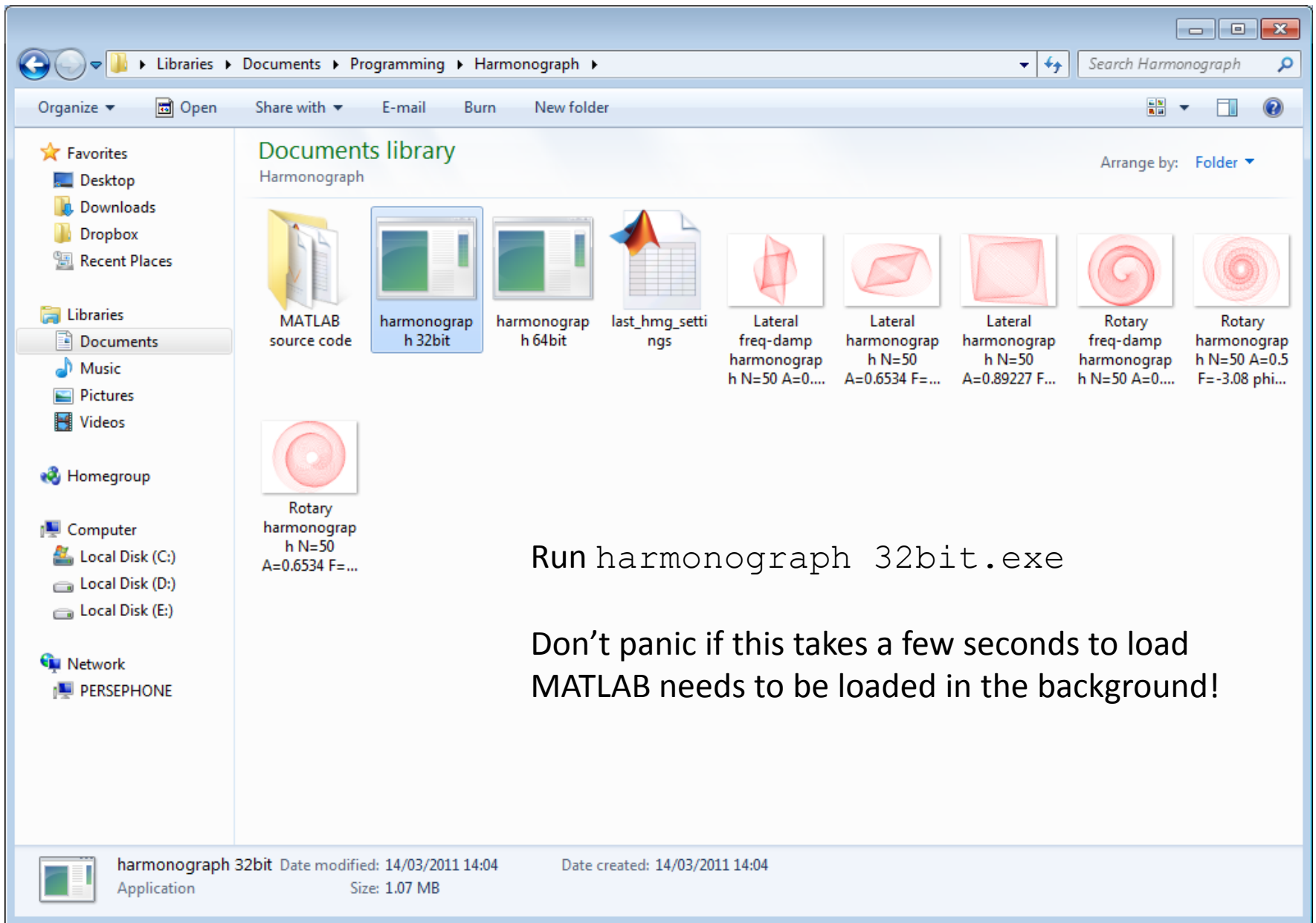
Save .PNG

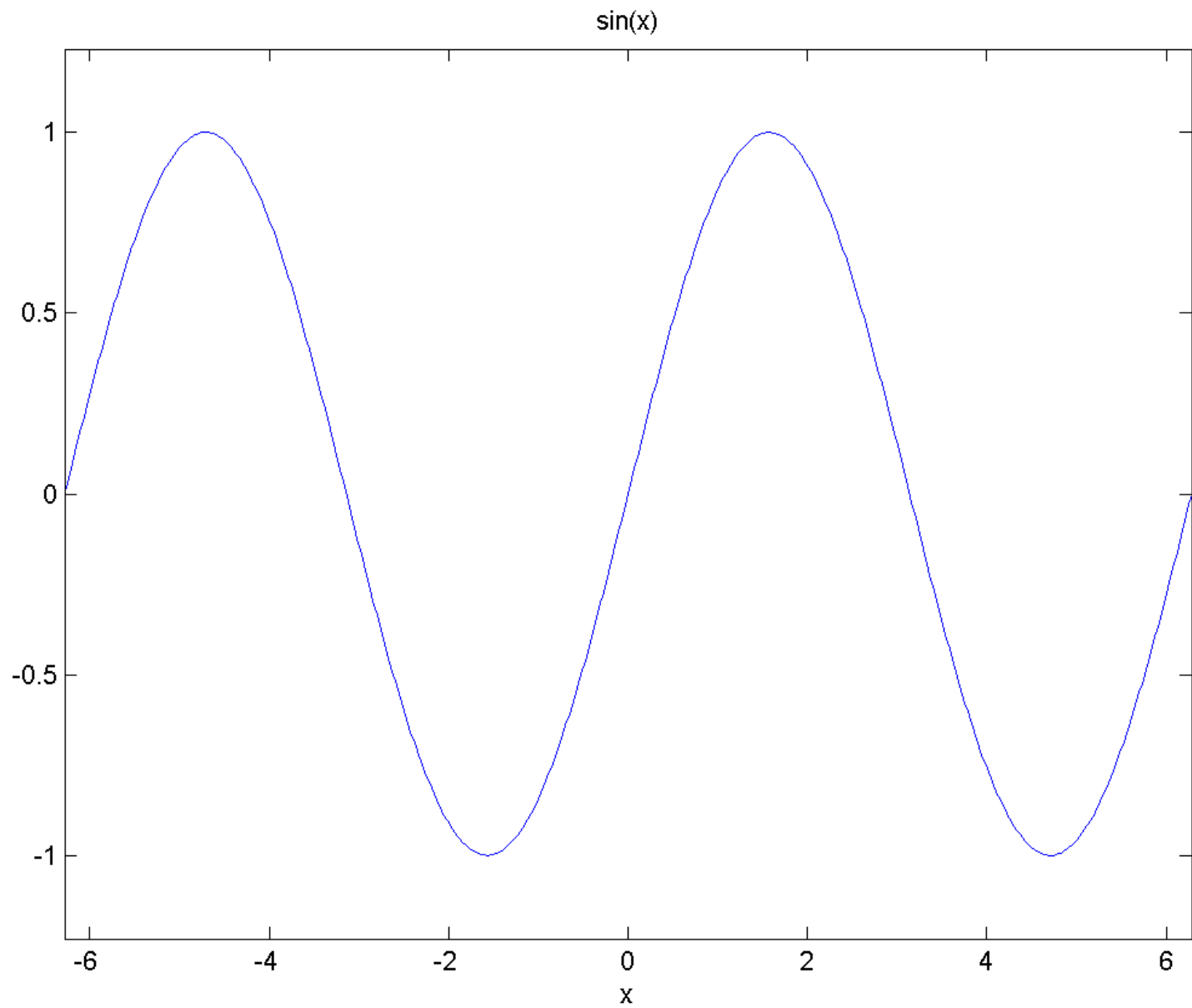
Harmonograph types

Lateral

# loops

50

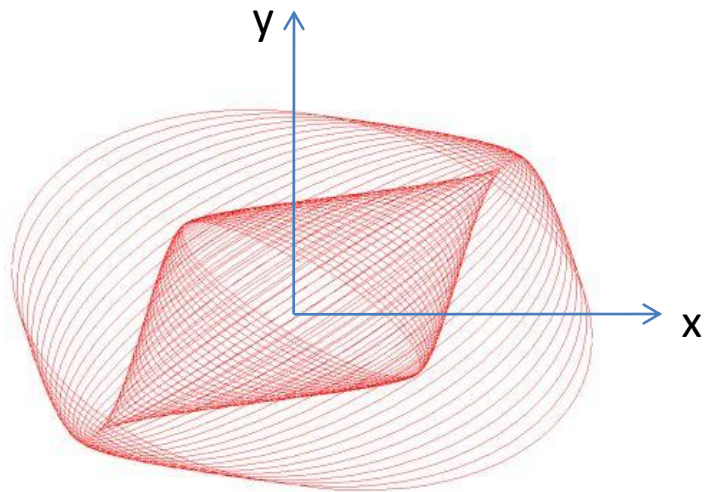




$$x = A_1 e^{-\frac{t}{T_1}} \sin(tW_1 + P_1) + A_2 e^{-\frac{t}{T_2}} \sin(tW_2 + P_2)$$

$$y = A_3 e^{-\frac{t}{T_3}} \sin(tW_3 + P_3) + A_4 e^{-\frac{t}{T_4}} \sin(tW_4 + P_4)$$

Lateral harmonograph



$$T = \frac{2\pi}{\omega \log\left(\frac{100}{100-D}\right)} [1, \infty, 1, \infty]$$

$$A = [1, 0 - a, 0]$$

$$W = [\omega, 0, F\omega, 0]$$

$$P = [0, 0, \phi, 0]$$

*Parameters*

$t$  is time /seconds

$\omega$  is  $2\pi$  times the first pendulum swing frequency /Hz

$a$  is the amplitude ratio

$F$  is the frequency ratio

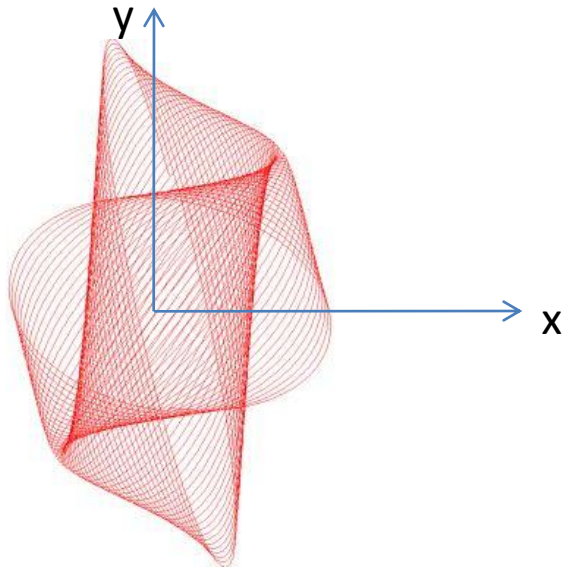
$D$  is the damping factor (typically between 0 and 5)

$\phi$  is the phase difference /radians between the pendula

$$x = A_1 e^{-\frac{t}{T_1}} \sin(tW_1 + P_1) + A_2 e^{-\frac{t}{T_2}} \sin(tW_2 + P_2)$$

$$y = A_3 e^{-\frac{t}{T_3}} \sin(tW_3 + P_3) + A_4 e^{-\frac{t}{T_4}} \sin(tW_4 + P_4)$$

Lateral harmonograph with frequency damping



$$T = \frac{2\pi}{\omega \log\left(\frac{100}{100-D}\right)} \left[1, \infty, \frac{1}{F}, \infty\right]$$

$$A = [1, 0 - a, 0]$$

$$W = [\omega, 0, F\omega, 0]$$

$$P = [0, 0, \phi, 0]$$

*Parameters*

$t$  is time /seconds

$\omega$  is  $2\pi$  times the first pendulum swing frequency /Hz

$a$  is the amplitude ratio

$F$  is the frequency ratio

$D$  is the damping factor (typically between 0 and 5)

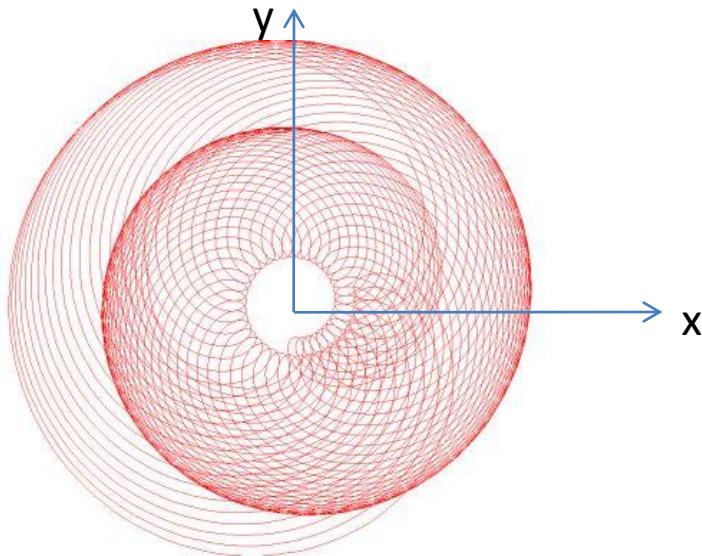
$\phi$  is the phase difference /radians between the pendula



$$x = A_1 e^{-\frac{t}{T_1}} \sin(tW_1 + P_1) + A_2 e^{-\frac{t}{T_2}} \sin(tW_2 + P_2)$$

$$y = A_3 e^{-\frac{t}{T_3}} \sin(tW_3 + P_3) + A_4 e^{-\frac{t}{T_4}} \sin(tW_4 + P_4)$$

