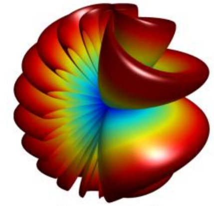
BRITISH COUNCIL

Science in Schools Workshop
Marseilles March 2011.



Spherium

Mathematics and the Machine

The utility and beauty of computer visualization

Dr Andrew French

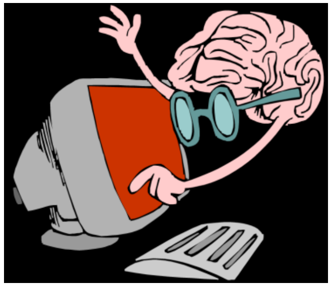
Summary

- Introduction
  - Computers and mathematics - a harmonious relationship!
- Software demonstrations
  - Harmonograph
  - Spherium
  - Julia
- Rene Descartes' theory of the rainbow

Computers and mathematics

- All computer operations are fundamentally the change of a set of binary (0 or 1) switches in *memory* and in a *microprocessor*
- Binary digits e.g. 11110101 are a representation of *numbers*. (In this case 245)
- A 3GHz PC with a 64 bit data bus implies 192 billion switches can potentially be altered every second!
- Despite the appearance of a modern multi-media machine, a computer is essentially a 'number cruncher'

Computers vs humans



- Computers can perform millions of calculations per second
  - They follow mathematical recipes perfectly as directed
  - They don't get bored
- The human brain is massively more powerful than any computer yet built. We can all perform incredible feats of computation (try programming a computer to recognise a face from any angle...)
- ...Yet most of us cannot add up two double digit numbers in less than a few seconds

Humans + computers

+

=

Use a computer to become a superhuman calculator!

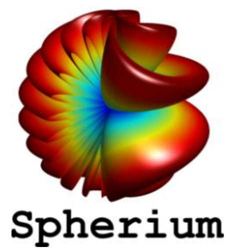
So why do we need all this computational power?


Computers are everywhere!



The utility and beauty of computation

- Computers perform fast arithmetic to enable many modern appliances to work
- In this workshop we will make use of this ‘super-power’ to create **art** from **mathematics**



- We will then look at how maths can be used as a **language to describe the physical world**, and how the use of computers can enable us to relate the maths to what we can readily measure.

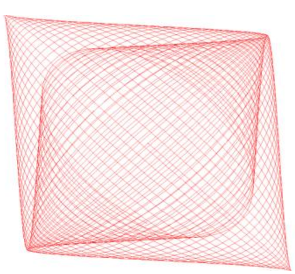
Summary

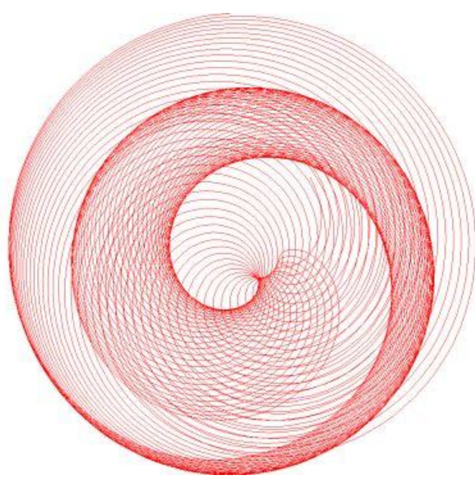
- Introduction ✓
  - Computers and mathematics - a harmonious relationship!
- **Software demonstrations**
  - **Harmonograph**
  - **Spherium**
  - **Julia**
- Rene Descartes' theory of the rainbow

Harmonograph

- The Harmonograph was a Victorian curiosity attributed to Professor Blackburn in 1844

- Use two or three pendulums to create strange and beautiful patterns





Example of a *lateral* harmonograph

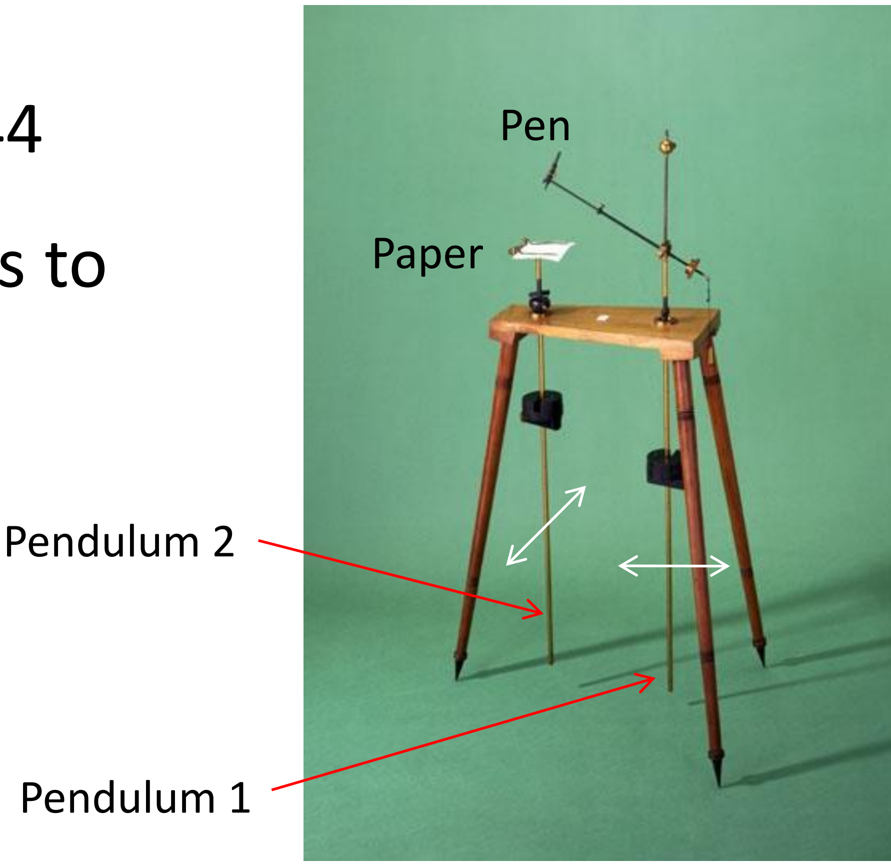
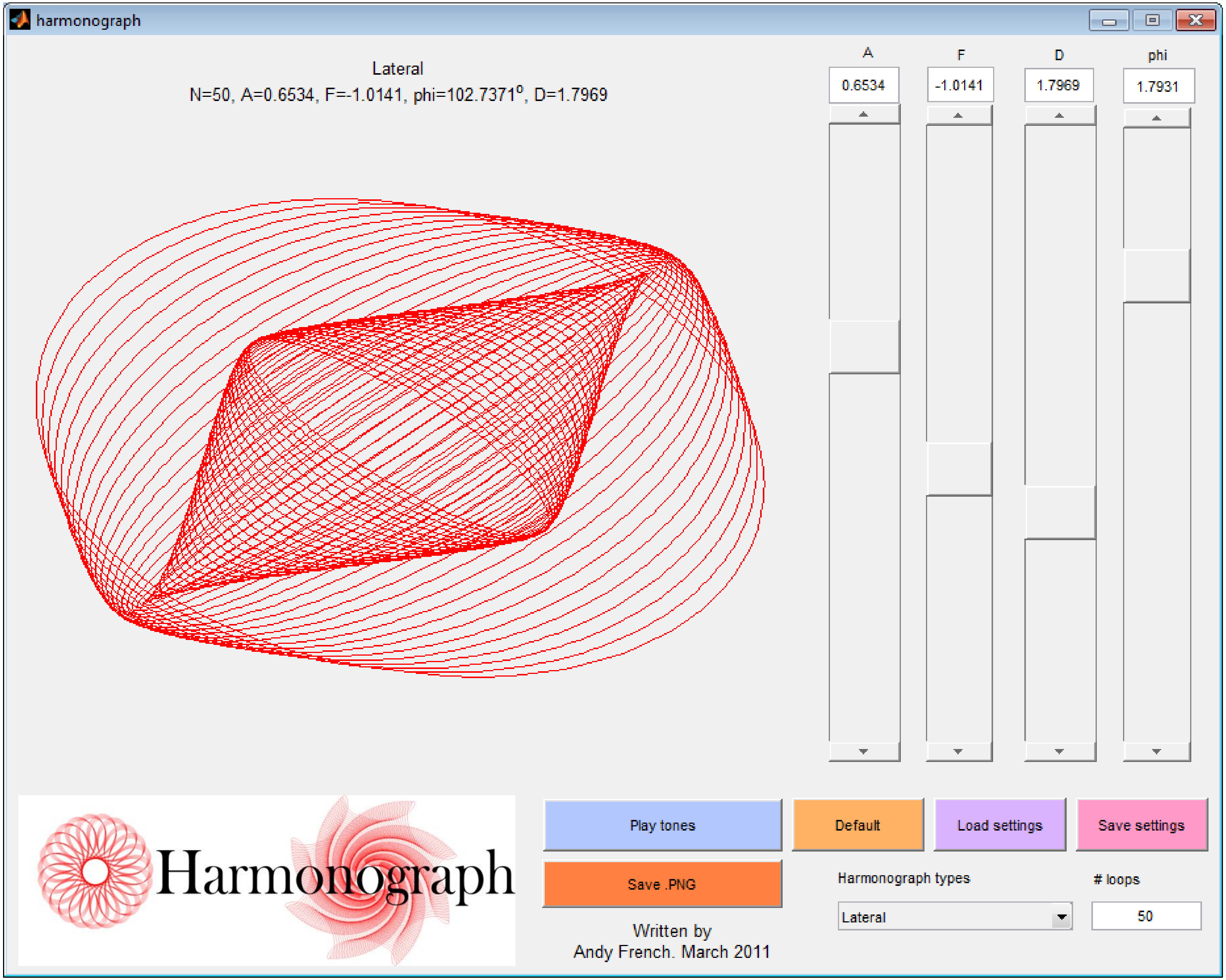
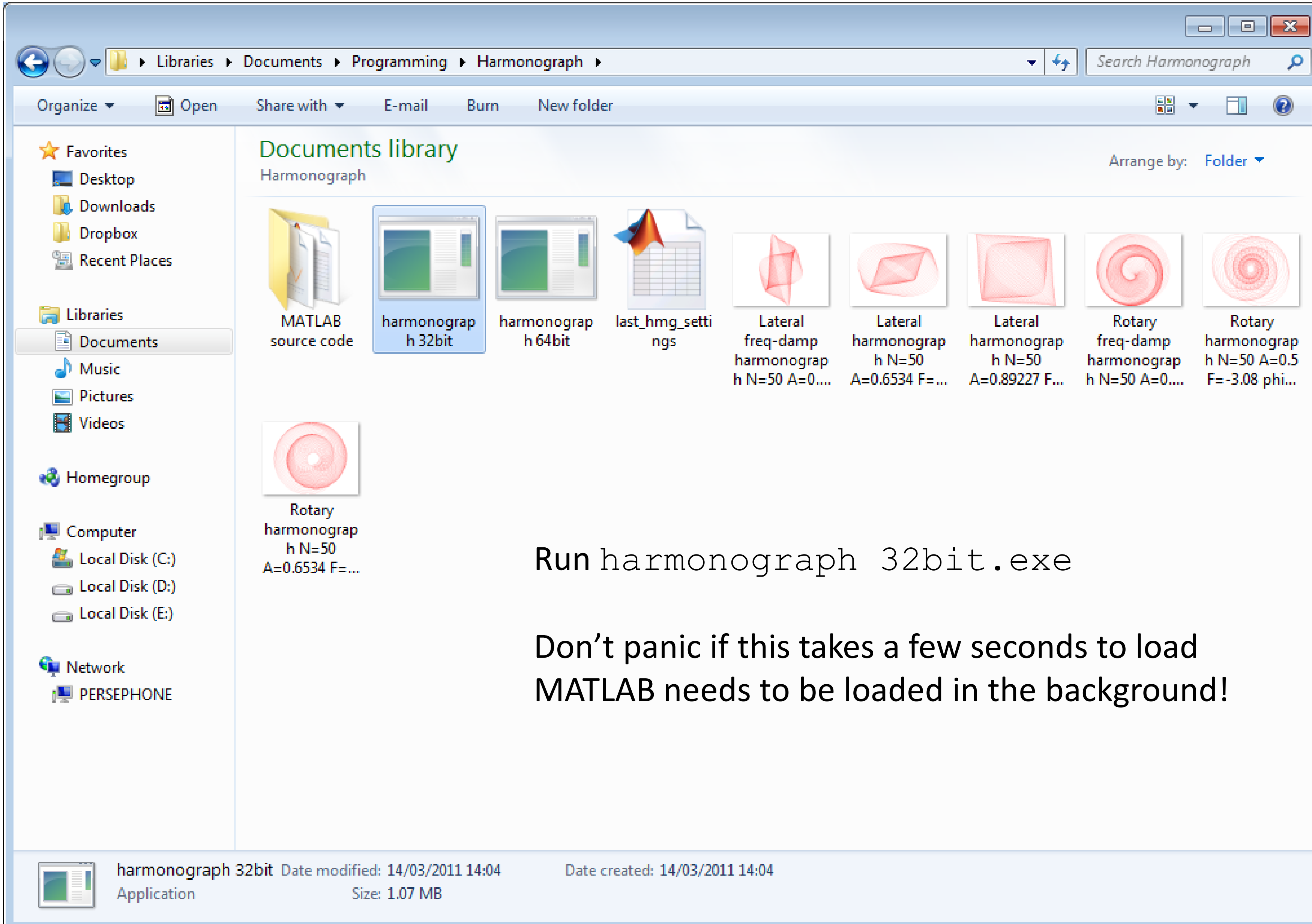