Rotary harmonograph



$$T = \frac{2\pi}{\omega \log\left(\frac{100}{100-D}\right)} [1, 1, 1, 1]$$

$$A = [1, a, 1, a]$$

$$W = [\omega, -F\omega, \omega, -F\omega]$$

$$P = \left[0, \phi, \frac{\pi}{2}, \frac{\pi}{2} + \phi\right]$$

*Parameters*

$t$  is time /seconds

$\omega$  is  $2\pi$  times the first pendulum swing frequency /Hz

$a$  is the amplitude ratio

$F$  is the frequency ratio

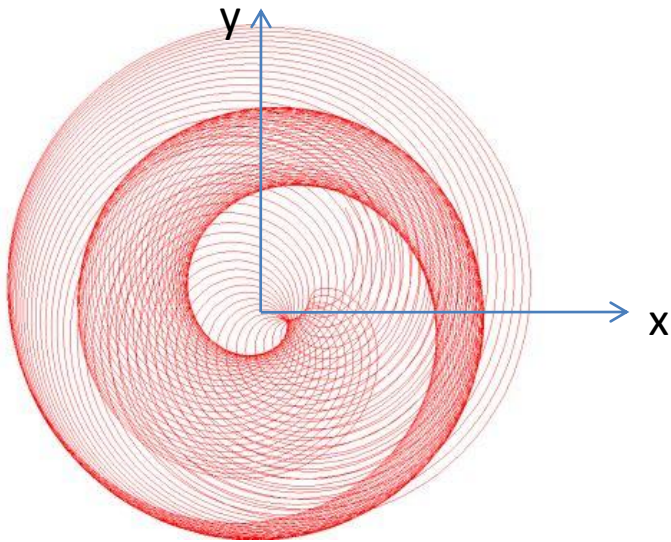
$D$  is the damping factor (typically between 0 and 5)

$\phi$  is the phase difference /radians between the pendula

$$x = A_1 e^{-\frac{t}{T_1}} \sin(tW_1 + P_1) + A_2 e^{-\frac{t}{T_2}} \sin(tW_2 + P_2)$$

$$y = A_3 e^{-\frac{t}{T_3}} \sin(tW_3 + P_3) + A_4 e^{-\frac{t}{T_4}} \sin(tW_4 + P_4)$$

Rotary harmonograph with frequency damping



$$T = \frac{2\pi}{\omega \log\left(\frac{100}{100-D}\right)} \left[1, \frac{1}{F}, 1, \frac{1}{F}\right]$$

$$A = [1, a, 1, a]$$

$$W = [\omega, -F\omega, \omega, -F\omega]$$

$$P = \left[0, \phi, \frac{\pi}{2}, \frac{\pi}{2} + \phi\right]$$

*Parameters*

$t$  is time /seconds

$\omega$  is  $2\pi$  times the first pendulum swing frequency /Hz

$a$  is the amplitude ratio

$F$  is the frequency ratio

$D$  is the damping factor (typically between 0 and 5)

$\phi$  is the phase difference /radians between the pendula

# Musical harmony



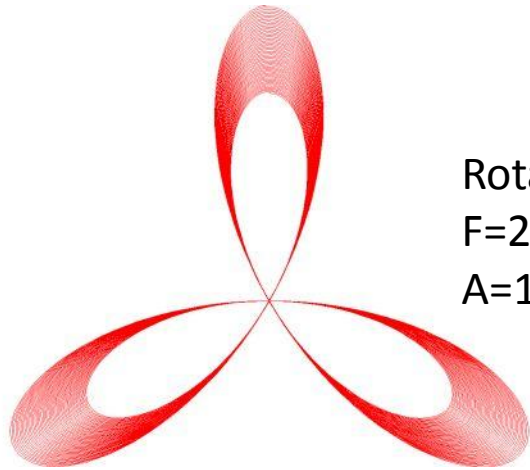
- The mathematics of music has been known since the time of Pythagoras, 2500 years ago
- Frequency intervals of simple fractions e.g. 3:2 (a fifth) yield 'harmonious' music
- An **octave** means a **frequency ratio of 2**. An octave above concert A (440Hz) is therefore 880Hz. An octave below is 220Hz.
- The modern 'equal-tempered scale' divides an octave (the frequency ratio 2) into twelve parts such that

$$F_n = 2^{n/12} = \sqrt[12]{2^n}$$

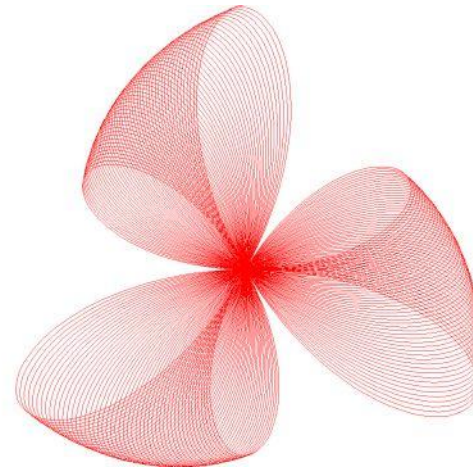
# Musical harmony

Name	Exact value in 12-TET	Decimal value in 12-TET	Cents	Just intonation interval
Unison (C)	$2^{0/12} = 1$	1.000000	0	$\frac{1}{1} = 1.000000$
Minor second (C#/D $\flat$ )	$2^{1/12} = \sqrt[12]{2}$	1.059463	100	$\frac{16}{15} = 1.066667$
Major second (D)	$2^{2/12} = \sqrt[6]{2}$	1.122462	200	$\frac{9}{8} = 1.125000$
Minor third (D#/E $\flat$ )	$2^{3/12} = \sqrt[4]{2}$	1.189207	300	$\frac{6}{5} = 1.200000$
Major third (E)	$2^{4/12} = \sqrt[3]{2}$	1.259921	400	$\frac{5}{4} = 1.250000$
Perfect fourth (F)	$2^{5/12} = \sqrt[12]{32}$	1.334840	500	$\frac{4}{3} = 1.333333$
Augmented fourth (F#/G $\flat$ )	$2^{6/12} = \sqrt{2}$	1.414214	600	$\frac{7}{5} = 1.400000$
Perfect fifth (G)	$2^{7/12} = \sqrt[12]{128}$	1.498307	700	$\frac{3}{2} = 1.500000$
Minor sixth (G#/A $\flat$ )	$2^{8/12} = \sqrt[3]{4}$	1.587401	800	$\frac{8}{5} = 1.600000$
Major sixth (A)	$2^{9/12} = \sqrt[4]{8}$	1.681793	900	$\frac{5}{3} = 1.666667$
Minor seventh (A#/B $\flat$ )	$2^{10/12} = \sqrt[6]{32}$	1.781797	1000	$\frac{7}{4} = 1.750000$
Major seventh (B)	$2^{11/12} = \sqrt[12]{2048}$	1.887749	1100	$\frac{15}{8} = 1.875000$
Octave (C)	$2^{12/12} = 2$	2.000000	1200	$\frac{2}{1} = 2.000000$

# Represent musical harmonies visually with the harmonograph!

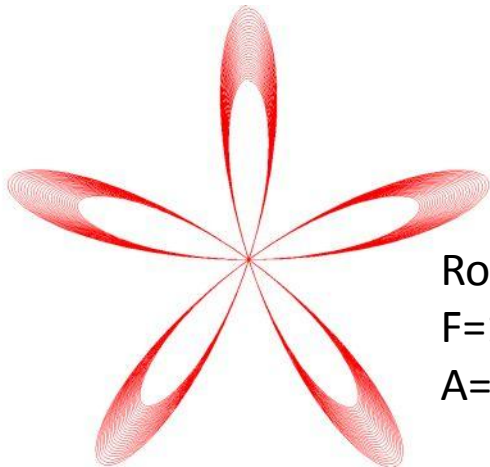


Rotary  
 $F=2, D=0.7,$   
 $A=1, \text{phi}=0$

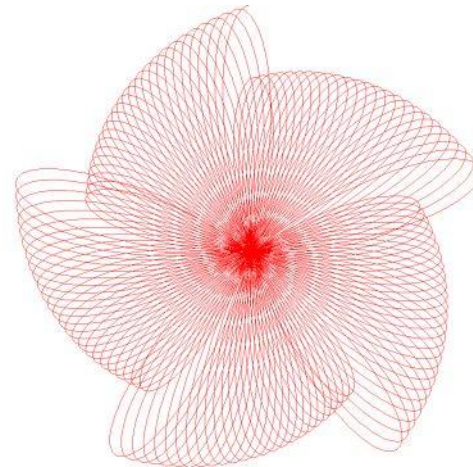


Rotary  
 $F=2.01, D=0.7,$   
 $A=1, \text{phi}=0$

*Note the  
difference a  
small change  
in  $F$  makes....*



Rotary  
 $F=1.5, D=0.7,$   
 $A=1, \text{phi}=0$



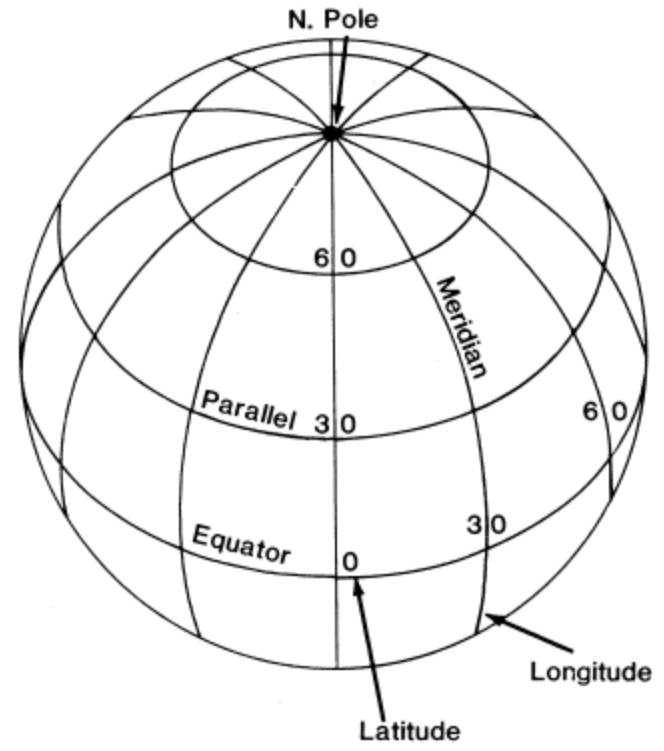
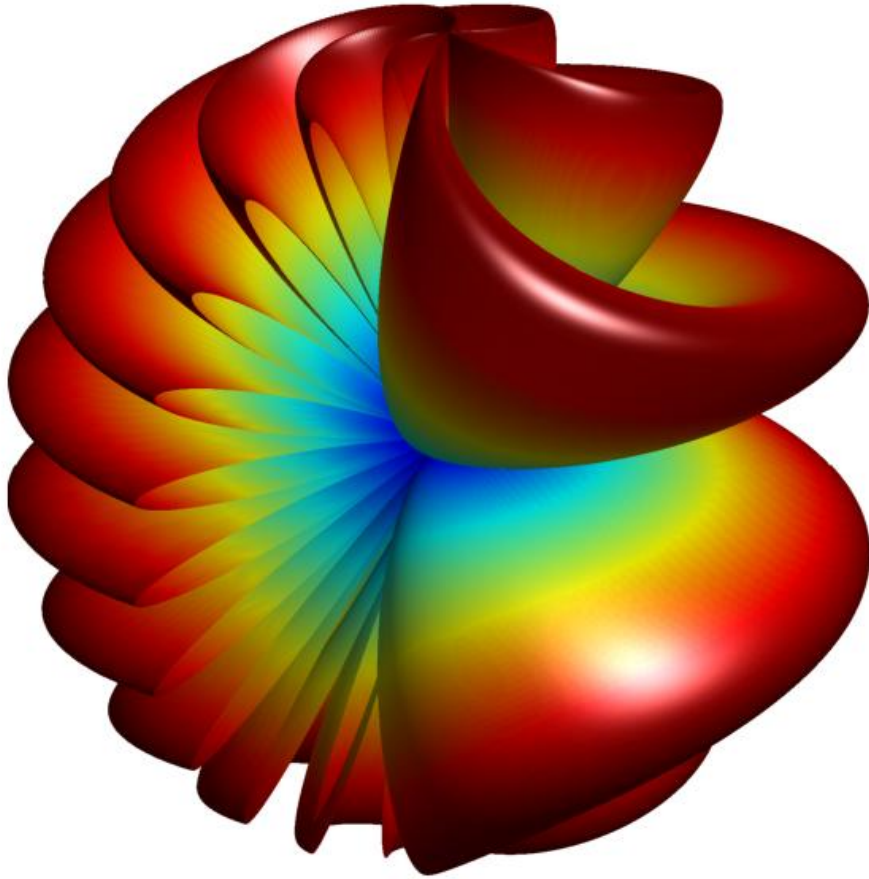
Rotary  
 $F=1.51, D=0.7,$   
 $A=1, \text{phi}=0$

# Summary

- Introduction ✓
  - Computers and mathematics - a harmonious relationship!
- **Software demonstrations**
  - **Harmonograph** ✓
  - **Spherium**
  - **Julia**
- Rene Descartes' theory of the rainbow



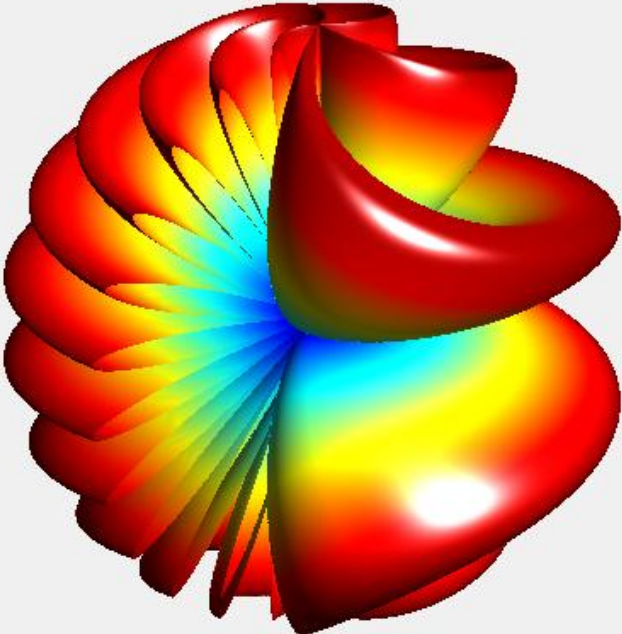
$$R = f(\text{azi}, \text{elev})$$



# Spherium

$R$  = radius  
azi = longitude  
elev = latitude

spherium



Ammonite options

Plot spiral?  Add ridges

Add bumps  Add ridges to colour

Add bumps to colour

Spiral type  
Logarithmic

# spiral turns: 5      # surface points per turn: 200

Ridge frequency: 10      Cross section ratio: 0.9

Bump amplitude: 0.01      Spiral bump amplitude: 0.5

Spiral bump frequency: 10

Colouring function: None

Default   Save settings   Load settings

azimuth /deg: 136.0001

23 elevation/deg   Save .PNG

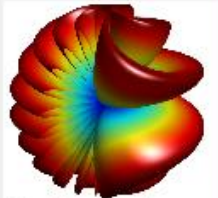
Output size: A4 landscape   Output DPI: 300   Sphere or surface?: Surface

Sphere function  $f(\text{azi}, \text{elev})$     **$\text{abs}(\cos(\text{elev} * \text{azi}) * \sin(\text{azi} * \text{elev}))$**

# surface points: 500   Renderer: OpenGL

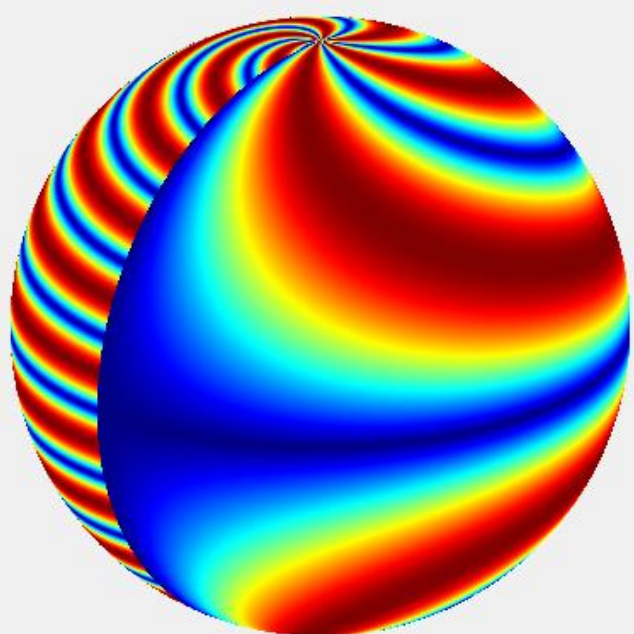
Add axis    Add colorbar

Written by Andy French. March 2011



Spherium

spherium



Ammonite options

Plot spiral?  Add ridges

Add bumps  Add ridges to colour

Add bumps to colour

Spiral type  
Logarithmic

# spiral turns: 5      # surface points per turn: 200

Ridge frequency: 10      Cross section ratio: 0.9

Bump amplitude: 0.01      Spiral bump amplitude: 0.5

Spiral bump frequency: 10

Colouring function: None

Default   Save settings   Load settings

azimuth /deg: 136.0001      elevation/deg: 23

Output size: A4 landscape      Output DPI: 300      Sphere or surface?: Sphere

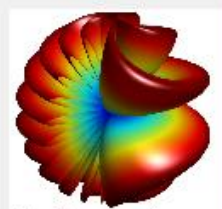
# surface points: 500      Renderer: OpenGL

Add axis    Add colorbar

Sphere function f( azi,elev )

$$\text{abs}( \cos(\text{elev} * \text{azi}) * \sin(\text{azi} * \text{elev}) )$$

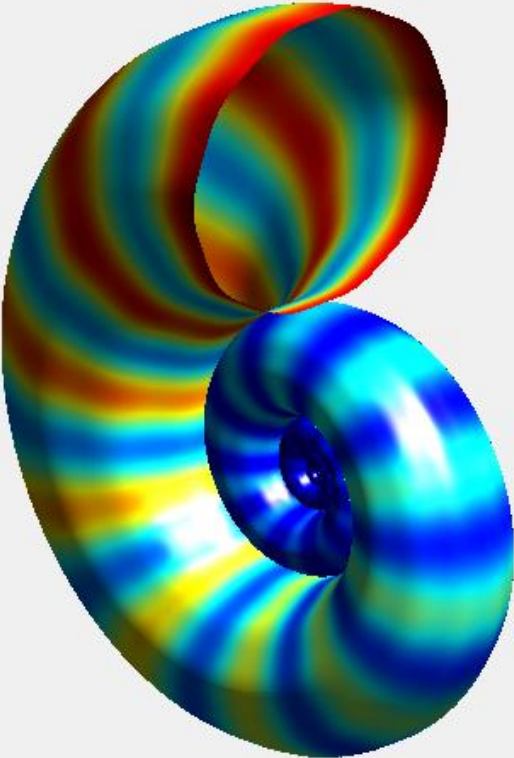
Save .PNG



Written by  
Andy French. March 2011

Spherium

spherium



Ammonite options

Plot spiral?  Add ridges

Add bumps  Add ridges to colour

Add bumps to colour

Spiral type  
Logarithmic

# spiral turns: 5      # surface points per turn: 200

Ridge frequency: 10      Cross section ratio: 0.9

Bump amplitude: 0.01      Spiral bump amplitude: 0.5

Spiral bump frequency: 10

Colouring function: None

Default   Save settings   Load settings

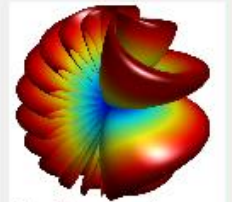
azimuth /deg: 136.0001      elevation/deg: 23

Output size: A4 landscape      Output DPI: 300      Sphere or surface?: Ammonite

# surface points: 500      Renderer: OpenGL

Sphere function f( azi,elev )  
**abs( cos(elev\*azi)\*sin(azi\*elev) )**

Save .PNG

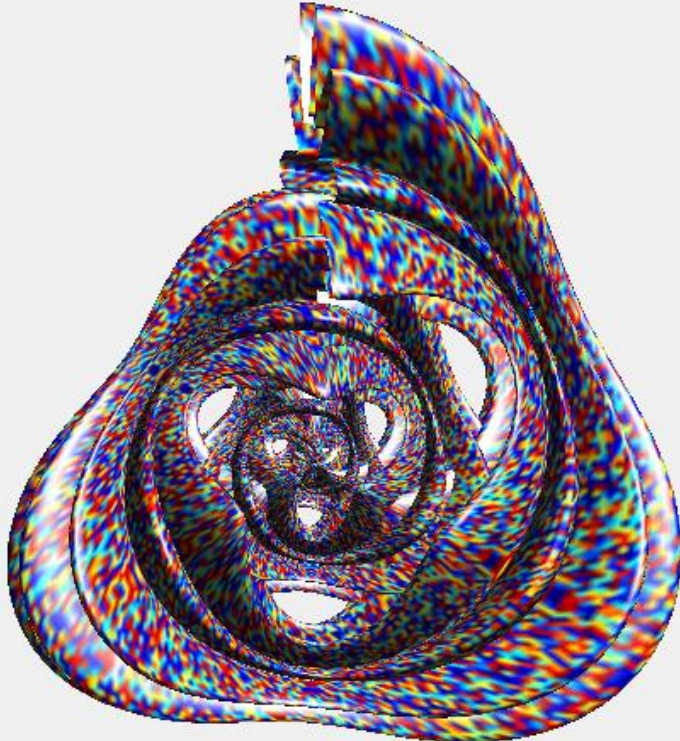


Written by  
Andy French. March 2011

**Spherium**



spherium



Ammonite options

Plot spiral?  Add ridges

Add bumps  Add ridges to colour

Add bumps to colour

Spiral type  
Archimedian

# spiral turns: 3      # surface points per turn: 300

Ridge frequency: 10      Cross section ratio: 0.9

Bump amplitude: 0.5      Spiral bump amplitude: 0.5

Spiral bump frequency: 3

Colouring function: Sine

Default   Save settings   Load settings

azimuth /deg: -98.487      elevation/deg: 2.5375

Save .PNG

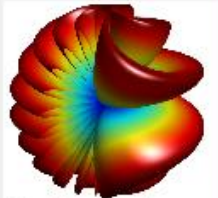
Output size: A4 landscape      Output DPI: 300      Sphere or surface?: Ammonite

Sphere function  $f(\text{azi}, \text{elev})$        $\text{abs}(\cos(\text{elev} * \text{azi}) * \sin(\text{azi} * \text{elev}))$

# surface points: 500      Renderer: OpenGL

Add axis    Add colorbar

Written by Andy French. March 2011



Spherium

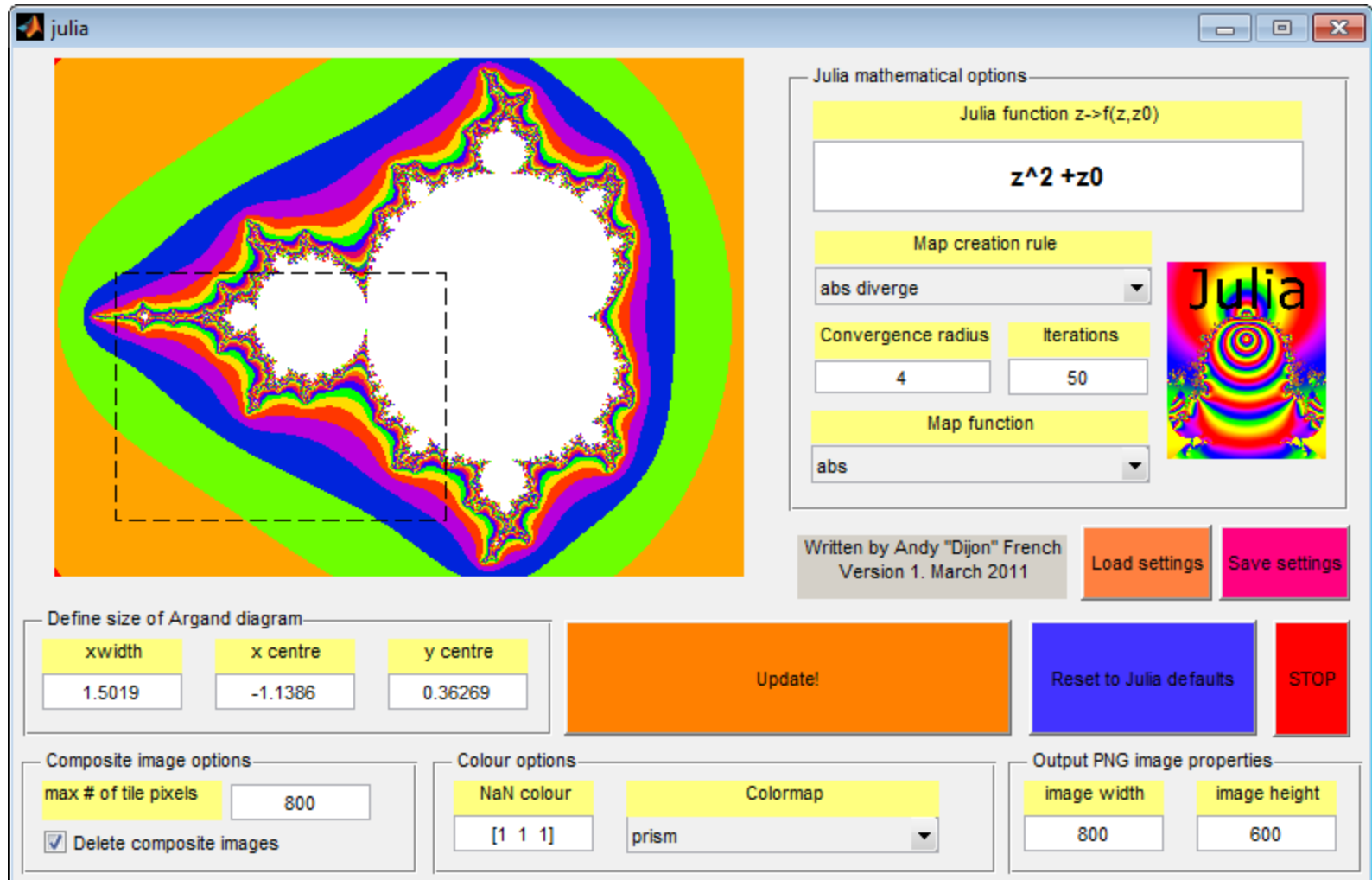
# Summary

- Introduction ✓
  - Computers and mathematics - a harmonious relationship!
- **Software demonstrations**
  - Harmonograph ✓
  - Spherium ✓
  - Julia
- Rene Descartes' theory of the rainbow