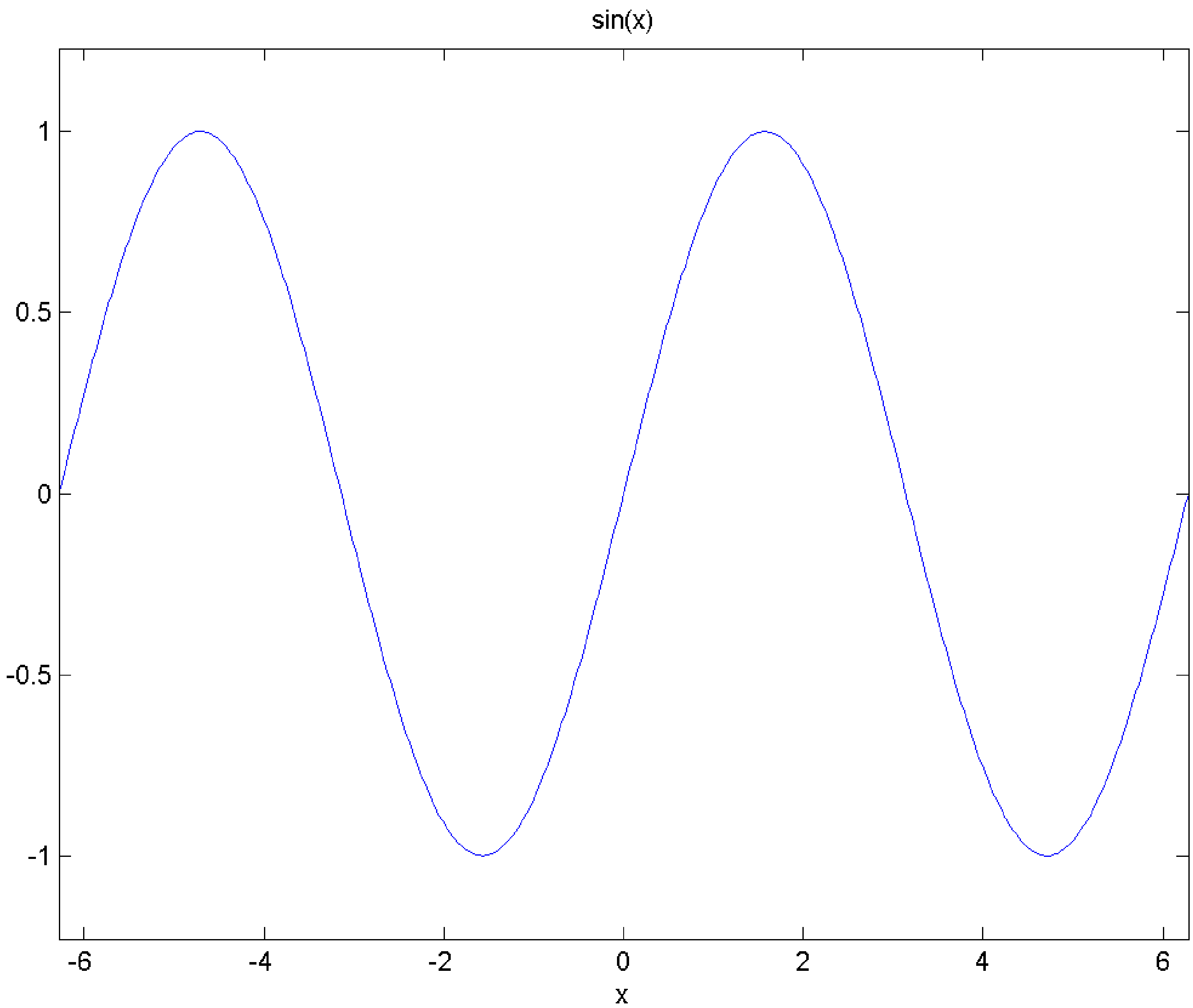
Photo from The Science Museum


harmonograph
Lateral
N=50, A=0.6534, F=-1.0141, phi=102.7371°, D=1.7969
A
0.6534
F
-1.0141
D
1.7969
phi
1.7931
Harmonograph
Play tones
Default
Load settings
Save settings
Save .PNG
Harmonograph types
Lateral
loops
50
Written by
Andy French. March 2011



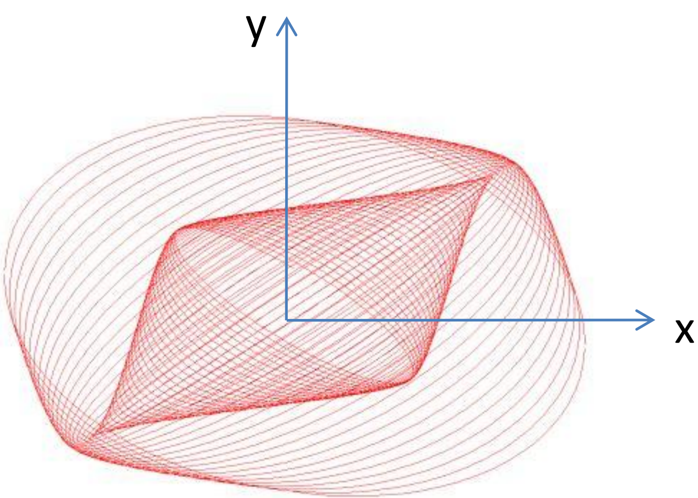
Run `harmonograph 32bit.exe`

Don’t panic if this takes a few seconds to load MATLAB needs to be loaded in the background!


sin(x)
1
0.5
0
-0.5
-1
-6
-4
-2
0
2
4
6
x

$$
\begin{aligned}
x &= A_1 e^{-\frac{t}{T_1}} \sin\left(tW_1 + P_1\right) + A_2 e^{-\frac{t}{T_2}} \sin\left(tW_2 + P_2\right) \\
y &= A_3 e^{-\frac{t}{T_3}} \sin\left(tW_3 + P_3\right) + A_4 e^{-\frac{t}{T_4}} \sin\left(tW_4 + P_4\right)
\end{aligned}
$$

Lateral harmonograph

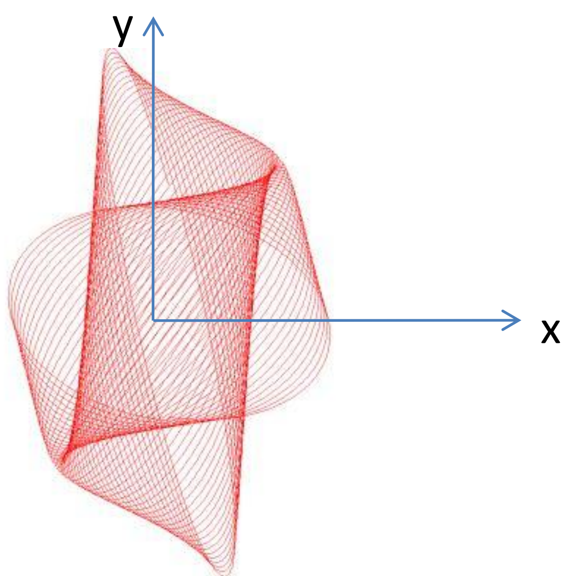


$$
\begin{aligned}
T &= \frac{2\pi}{\omega \log\left(\frac{100}{100-D}\right)} [1, \infty, 1, \infty] \\
A &= [1, 0-a, 0] \\
W &= [\omega, 0, F\omega, 0] \\
P &= [0, 0, \phi, 0]
\end{aligned}
$$

Parameters
t is time /seconds
ω is 2π times the first pendulum swing frequency /Hz
a is the amplitude ratio
F is the frequency ratio
D is the damping factor (typically between 0 and 5)
ϕ is the phase difference /radians between the pendula

$$
\begin{aligned}
x &= A_1 e^{-\frac{t}{T_1}} \sin(tW_1 + P_1) + A_2 e^{-\frac{t}{T_2}} \sin(tW_2 + P_2) \\
y &= A_3 e^{-\frac{t}{T_3}} \sin(tW_3 + P_3) + A_4 e^{-\frac{t}{T_4}} \sin(tW_4 + P_4)
\end{aligned}
$$

Lateral harmonograph with frequency damping



$$
\begin{aligned}
T &= \frac{2\pi}{\omega \log\left(\frac{100}{100-D}\right)} \left[1, \infty, \frac{1}{F}, \infty\right] \\
A &= [1, 0-a, 0] \\
W &= [\omega, 0, F\omega, 0] \\
P &= [0, 0, \phi, 0]
\end{aligned}
$$

Parameters

t is time /seconds

ω is 2π times the first pendulum swing frequency /Hz

a is the amplitude ratio

F is the frequency ratio

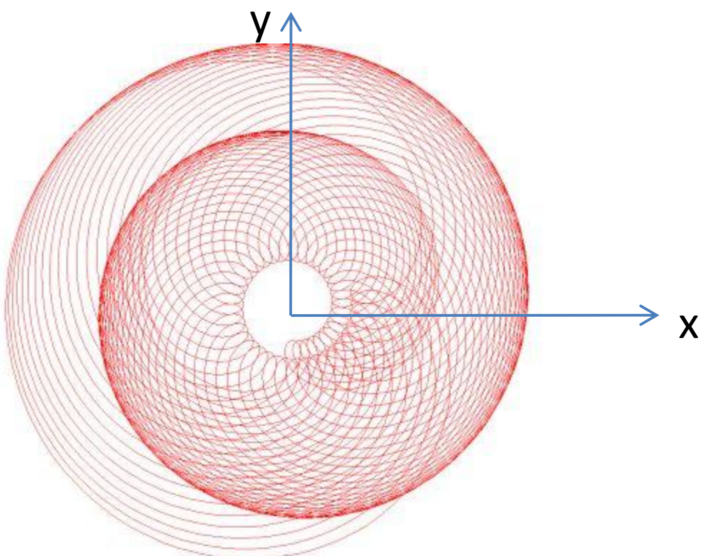
D is the damping factor (typically between 0 and 5)

ϕ is the phase difference /radians between the pendula

Mathematics and the Machine. Copyright A French. Marseilles 2011.

$$
\begin{aligned}
x &= A_1 e^{-\frac{t}{T_1}} \sin(tW_1 + P_1) + A_2 e^{-\frac{t}{T_2}} \sin(tW_2 + P_2) \\
y &= A_3 e^{-\frac{t}{T_3}} \sin(tW_3 + P_3) + A_4 e^{-\frac{t}{T_4}} \sin(tW_4 + P_4)
\end{aligned}
$$

Rotary harmonograph



$$
\begin{aligned}
T &= \frac{2\pi}{\omega \log\left(\frac{100}{100-D}\right)}[1,1,1,1] \\
A &= [1, a, 1, a] \\
W &= [\omega, -F\omega, \omega, -F\omega] \\
P &= [0, \phi, \tfrac{\pi}{2}, \tfrac{\pi}{2} + \phi]
\end{aligned}
$$

Parameters
t is time /seconds
ω is 2π times the first pendulum swing frequency /Hz
a is the amplitude ratio
F is the frequency ratio
D is the damping factor (typically between 0 and 5)
ϕ is the phase difference /radians between the pendula

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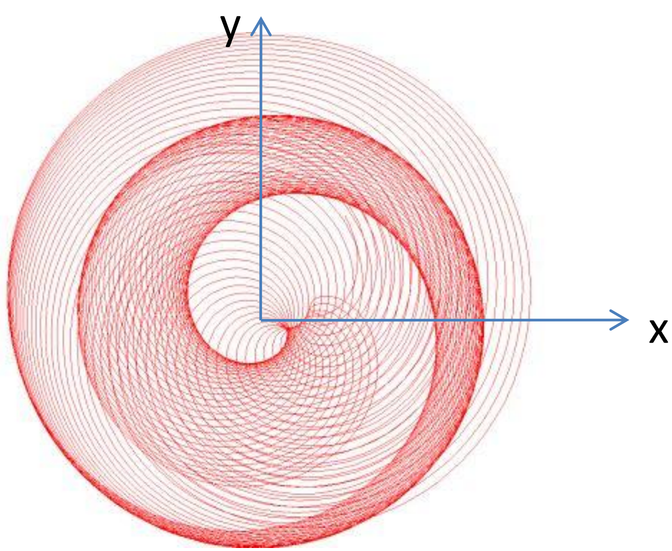
$$
\begin{aligned}
x &= A_1 e^{-\frac{t}{T_1}} \sin\left(tW_1 + P_1\right) + A_2 e^{-\frac{t}{T_2}} \sin\left(tW_2 + P_2\right) \\
y &= A_3 e^{-\frac{t}{T_3}} \sin\left(tW_3 + P_3\right) + A_4 e^{-\frac{t}{T_4}} \sin\left(tW_4 + P_4\right)
\end{aligned}
$$

Rotary harmonogrph with frequency damping



$$
\begin{aligned}
T &= \frac{2\pi}{\omega \log\left(\frac{100}{100-D}\right)} \left[1, \frac{1}{F}, 1, \frac{1}{F}\right] \\
A &= [1, a, 1, a] \\
W &= [\omega, -F\omega, \omega, -F\omega] \\
P &= \left[0, \phi, \tfrac{\pi}{2}, \tfrac{\pi}{2} + \phi\right]
\end{aligned}
$$

Parameters
t is time /seconds
ω is 2π times the first pendulum swing frequency /Hz
a is the amplitude ratio
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D is the damping factor (typically between 0 and 5)
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Musical harmony

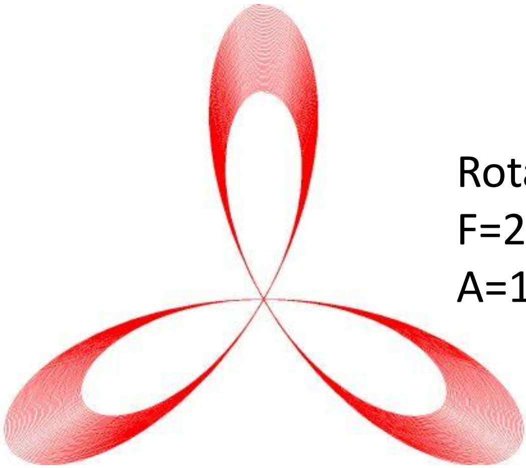
- The mathematics of music has been known since the time of Pythagoras, 2500 years ago
- Frequency intervals of simple fractions e.g. 3:2 (a fifth) yield 'harmonious' music
- An **octave** means a **frequency ratio of 2**. An octave above concert A (440Hz) is therefore 880Hz. An octave below is 220Hz.
- The modern 'equal-tempered scale' divides an octave (the frequency ratio 2) into twelve parts such that

$$F_n = 2^{n/12} = \sqrt[\frac{12}{n}]{2}$$

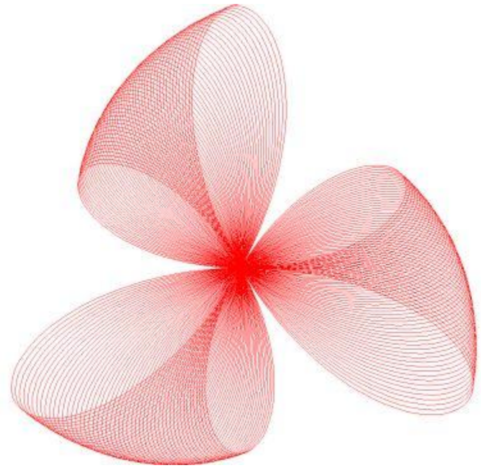
Musical harmony

Name	Exact value in 12-TET	Decimal value in 12-TET	Cents	Just intonation interval
Unison (C)	$2^{0/12} = 1$	1.000000	0	$\frac{1}{1}$ = 1.000000
Minor second (C♯/D♭)	$2^{1/12} = \sqrt[12]{2}$	1.059463	100	$\frac{16}{15}$ = 1.066667
Major second (D)	$2^{2/12} = \sqrt[6]{2}$	1.122462	200	$\frac{9}{8}$ = 1.125000
Minor third (D♯/E♭)	$2^{3/12} = \sqrt[4]{2}$	1.189207	300	$\frac{6}{5}$ = 1.200000
Major third (E)	$2^{4/12} = \sqrt[3]{2}$	1.259921	400	$\frac{5}{4}$ = 1.250000
Perfect fourth (F)	$2^{5/12} = \sqrt[12]{32}$	1.334840	500	$\frac{4}{3}$ = 1.333333
Augmented fourth (F♯/G♭)	$2^{6/12} = \sqrt{2}$	1.414214	600	$\frac{7}{5}$ = 1.400000
Perfect fifth (G)	$2^{7/12} = \sqrt[12]{128}$	1.498307	700	$\frac{3}{2}$ = 1.500000
Minor sixth (G♯/A♭)	$2^{8/12} = \sqrt[3]{4}$	1.587401	800	$\frac{8}{5}$ = 1.600000
Major sixth (A)	$2^{9/12} = \sqrt[4]{8}$	1.681793	900	$\frac{5}{3}$ = 1.666667
Minor seventh (A♯/B♭)	$2^{10/12} = \sqrt[6]{32}$	1.781797	1000	$\frac{7}{4}$ = 1.750000
Major seventh (B)	$2^{11/12} = \sqrt[12]{2048}$	1.887749	1100	$\frac{15}{8}$ = 1.875000
Octave (C)	$2^{12/12} = 2$	2.000000	1200	$\frac{2}{1}$ = 2.000000

Represent musical harmonies visually with the harmonograph!

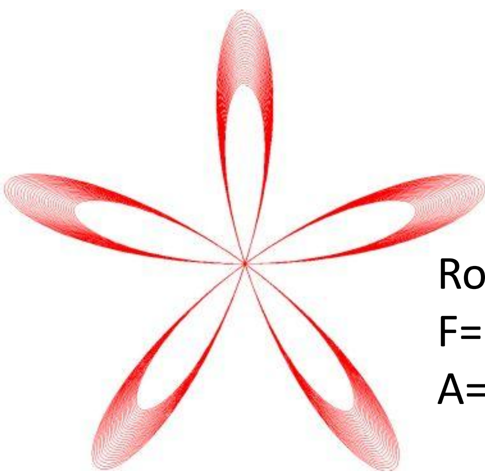


Rotary
F=2, D=0.7,
A=1, phi=0

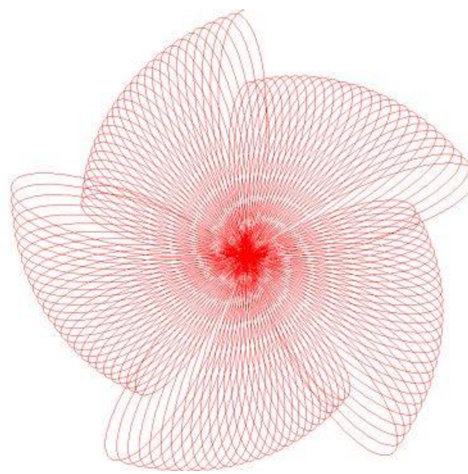


Rotary
F=2.01, D=0.7,
A=1, phi=0

Note the difference a small change in F makes....