# Julia default: The Mandlebrot set



The screenshot displays the Julia software interface. The main window shows a large, colorful fractal image of the Mandelbrot set. A dashed black rectangle is overlaid on the fractal, indicating a zoomed-in view. To the right of the main image is a control panel titled "Julia mathematical options". This panel includes a text box for the Julia function  $z \rightarrow f(z, z_0)$  containing  $z^2 + z_0$ . Below this is a "Map creation rule" dropdown menu set to "abs diverge". There are two input fields: "Convergence radius" with the value 4, and "Iterations" with the value 50. A "Map function" dropdown menu is set to "abs". To the right of these controls is a small thumbnail image of the Mandelbrot set with the word "Julia" written over it. Below the main image and controls are several other panels: "Define size of Argand diagram" with fields for xwidth (1.5019), x centre (-1.1386), and y centre (0.36269); "Composite image options" with a field for max # of tile pixels (800) and a checked checkbox for "Delete composite images"; "Colour options" with a field for NaN colour ([1 1 1]) and a dropdown for Colormap (prism); and "Output PNG image properties" with fields for image width (800) and image height (600). There are also several buttons: "Update!" (orange), "Reset to Julia defaults" (blue), "STOP" (red), "Load settings" (orange), and "Save settings" (pink). A small text box at the bottom right of the main panel reads "Written by Andy 'Dijon' French Version 1. March 2011".

Julia mathematical options

Julia function  $z \rightarrow f(z, z_0)$

$z^2 + z_0$

Map creation rule

abs diverge

Convergence radius: 4

Iterations: 50

Map function

abs

Define size of Argand diagram

xwidth: 1.5019

x centre: -1.1386

y centre: 0.36269

Update!

Reset to Julia defaults

STOP

Composite image options

max # of tile pixels: 800

Delete composite images

Colour options

NaN colour: [1 1 1]

Colormap: prism

Output PNG image properties

image width: 800

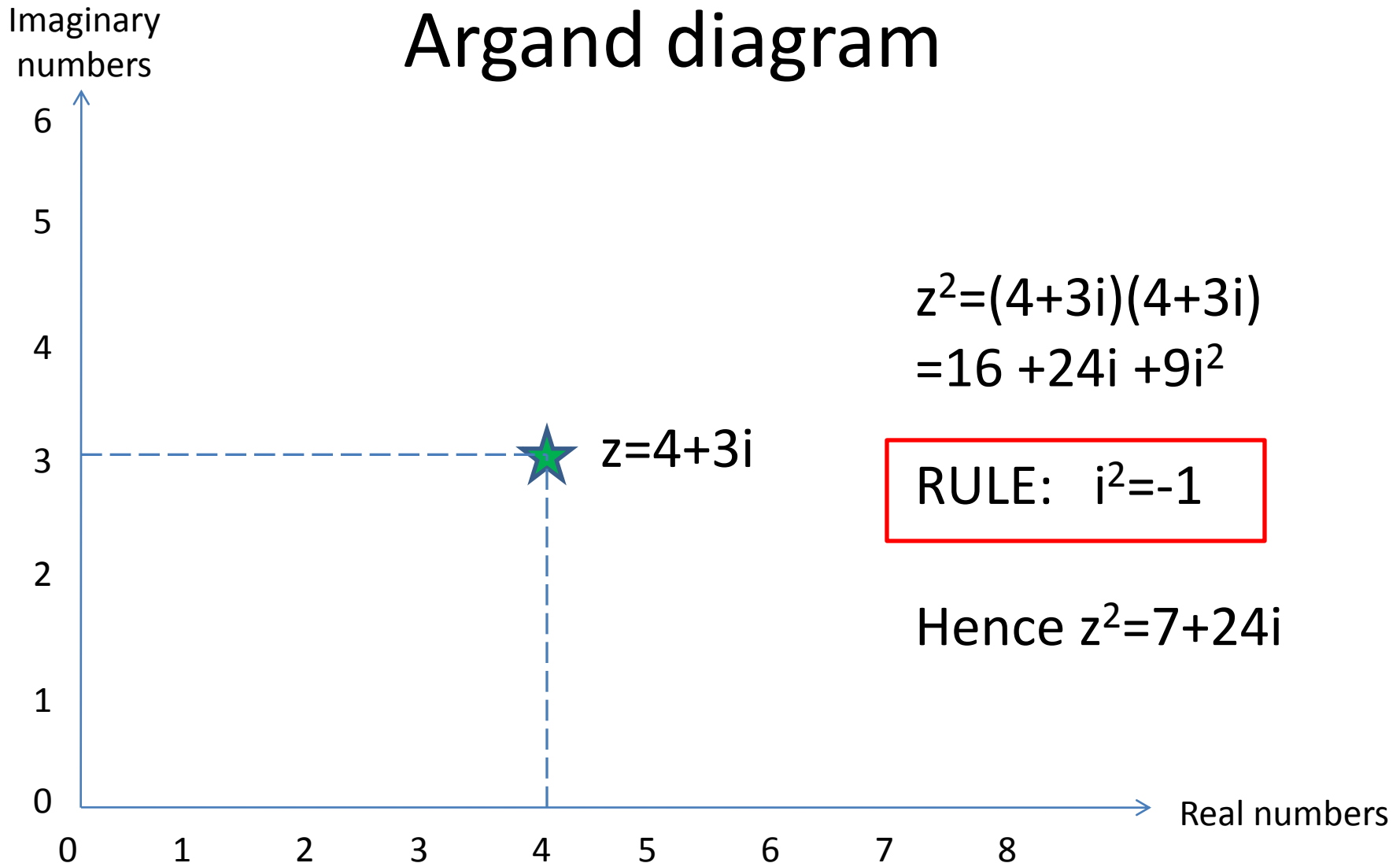
image height: 600

Written by Andy "Dijon" French  
Version 1. March 2011

Load settings

Save settings

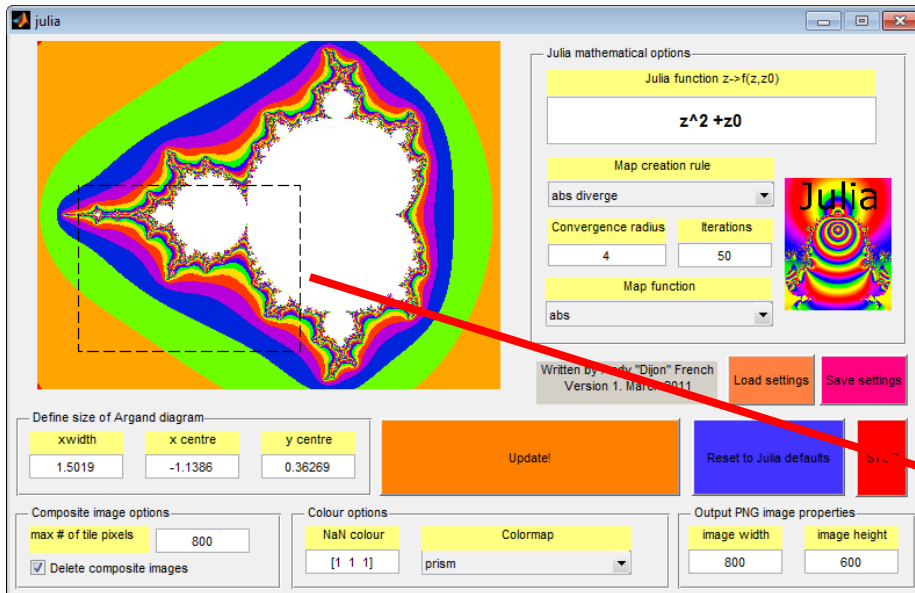
# Complex numbers and the Argand diagram



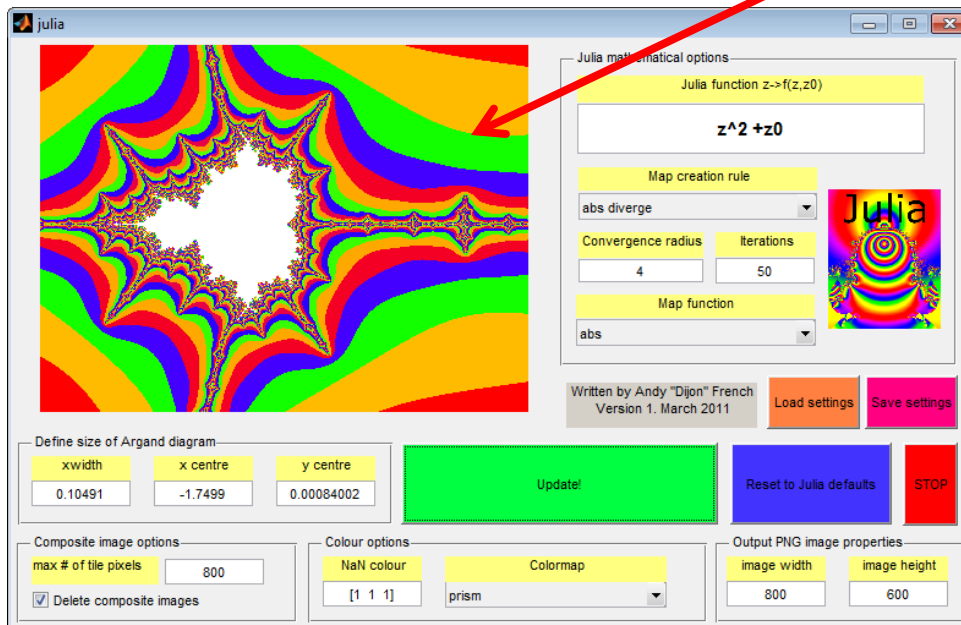
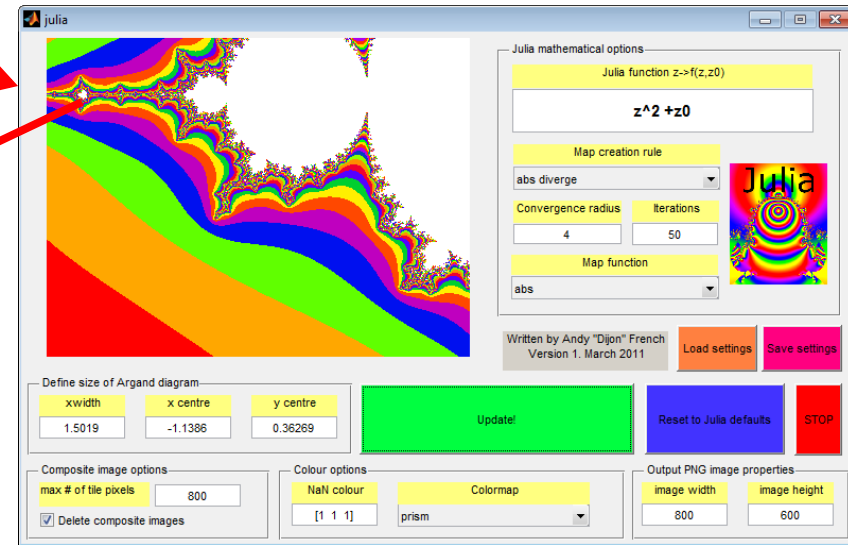
$$z^2 = (4+3i)(4+3i)$$
$$= 16 + 24i + 9i^2$$