Rotary
F=1.5, D=0.7,
A=1, phi=0



Rotary
F=1.51, D=0.7,
A=1, phi=0

Summary

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  - **Spherium**
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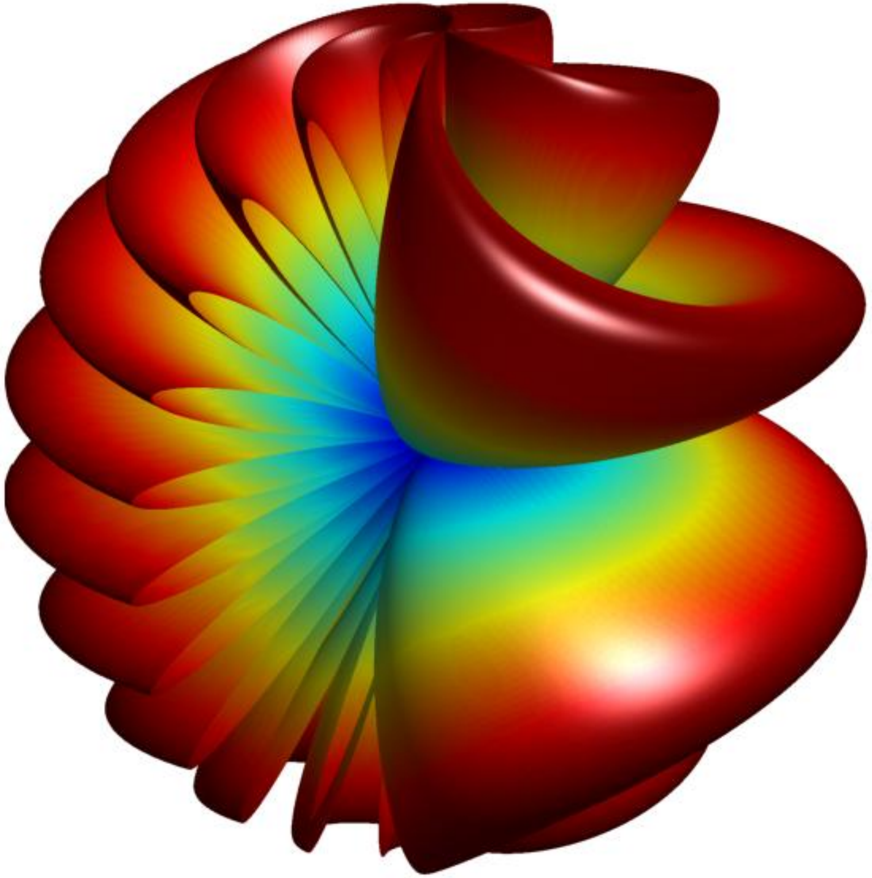
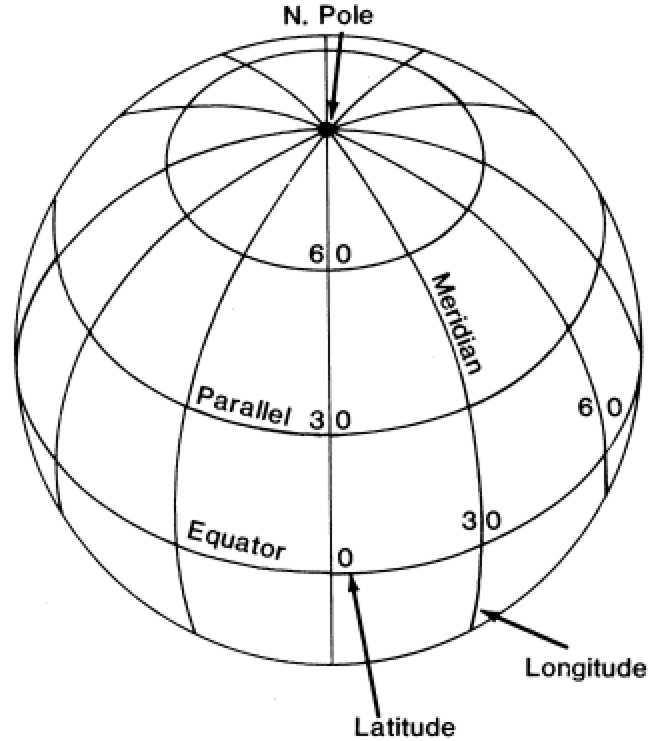
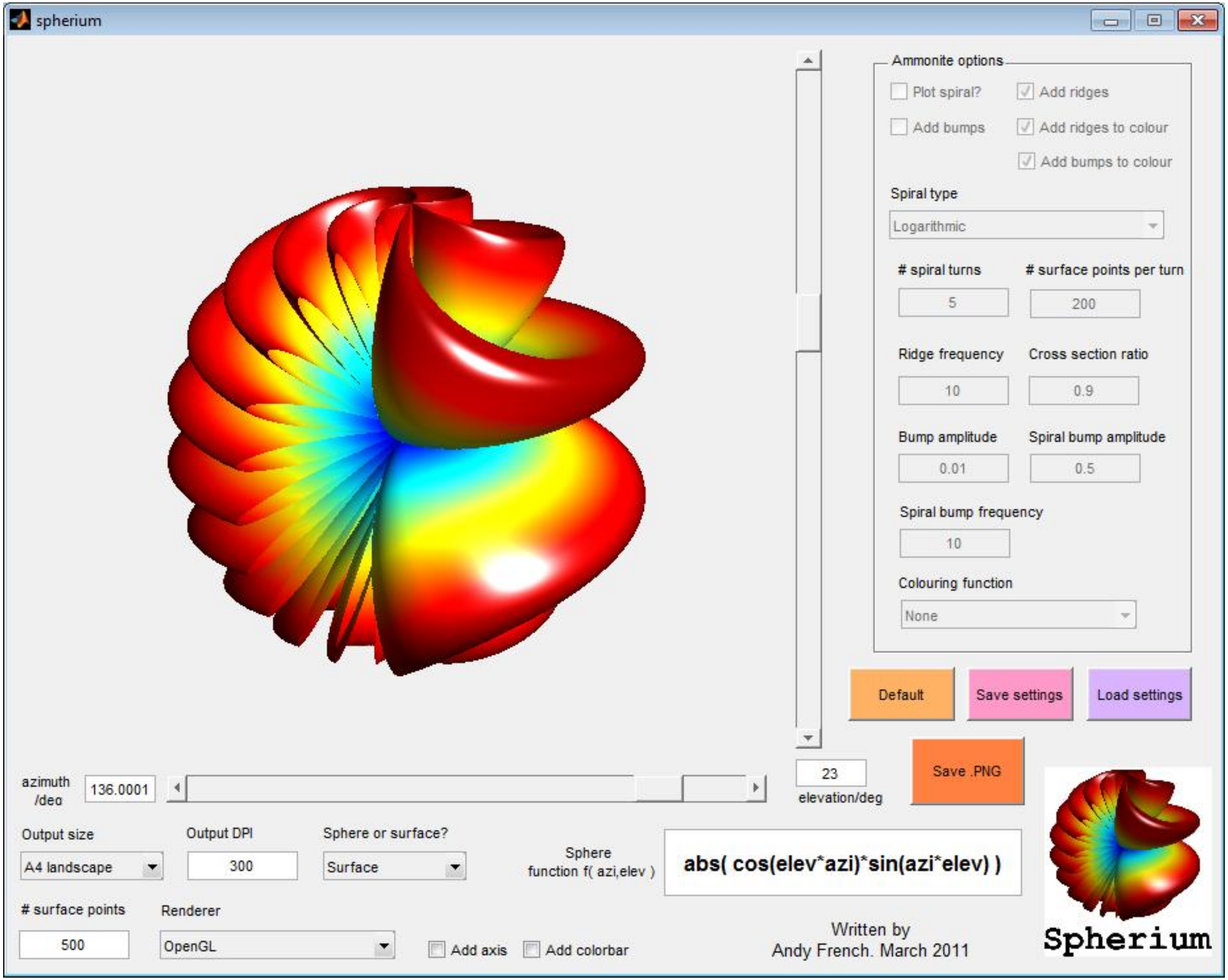
$$R = f(\mathrm{azi}, \mathrm{elev})$$