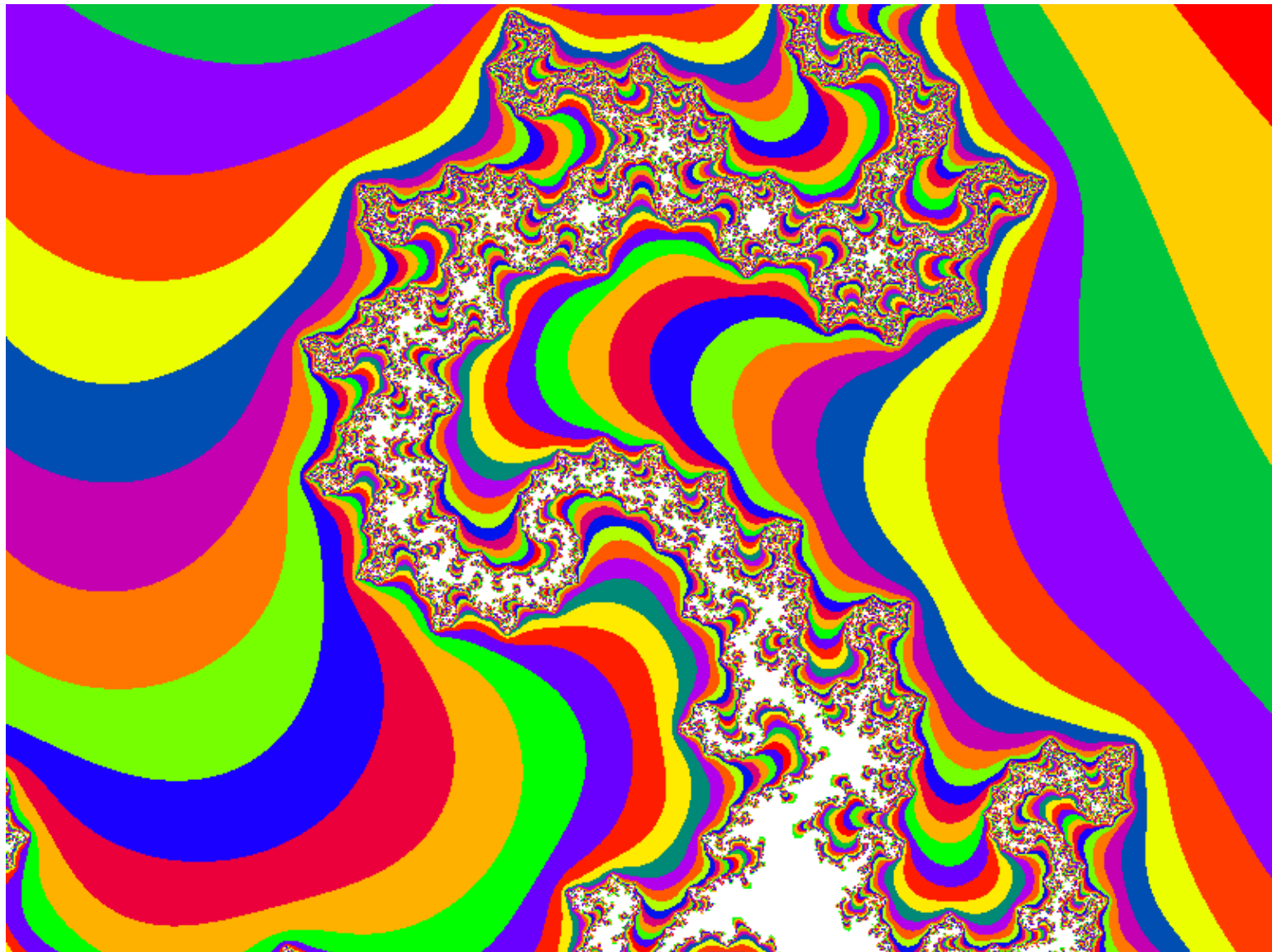
**RULE:  $i^2 = -1$**

$$\text{Hence } z^2 = 7 + 24i$$



Zooming in. The Mandelbrot set has *infinite complexity*

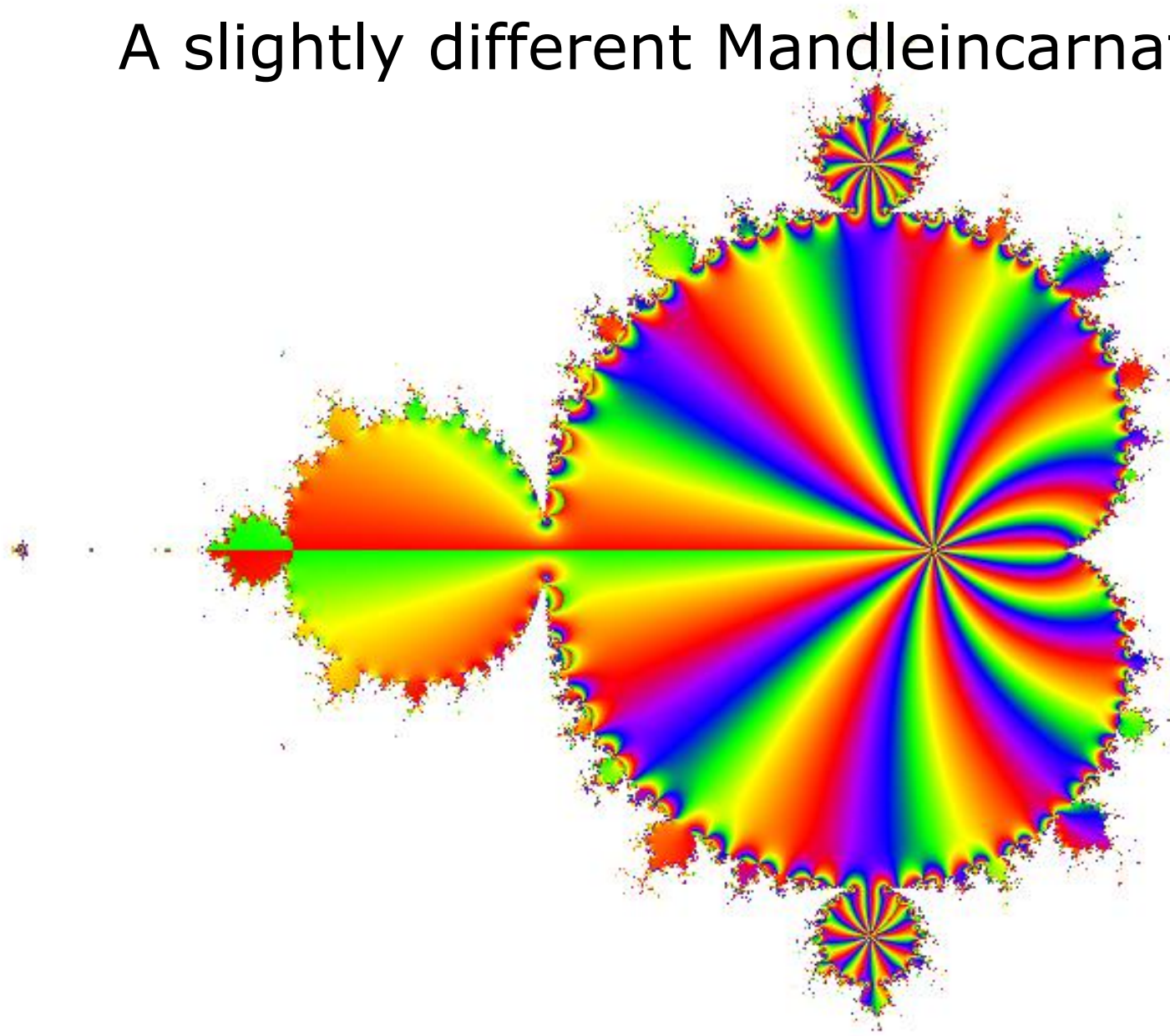




The Mandelbrot slurping complexity from the Argand plane!



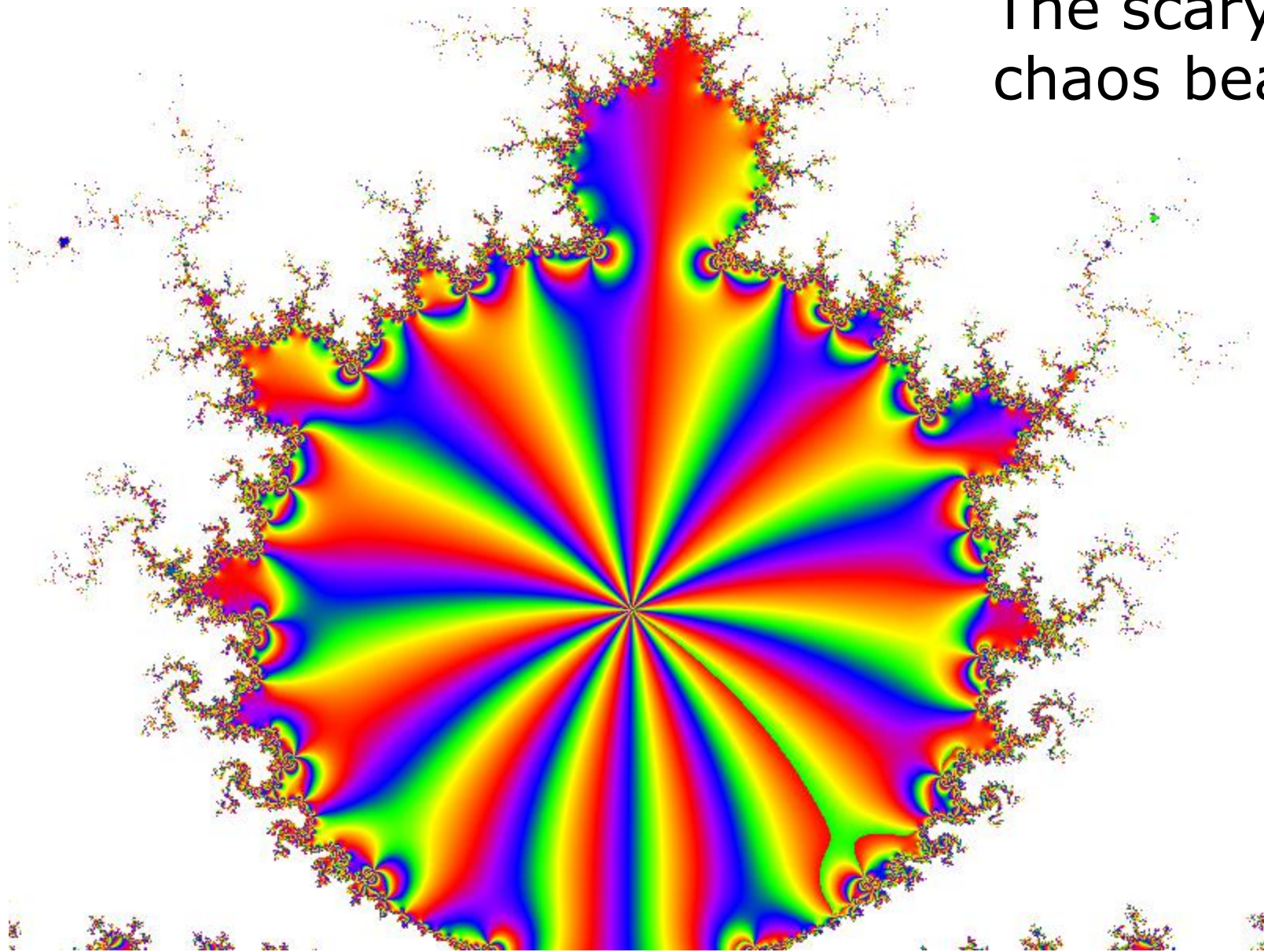
# A slightly different Mandleincarnation!

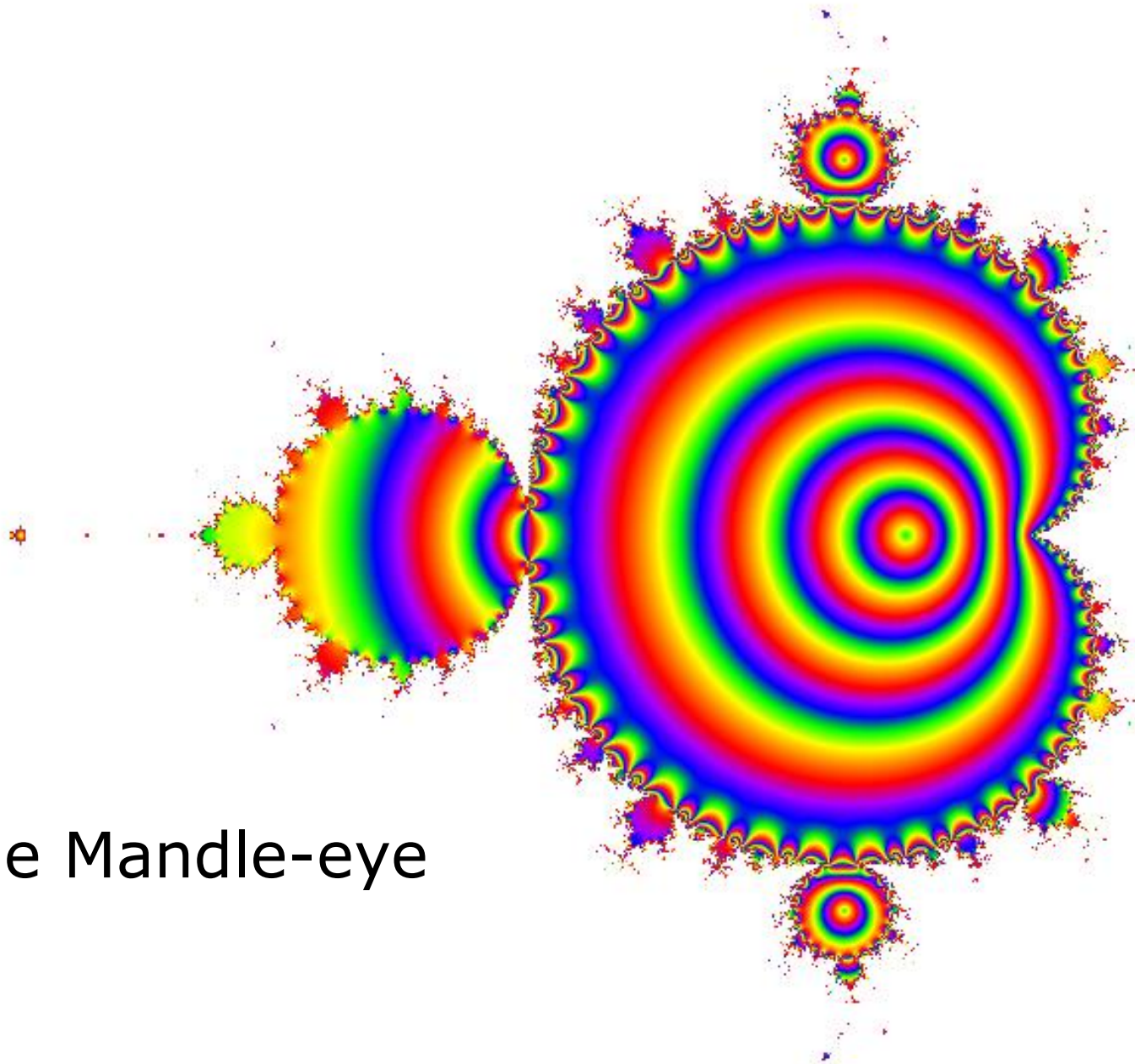


In this case plot  $\arg(z)$  following 25 iterations of  $z \rightarrow z^2 + z_0$



The scary  
chaos beast!

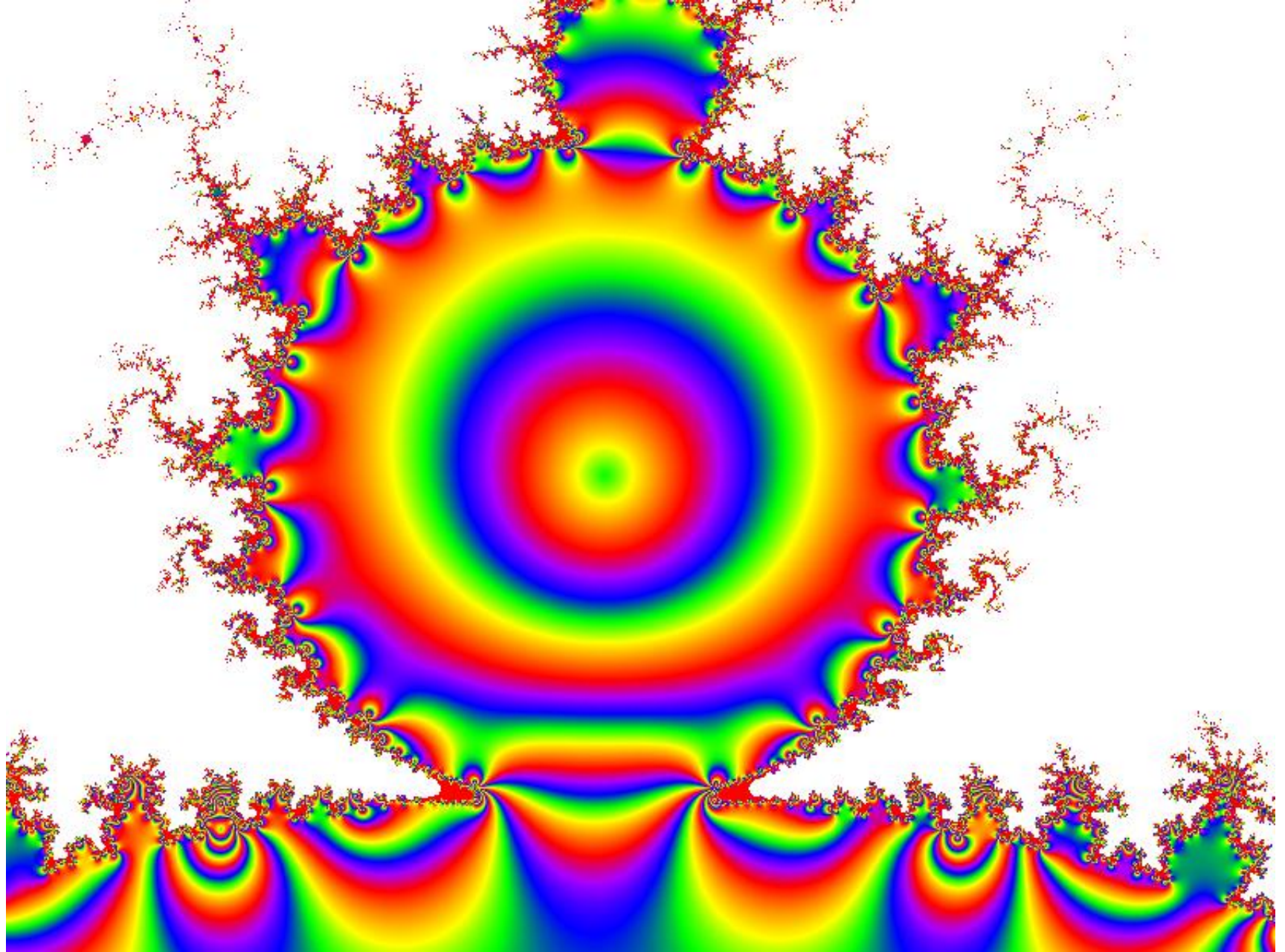




The Mandle-eye

In this case plot  $\exp(-|z|)$  following 25 iterations of  $z \rightarrow z^2 + z_0$





'Homering' in on the Cyclops (!)



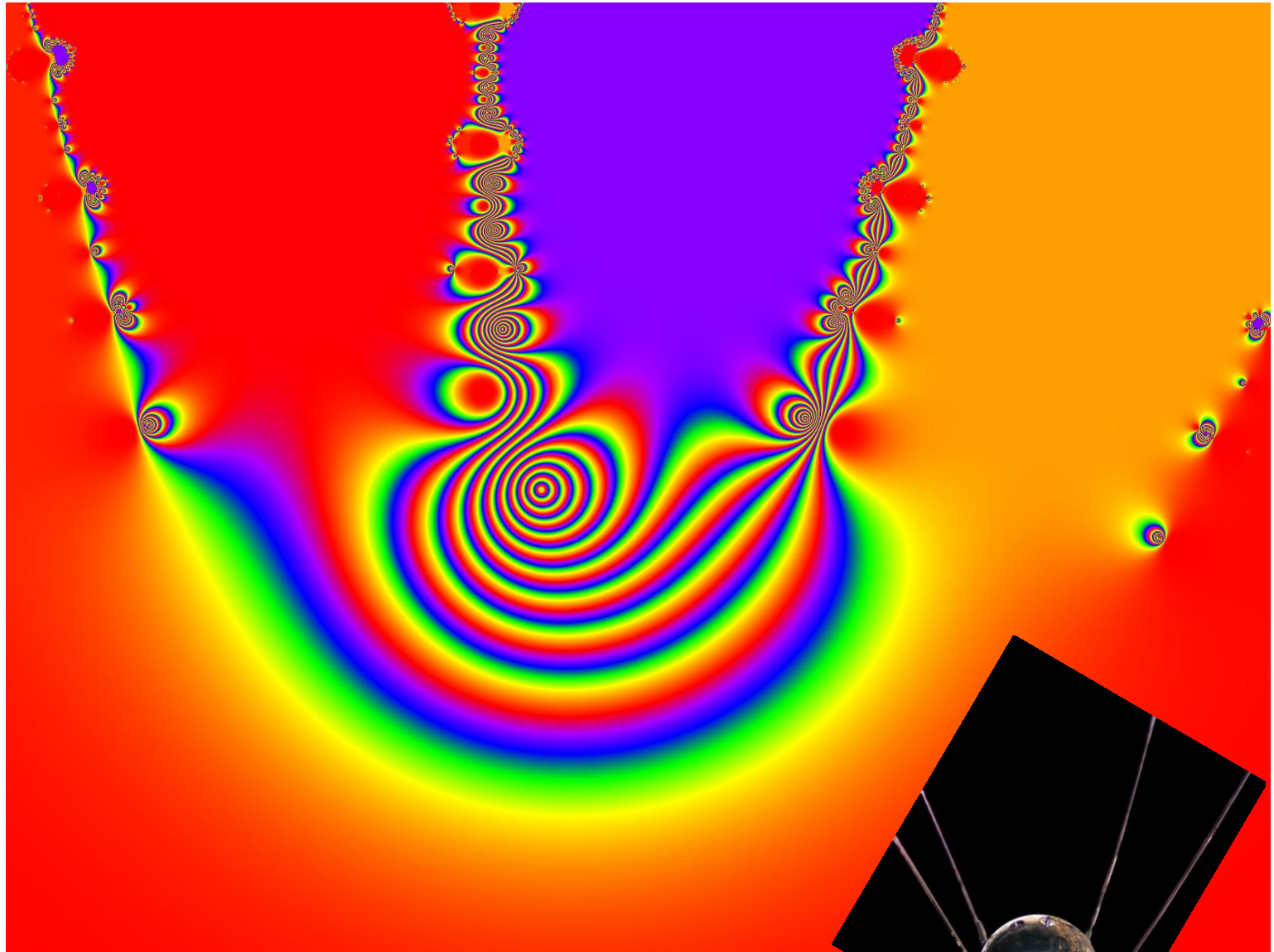


The

Mandlebrot

Variations





Sputnik interference

Julia mathematical options

Julia function  $z \rightarrow f(z, z_0)$

$$z - \frac{\sin(z) - \cos(z)}{\sin(z) + \cos(z)}$$

Map creation rule

plot z

Convergence radius: 4      Iterations: 30

Map function

exp\_decay

Written by Andy "Dijon" French  
Version 1. March 2011

Load settings      Save settings

Define size of Argand diagram

xwidth: 0.013789      x centre: -0.78372      y centre: -0.034464

Update!