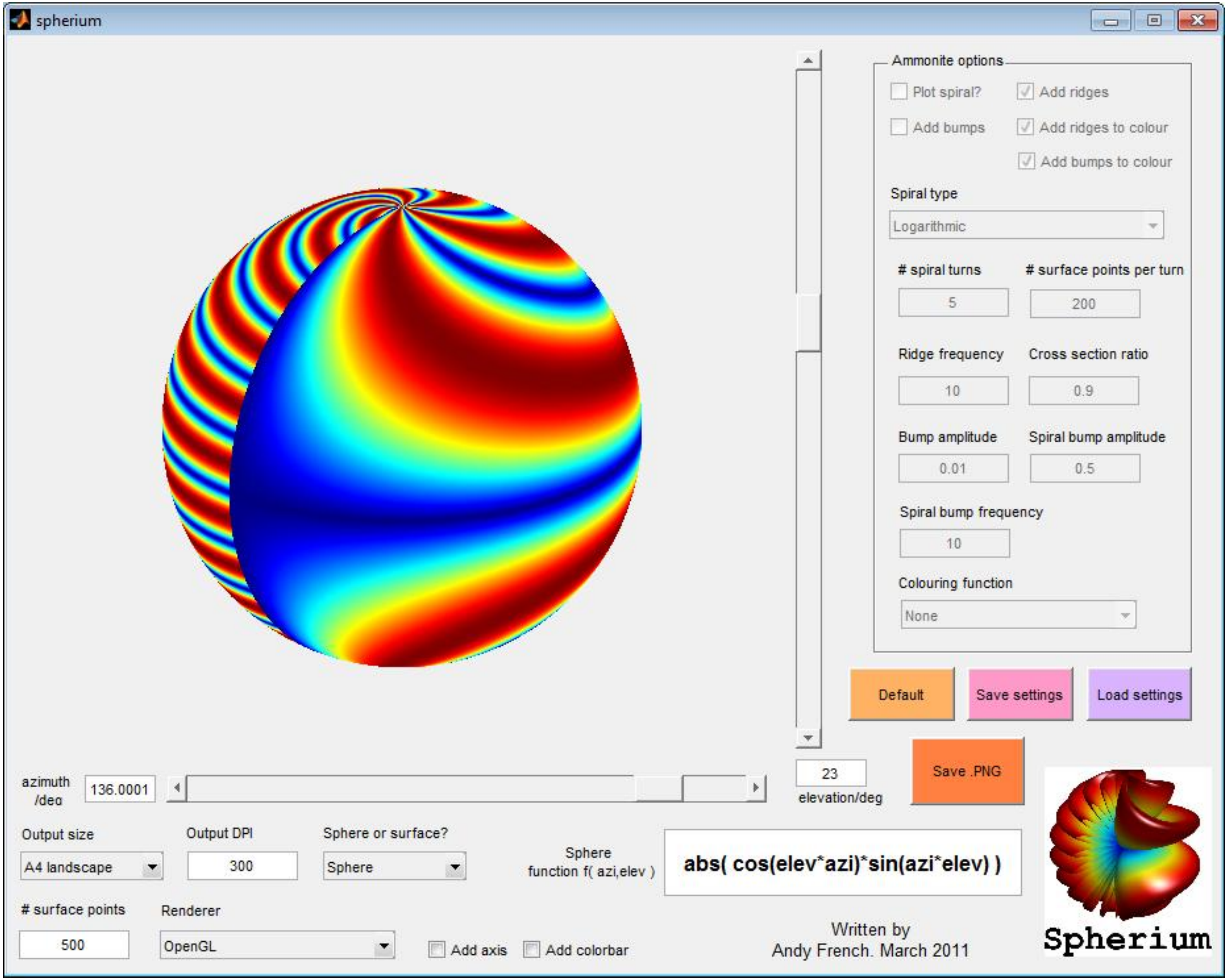


Spherium



R = radius
azi = longitude
elev = latitude


spherium
Ammonite options
Plot spiral?
Add ridges
Add bumps
Add ridges to colour
Add bumps to colour
Spiral type
Logarithmic
spiral turns
surface points per turn
5
200
Ridge frequency
Cross section ratio
10
0.9
Bump amplitude
Spiral bump amplitude
0.01
0.5
Spiral bump frequency
10
Colouring function
None
Default
Save settings
Load settings
23
elevation/deg
Save .PNG
azimuth /deg
136.0001
Output size
A4 landscape
Output DPI
300
Sphere or surface?
Surface
Sphere function f(azi,elev)
abs(cos(elev*azi)*sin(azi*elev))
# surface points
500
Renderer
OpenGL
Add axis
Add colorbar
Written by
Andy French. March 2011
Spherium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200
Ridge frequency
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10
0.9
Bump amplitude
Spiral bump amplitude
0.01
0.5
Spiral bump frequency
10
Colouring function
None
Default
Save settings
Load settings
23
elevation/deg
Save .PNG
azimuth /deg
136.0001
Output size
A4 landscape
Output DPI
300
Sphere or surface?
Sphere
Sphere function f(azi,elev)
abs(cos(elev*azi)*sin(azi*elev))
# surface points
500
Renderer
OpenGL
Add axis
Add colorbar
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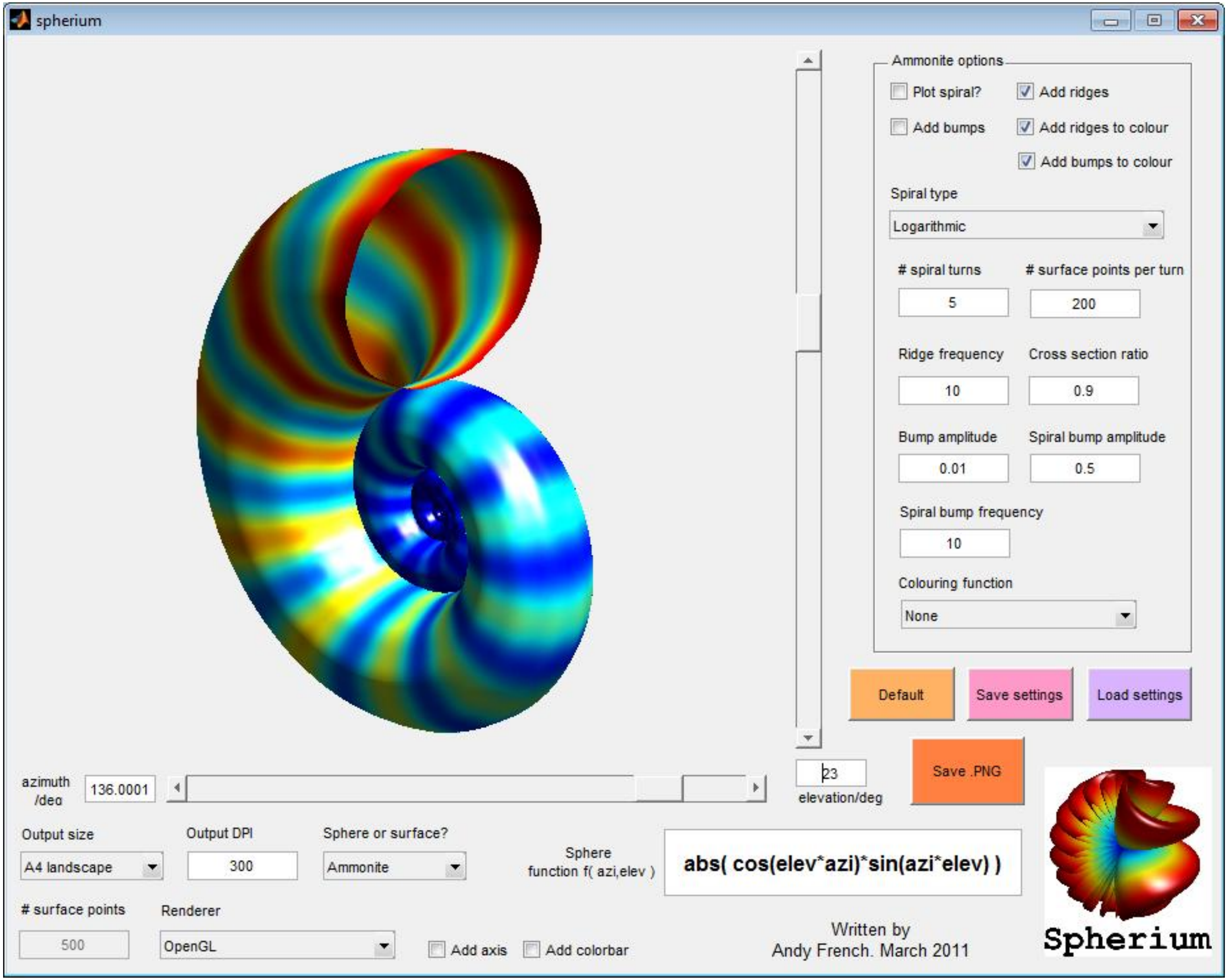
spherium
Ammonite options
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Add ridges
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0.01
0.5
Spiral bump frequency
10
Colouring function
None
Default
Save settings
Load settings
Save .PNG
23
elevation/deg
azimuth /deg
136.0001
Output size
A4 landscape
Output DPI
300
Sphere or surface?
Ammonite
Sphere function f(azi,elev)
abs(cos(elev*azi)*sin(azi*elev))
# surface points
500
Renderer
OpenGL
Add axis
Add colorbar
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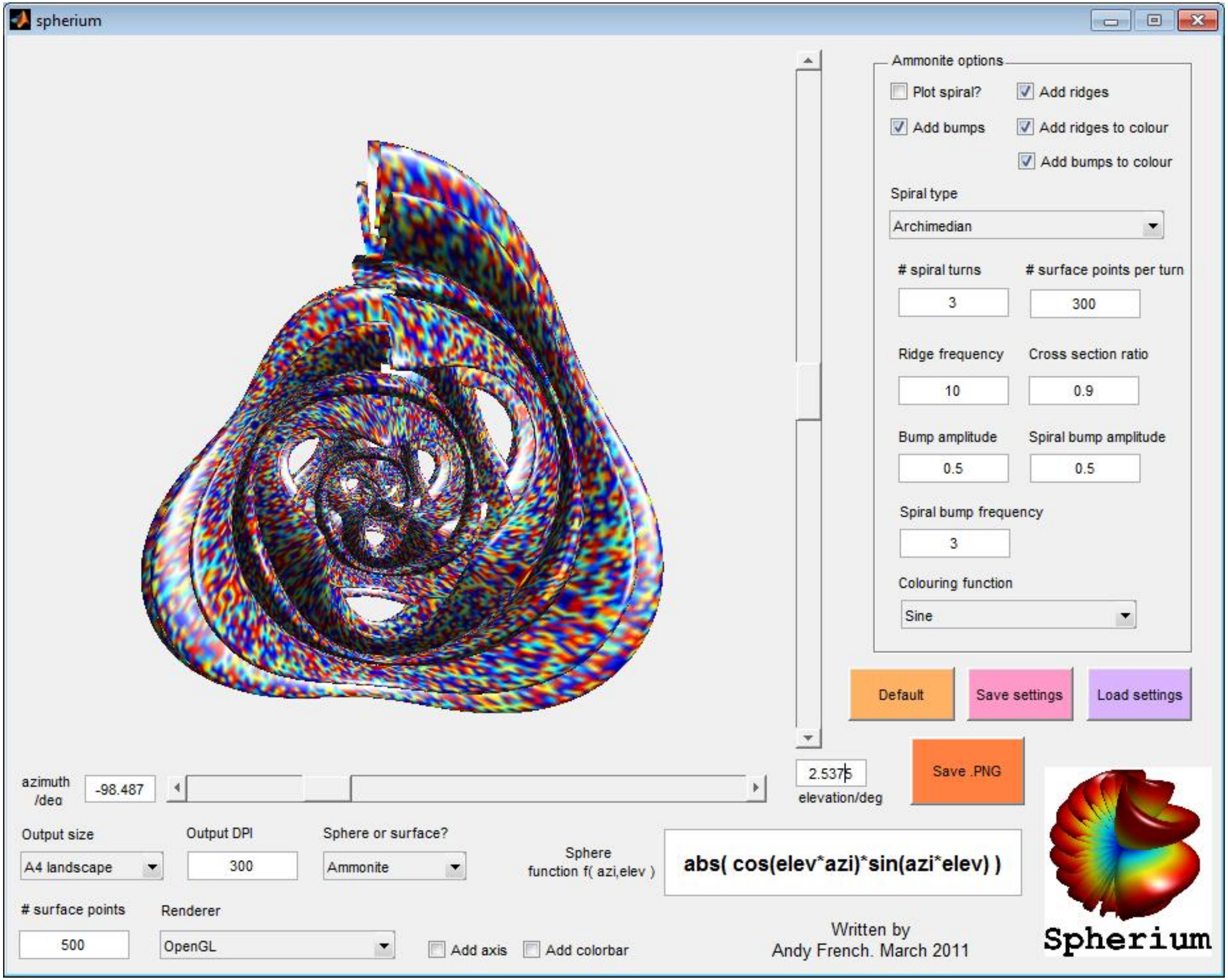
Spherium


spherium
Ammonite options
Plot spiral?
Add ridges
Add bumps
Add ridges to colour
Add bumps to colour
Spiral type
Archimedian
spiral turns
surface points per turn
3
300
Ridge frequency
Cross section ratio
10
0.9
Bump amplitude
Spiral bump amplitude
0.5
0.5
Spiral bump frequency
3
Colouring function
Sine
Default
Save settings
Load settings
Save .PNG
2.5375
elevation/deg
azimuth /deg
-98.487
Output size
A4 landscape
Output DPI
300
Sphere or surface?
Ammonite
Sphere function f(azi,elev)
abs(cos(elev*azi)*sin(azi*elev))
# surface points
500
Renderer
OpenGL
Add axis
Add colorbar
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Spherium

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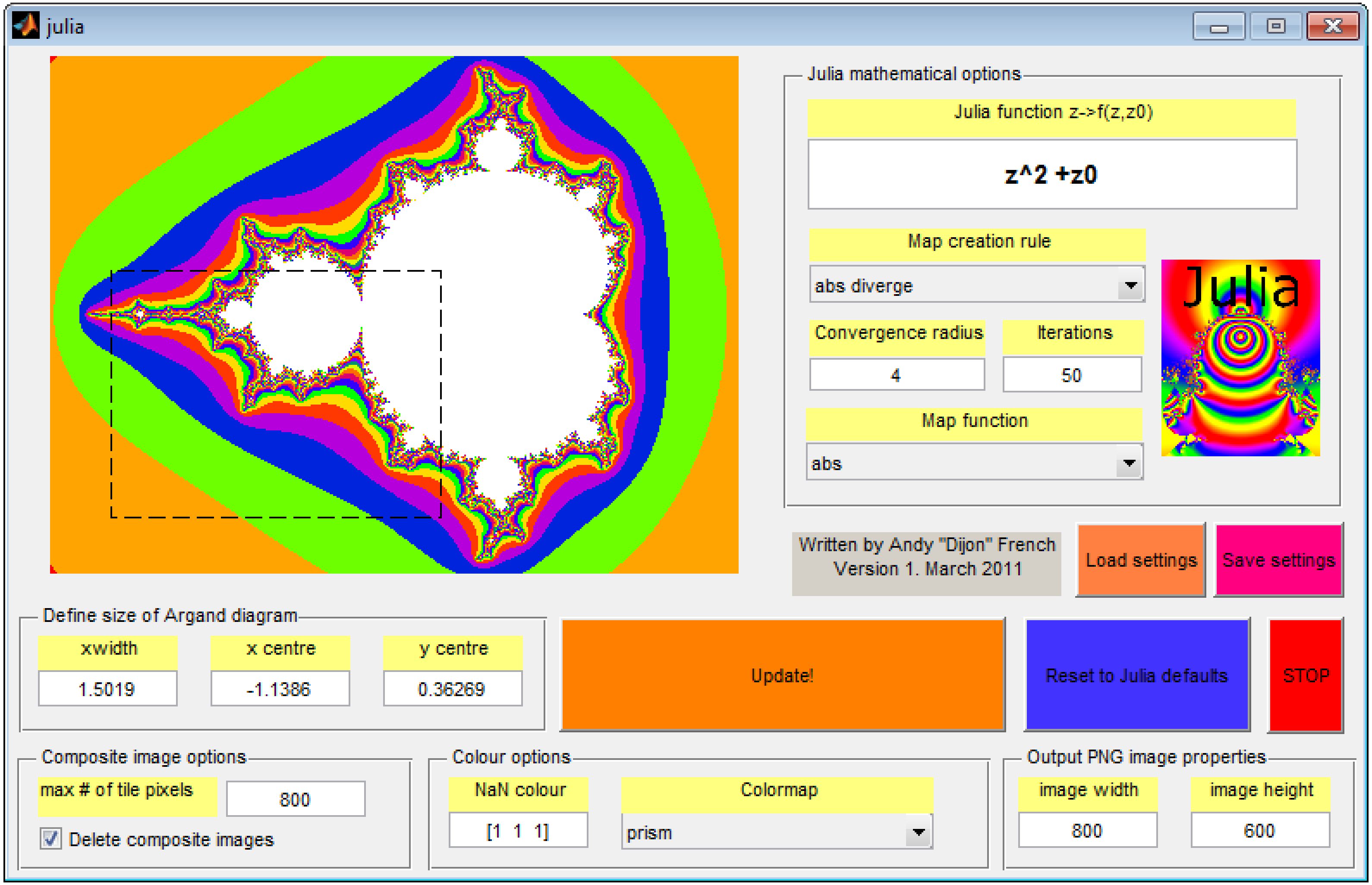
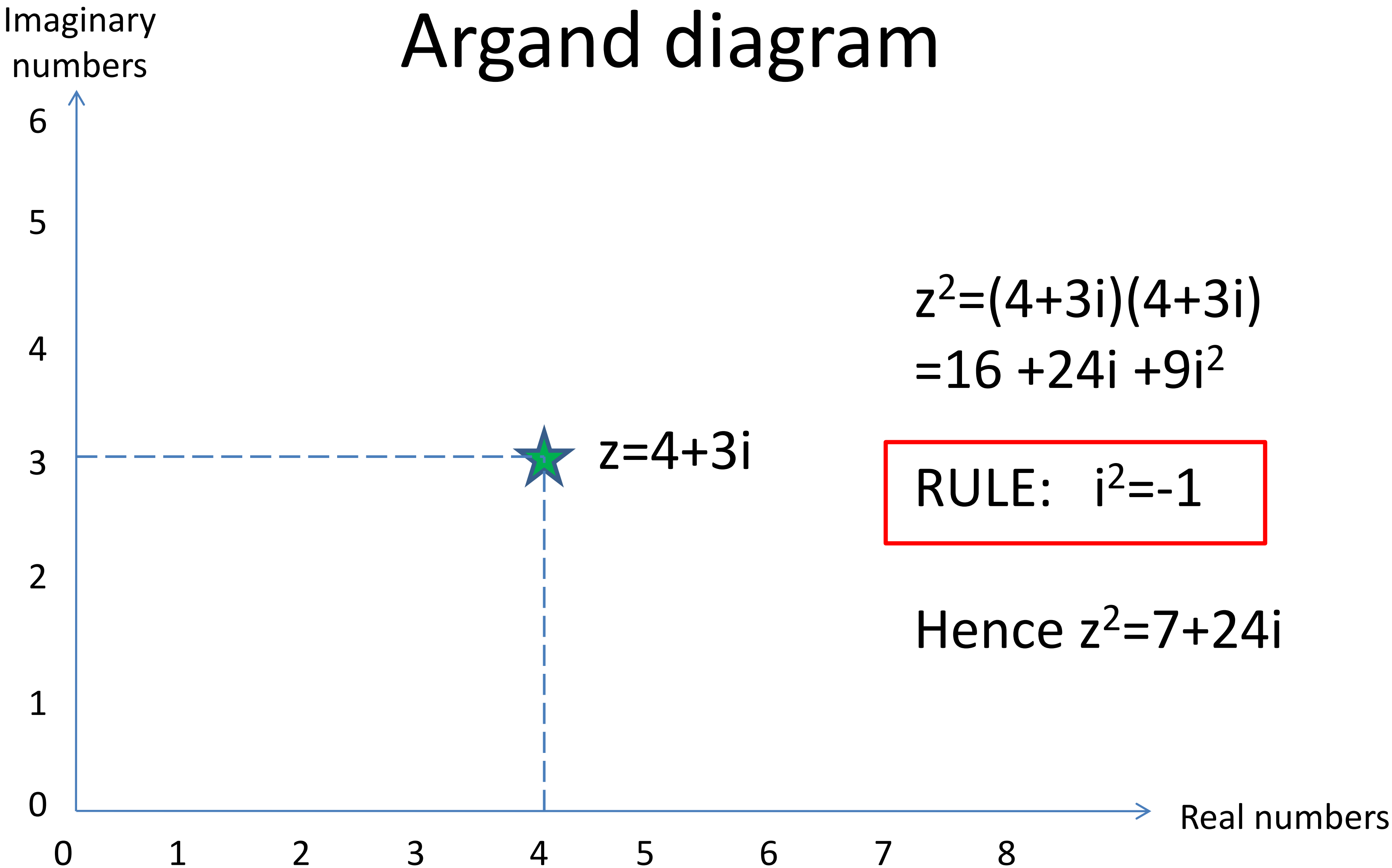
Julia