Reset to Julia defaults      STOP

Composite image options

max # of tile pixels: 800

Delete composite images

Colour options

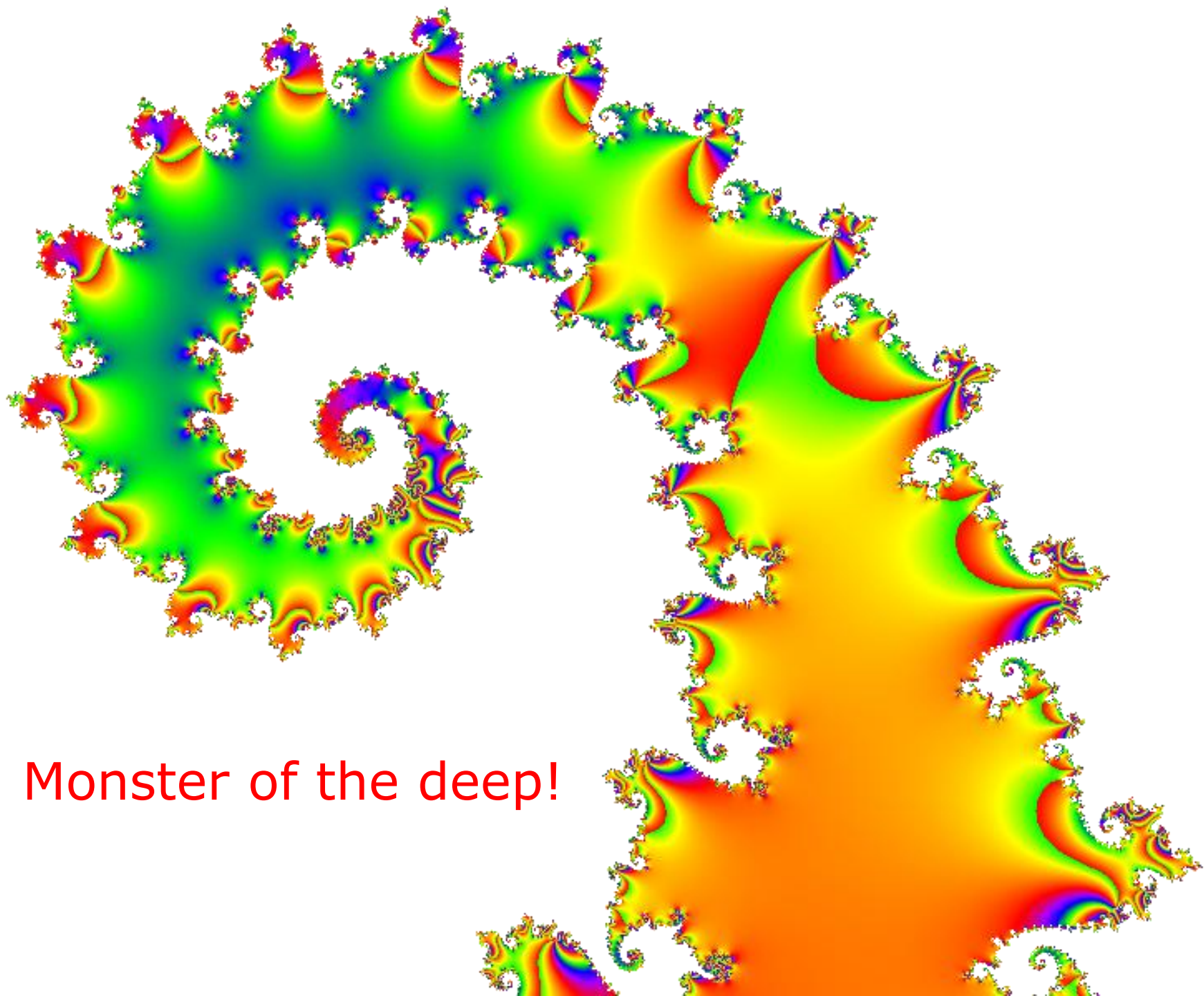
NaN colour: [1 1 1]      Colormap: prism

Output PNG image properties

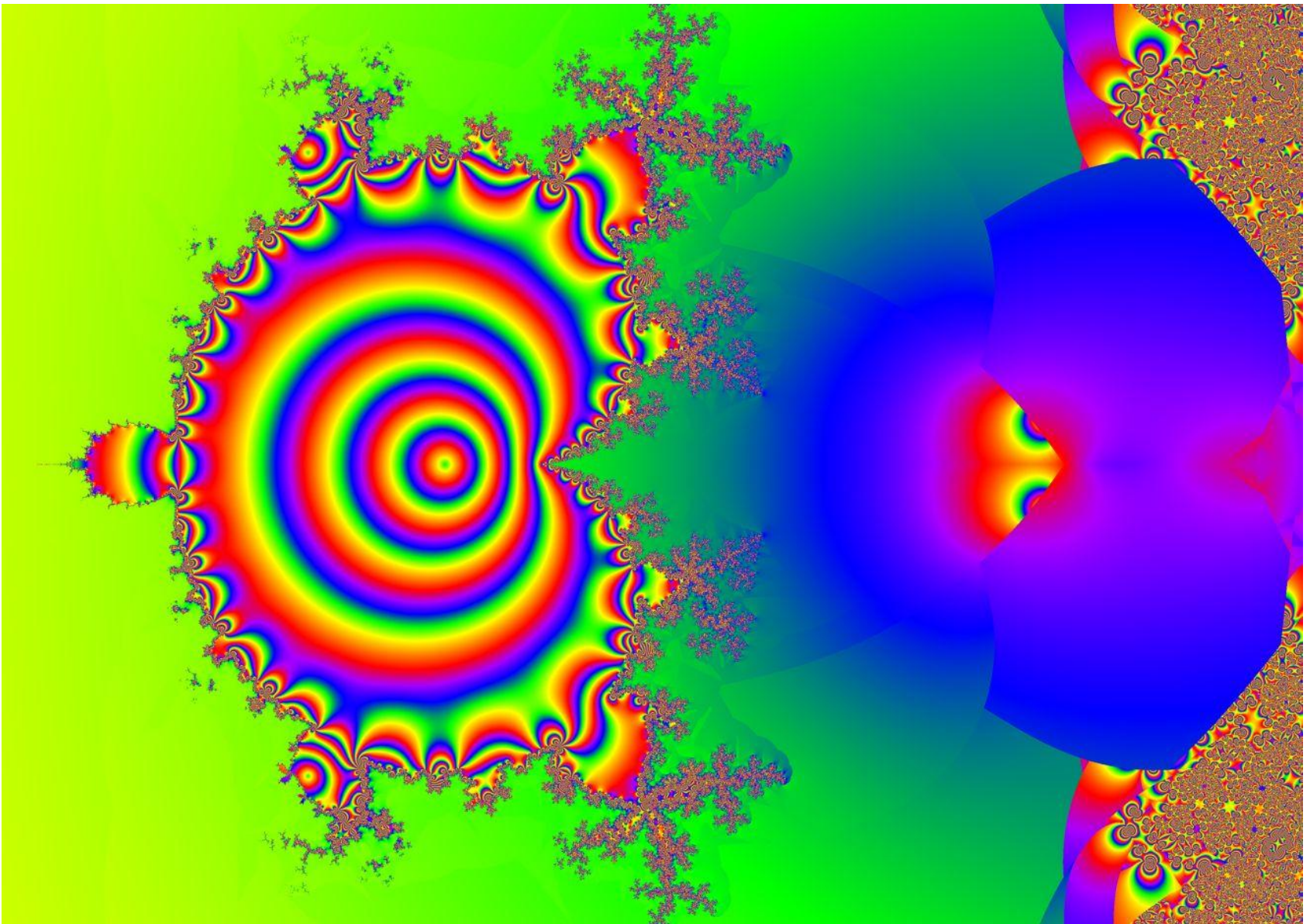
image width: 800      image height: 600

$$z \rightarrow z - \frac{\sin z - \cos z}{\sin z + \cos z}$$





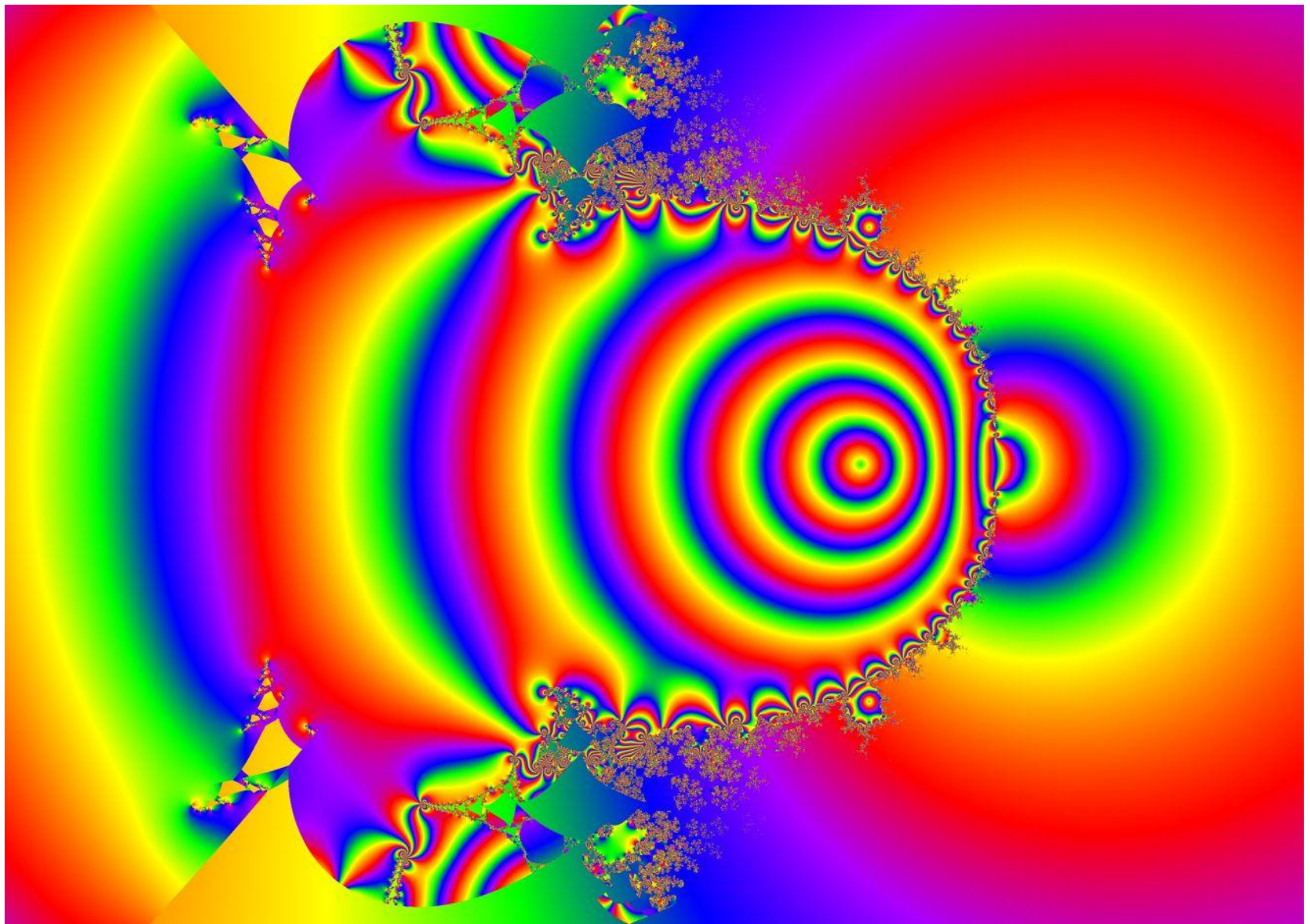
Monster of the deep!



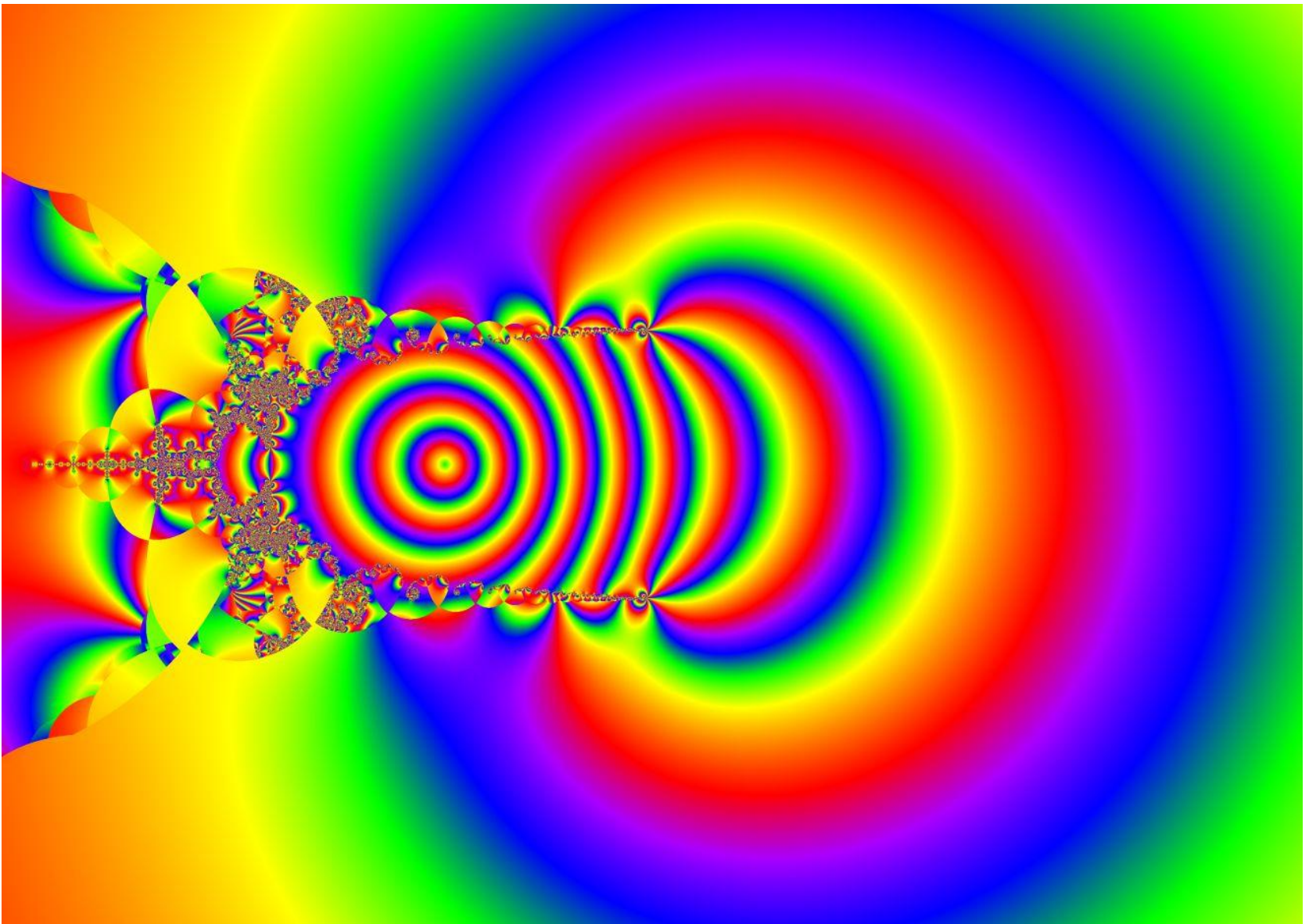
The Mandlerocket!