Julia default: The Mandlebrot set



Complex numbers and the Argand diagram

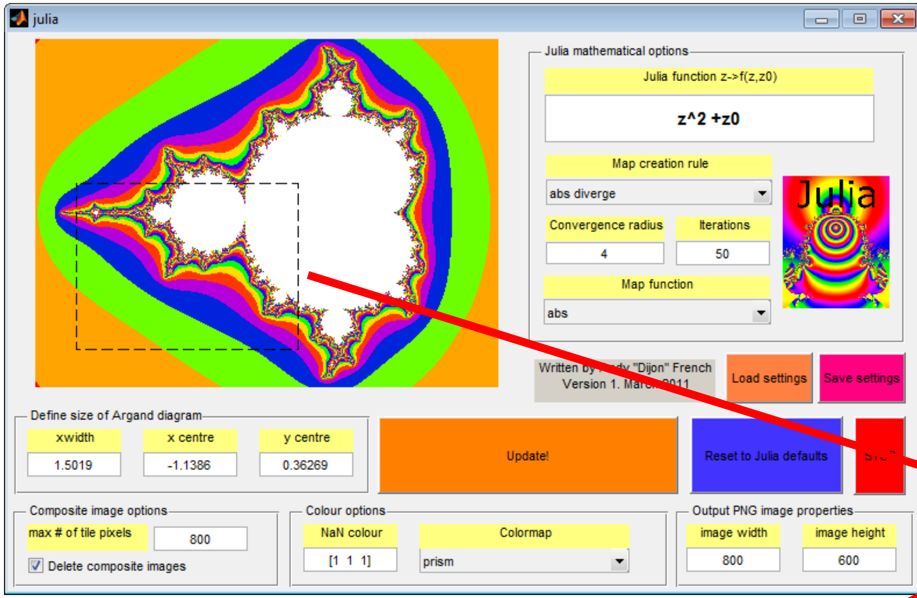


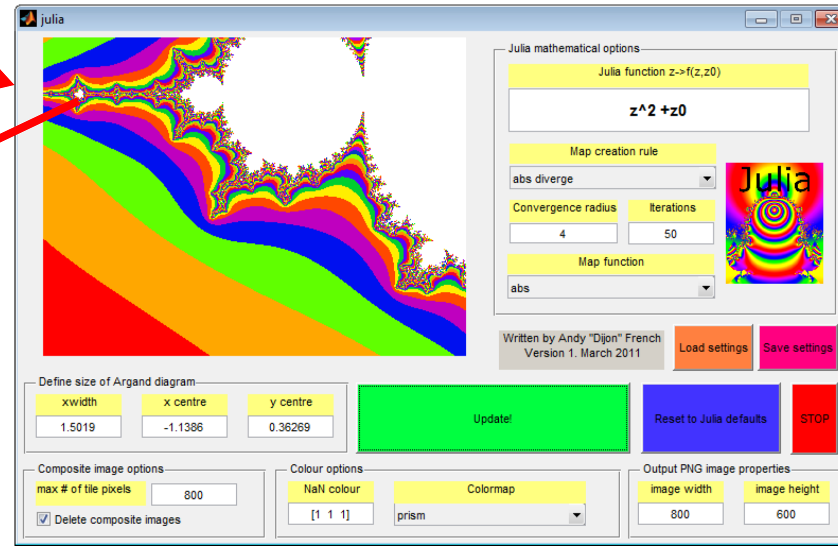
$z^2=(4+3i)(4+3i)$
$=16 +24i +9i^2$

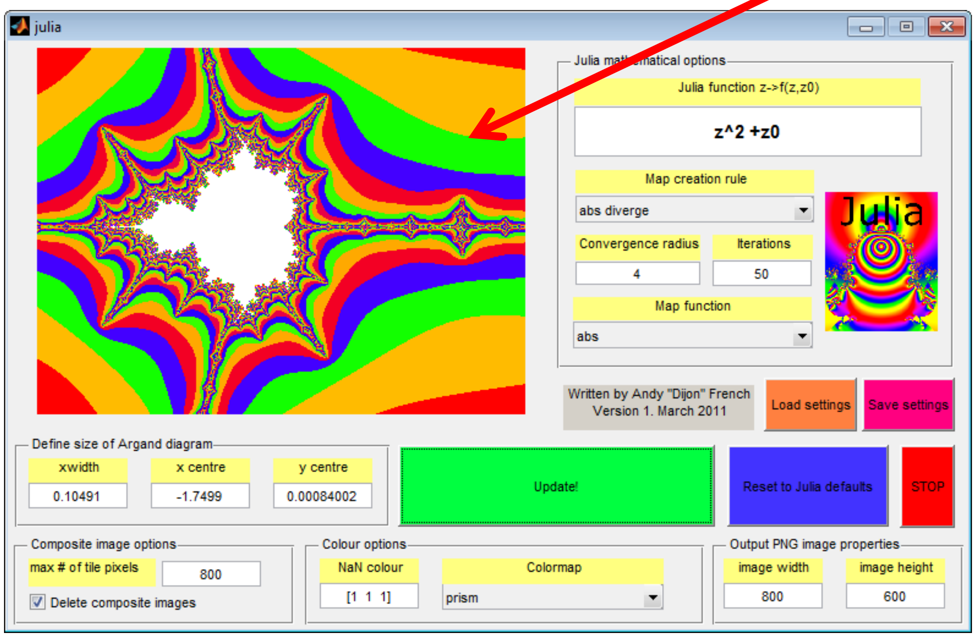
$\boxed{\text{RULE: } i^2=-1}$

Hence $z^2=7+24i$

Zooming in. The Mandlebrot set has *infinite complexity*

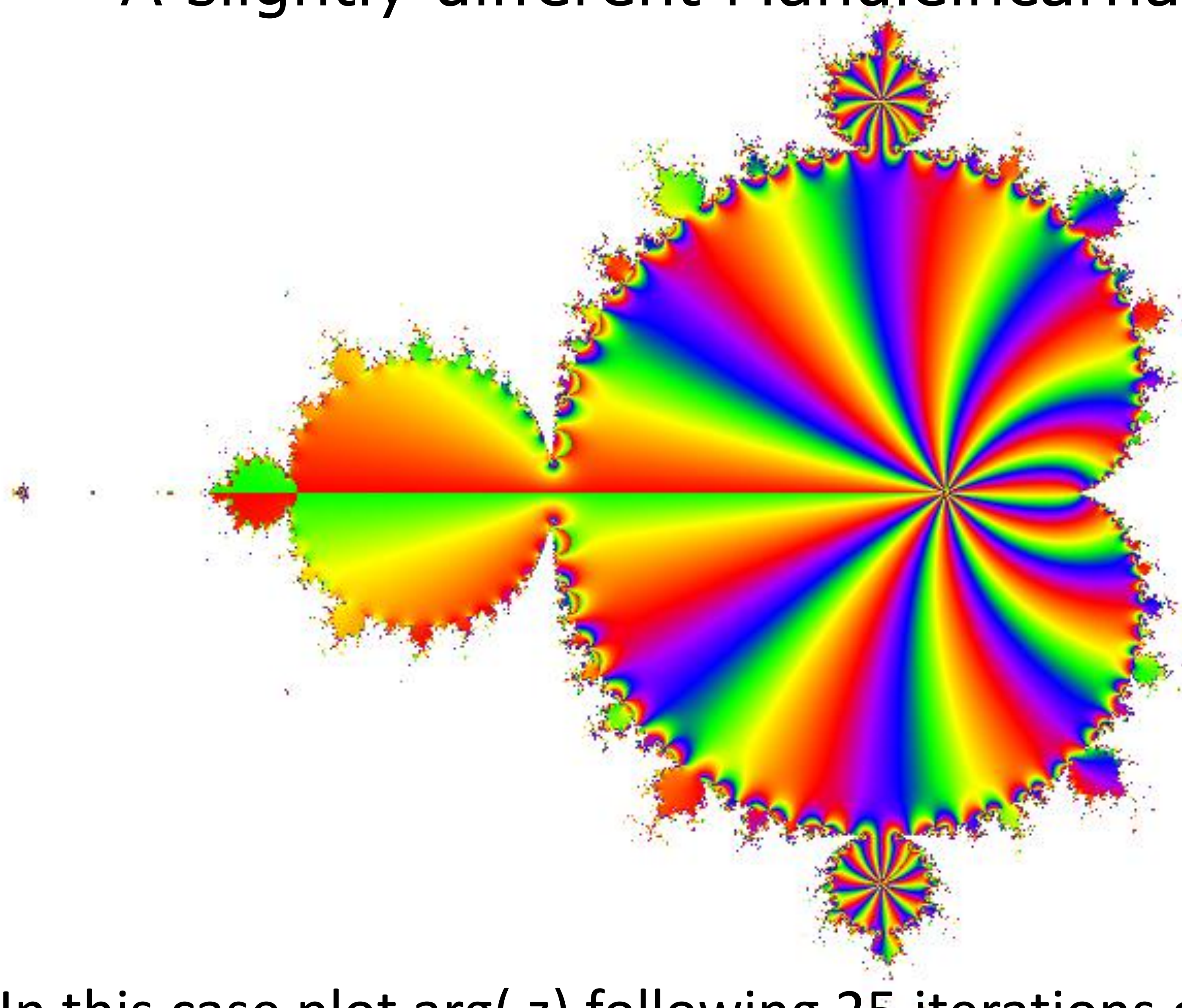




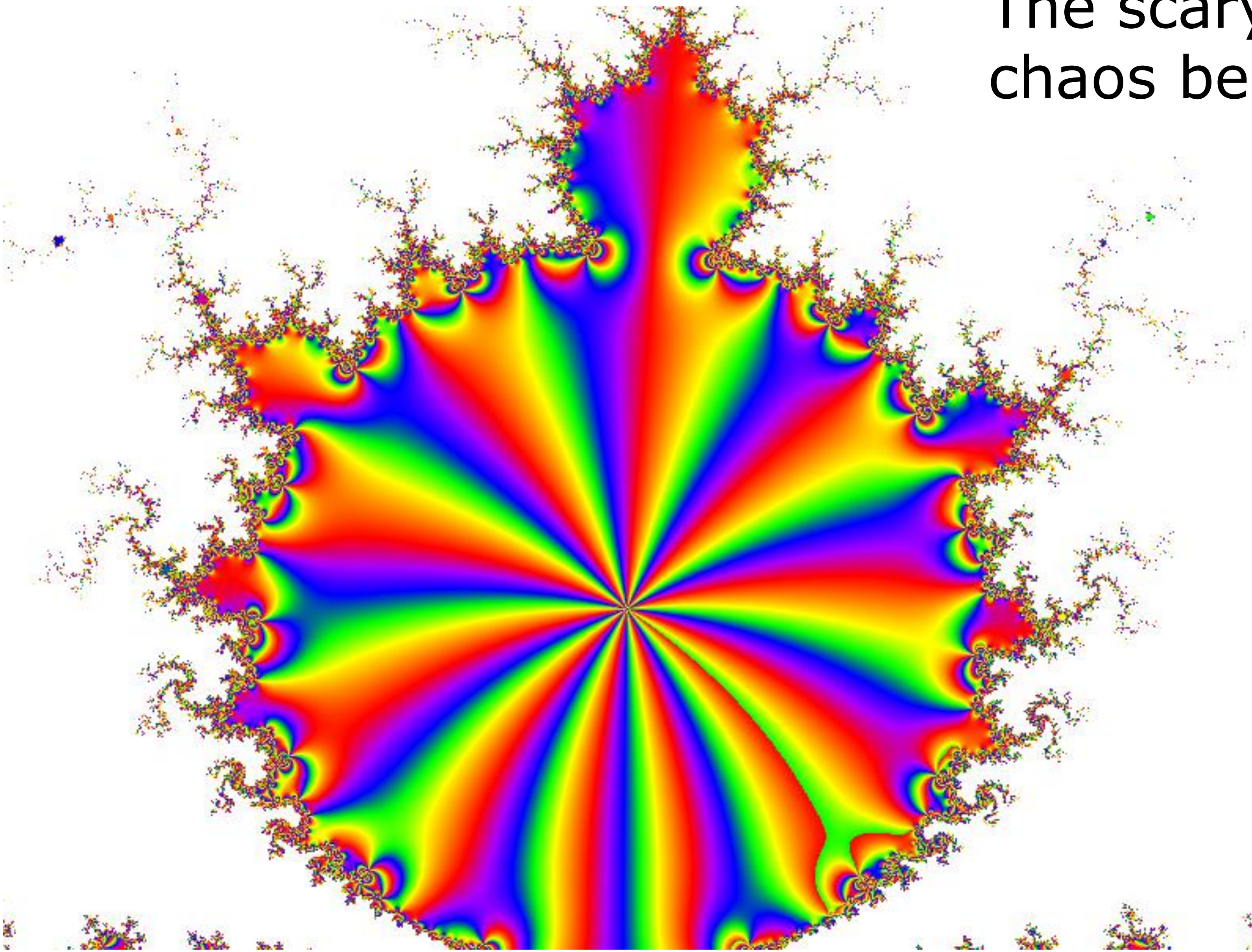


The Mandleplant slurping complexity from the Argand plane!

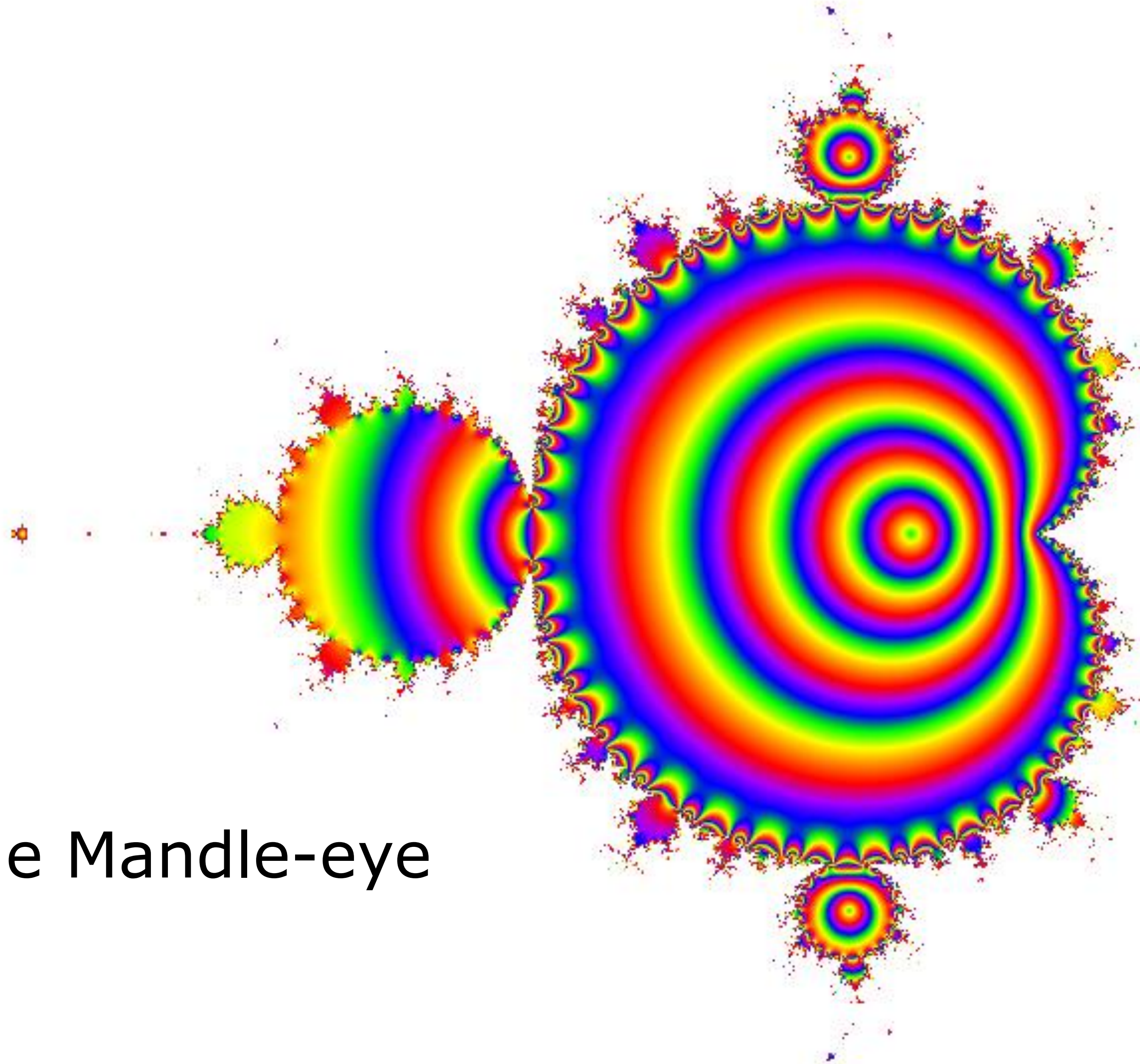
A slightly different Mandleincarnation!



In this case plot arg(z) following 25 iterations of $z \rightarrow z^2 + z_0$

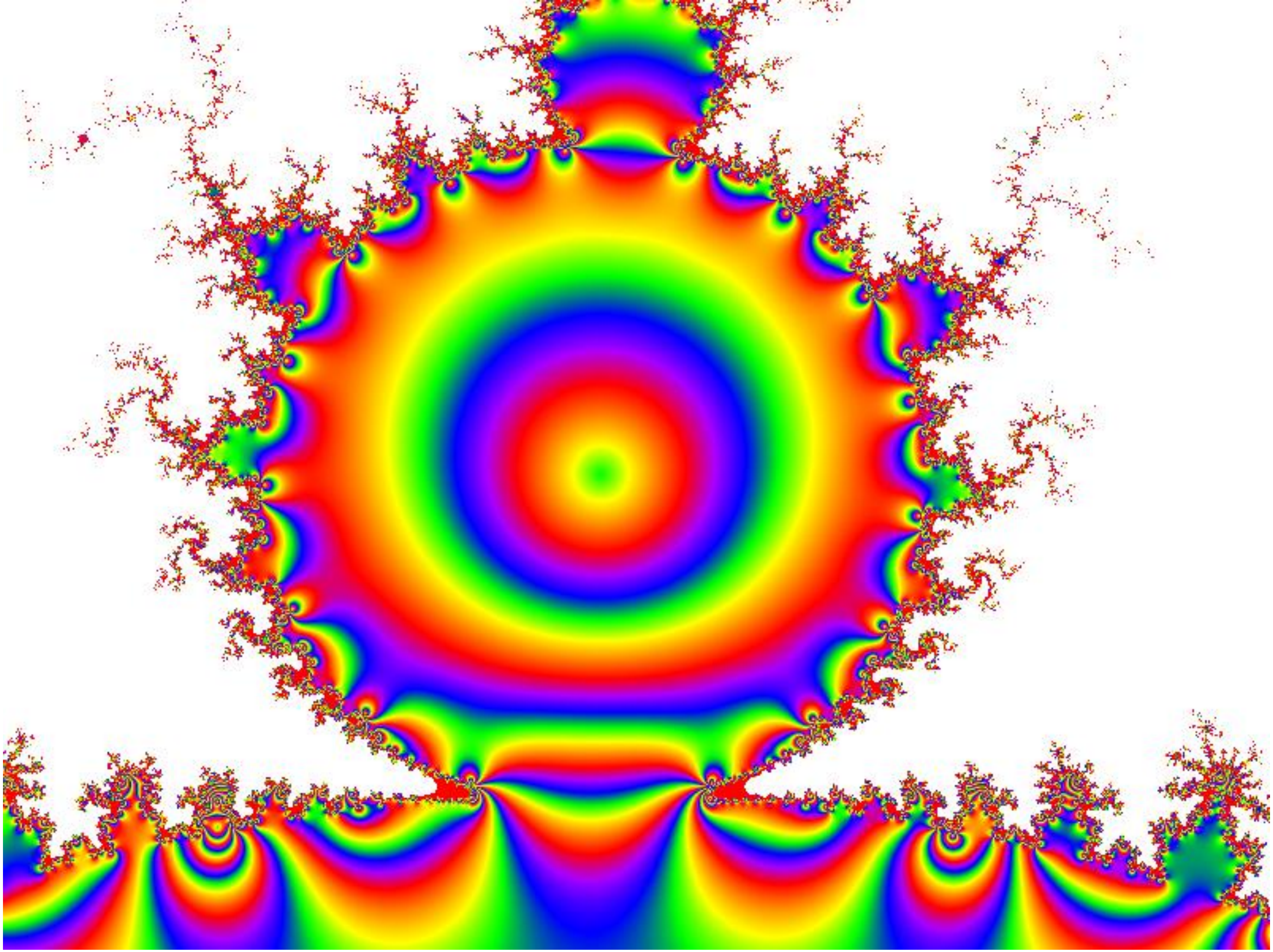


The scary chaos beast!