$$z \rightarrow a \sin(z^2 + z_0)$$





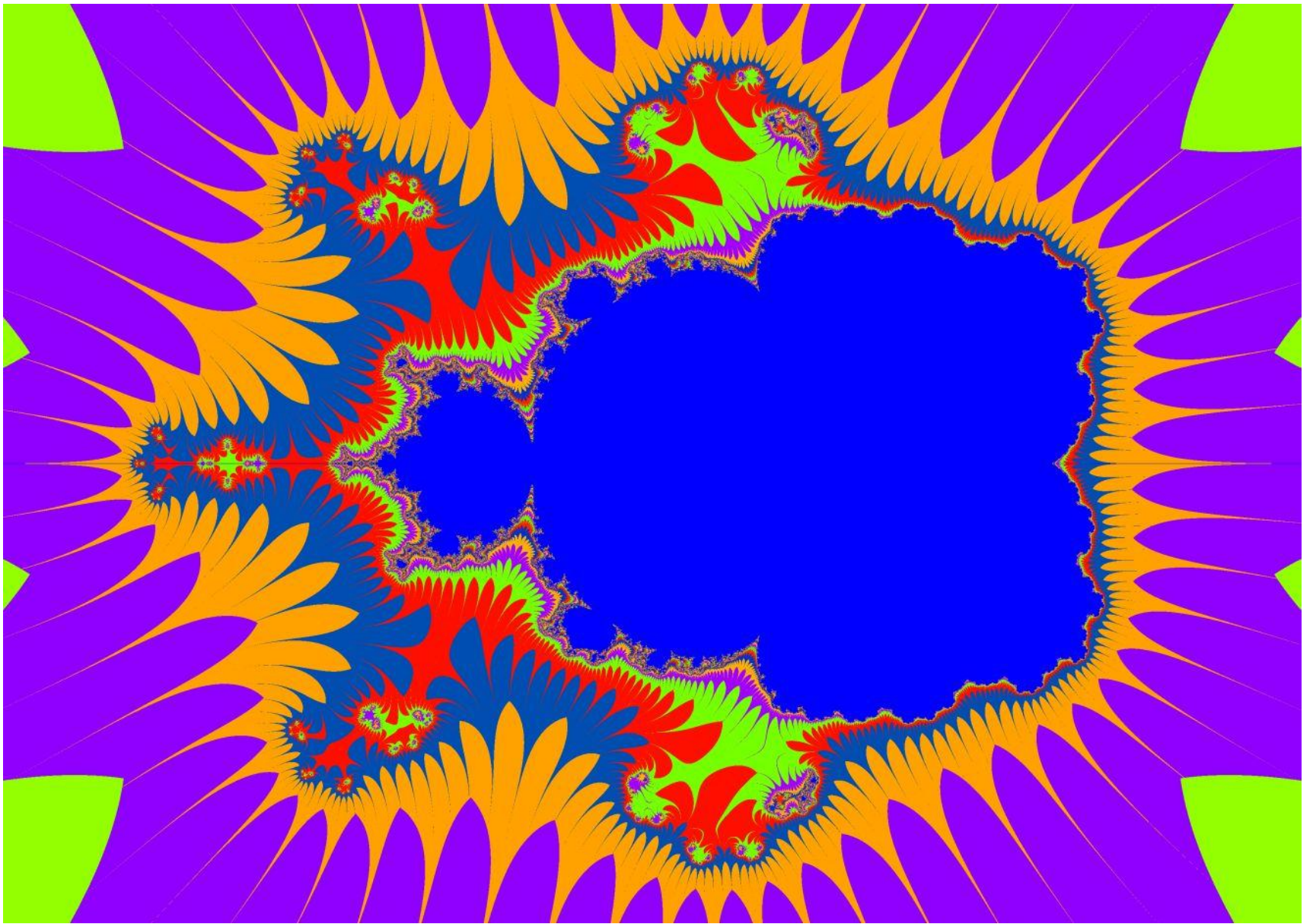
7 steps to enlightenment  $z \rightarrow \text{atan}(z^2 + z_0)$



The light bulb

$$z \rightarrow \log(z^2 + z_0)$$

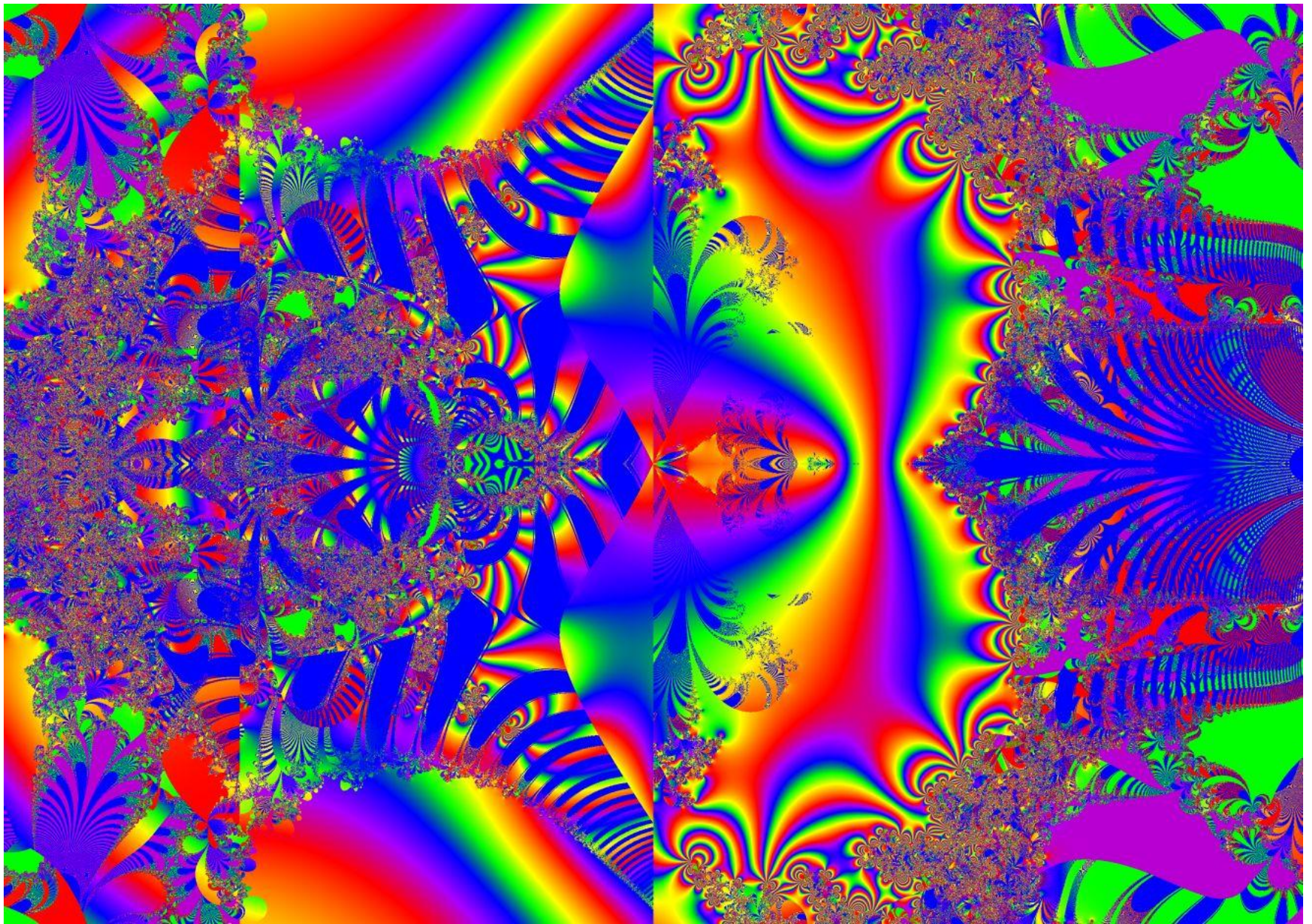




Micro mandlebeast

$$z \rightarrow (z^2 + z_0)^2$$

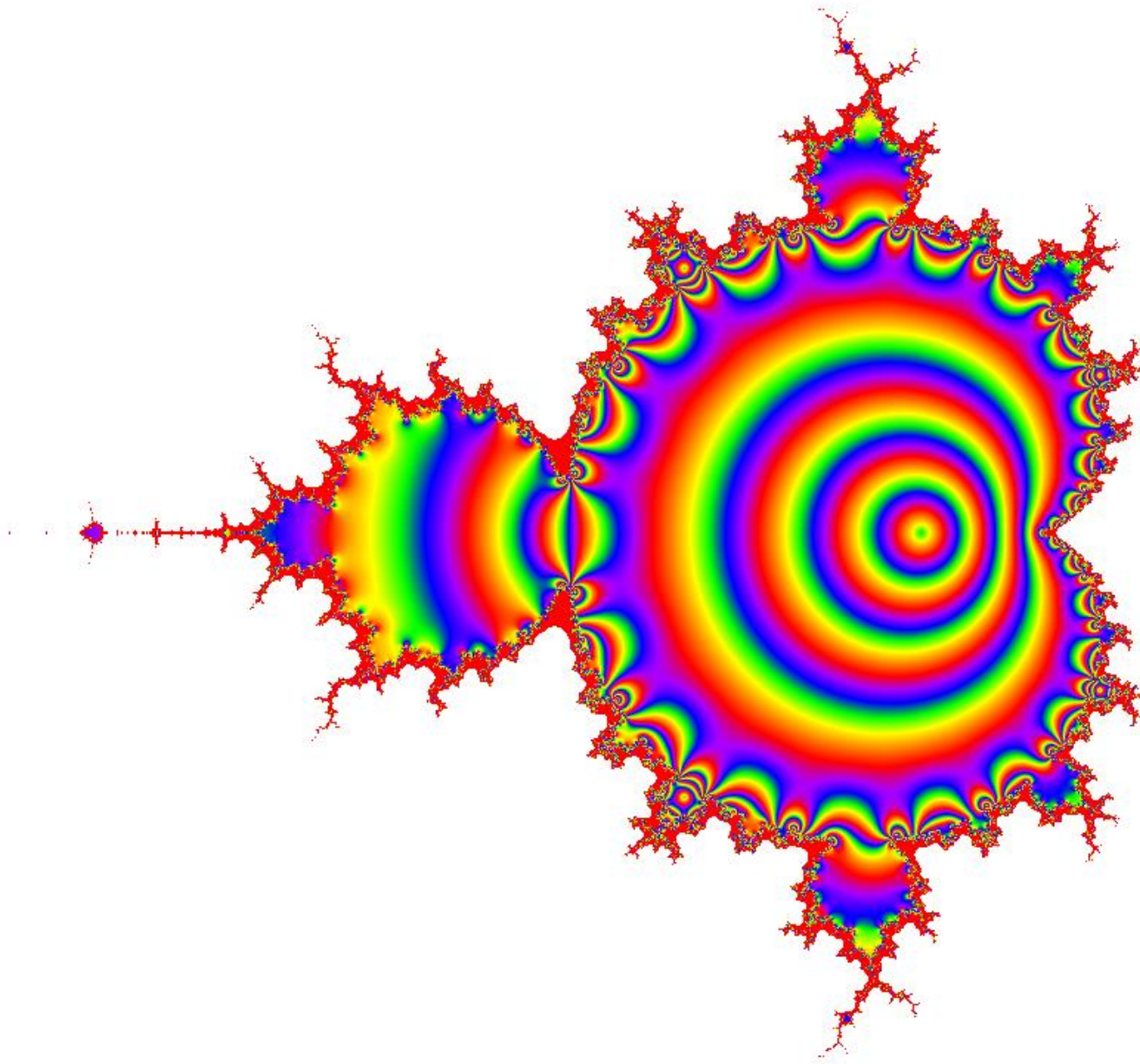




The profusion of power

$$z \rightarrow (z^2 + z_0)^z$$





Day of Julia

# Summary

- Introduction ✓
  - Computers and mathematics - a harmonious relationship!
- Software demonstrations
  - Harmonograph ✓
  - Spherium ✓
  - Julia ✓
- **Rene Descartes' theory of the rainbow**  
(separate presentation)