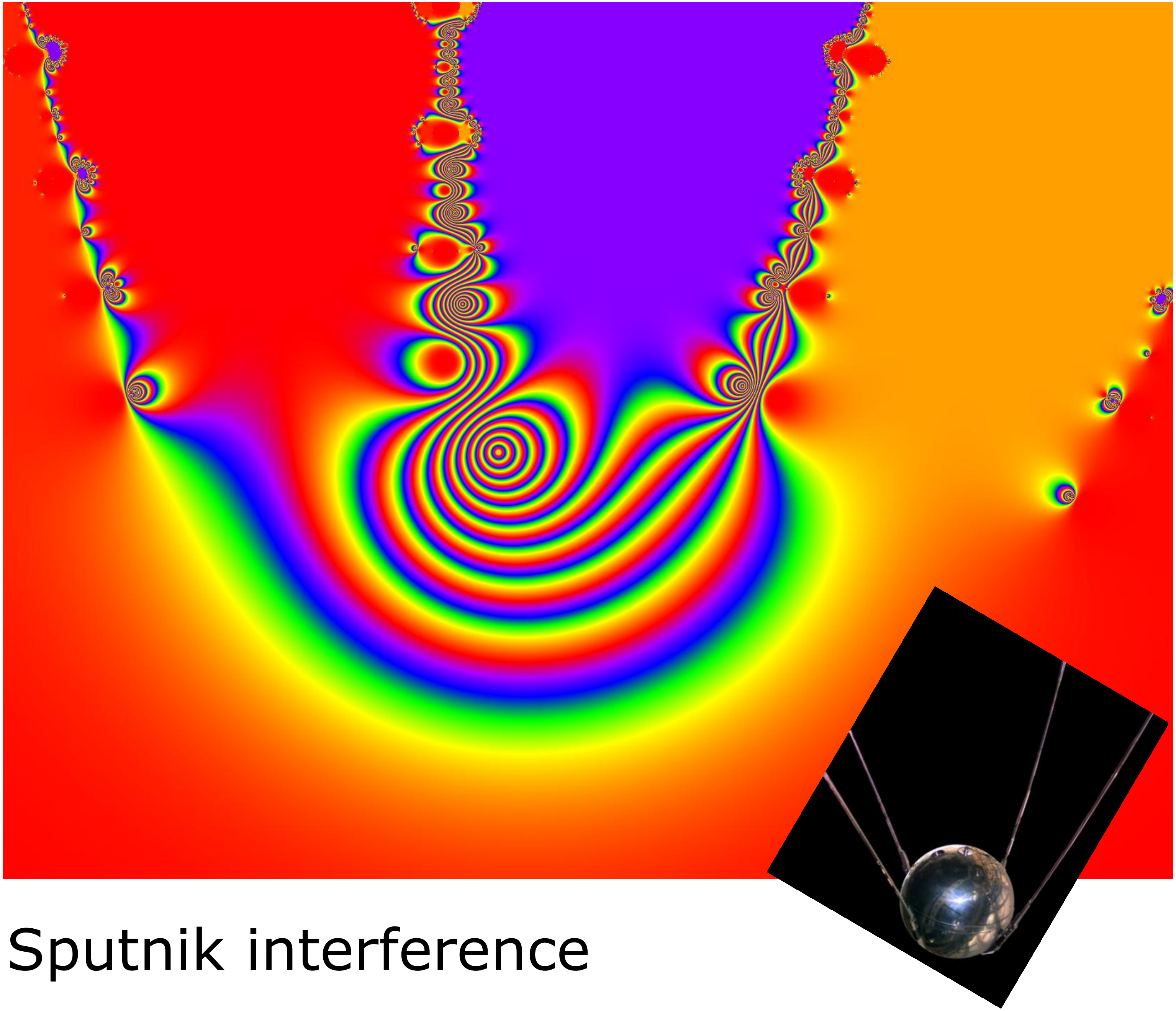
The Mandle-eye

In this case plot $\exp(-|z|)$ following 25 iterations of $z \rightarrow z^2 + z_0$

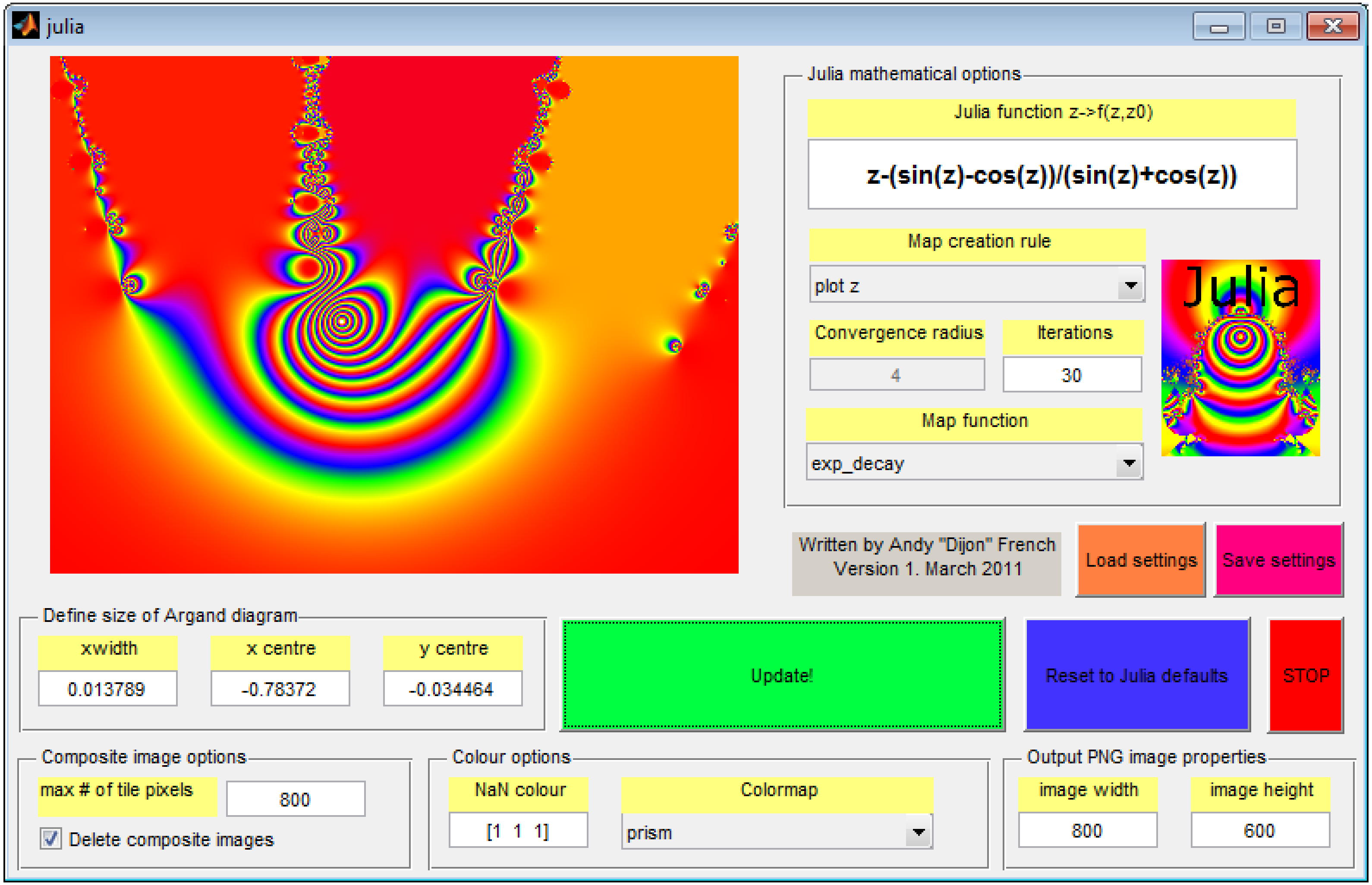


‘Homering’ in on the Cyclops (!)

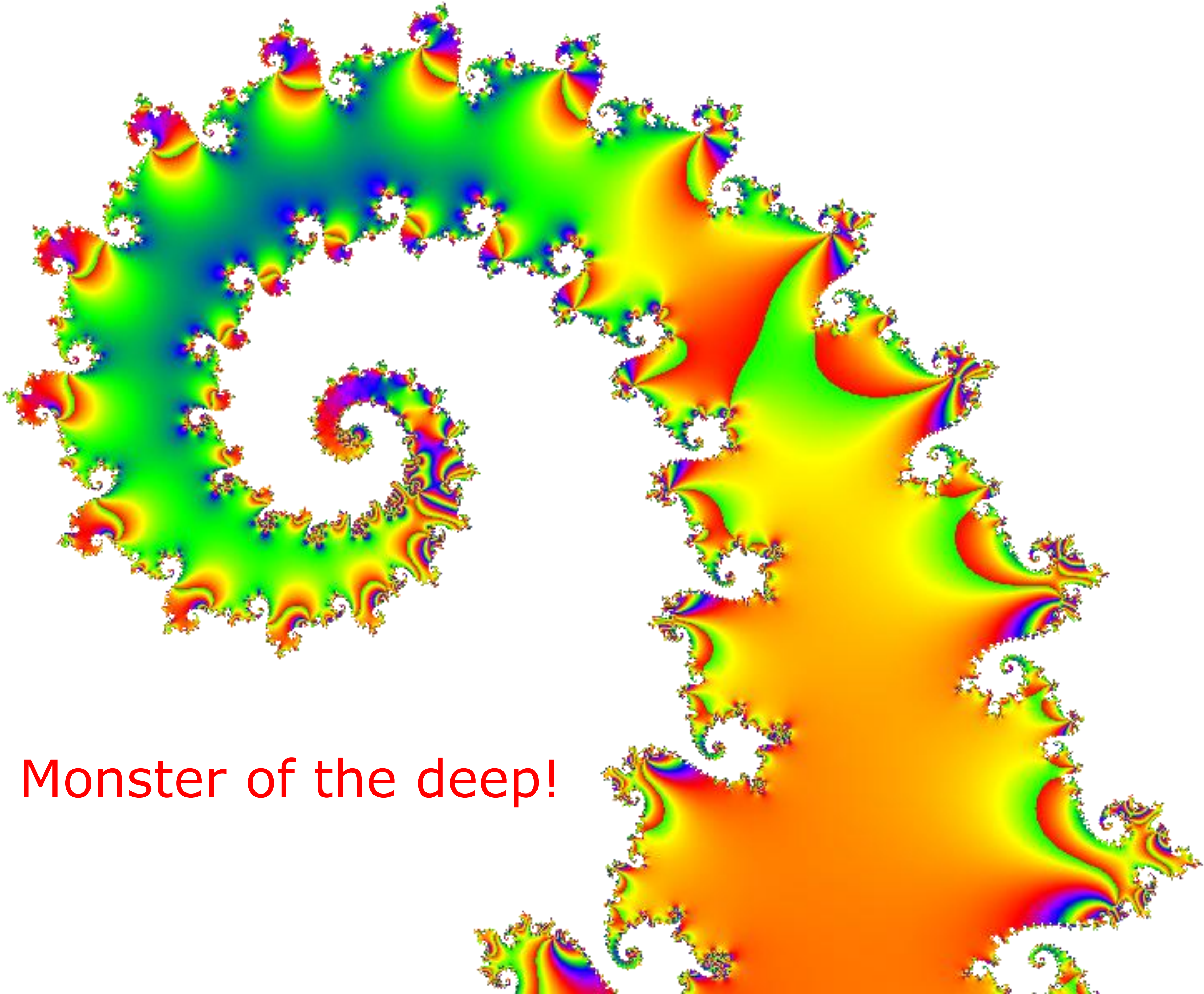

The Mandlebrot Variations



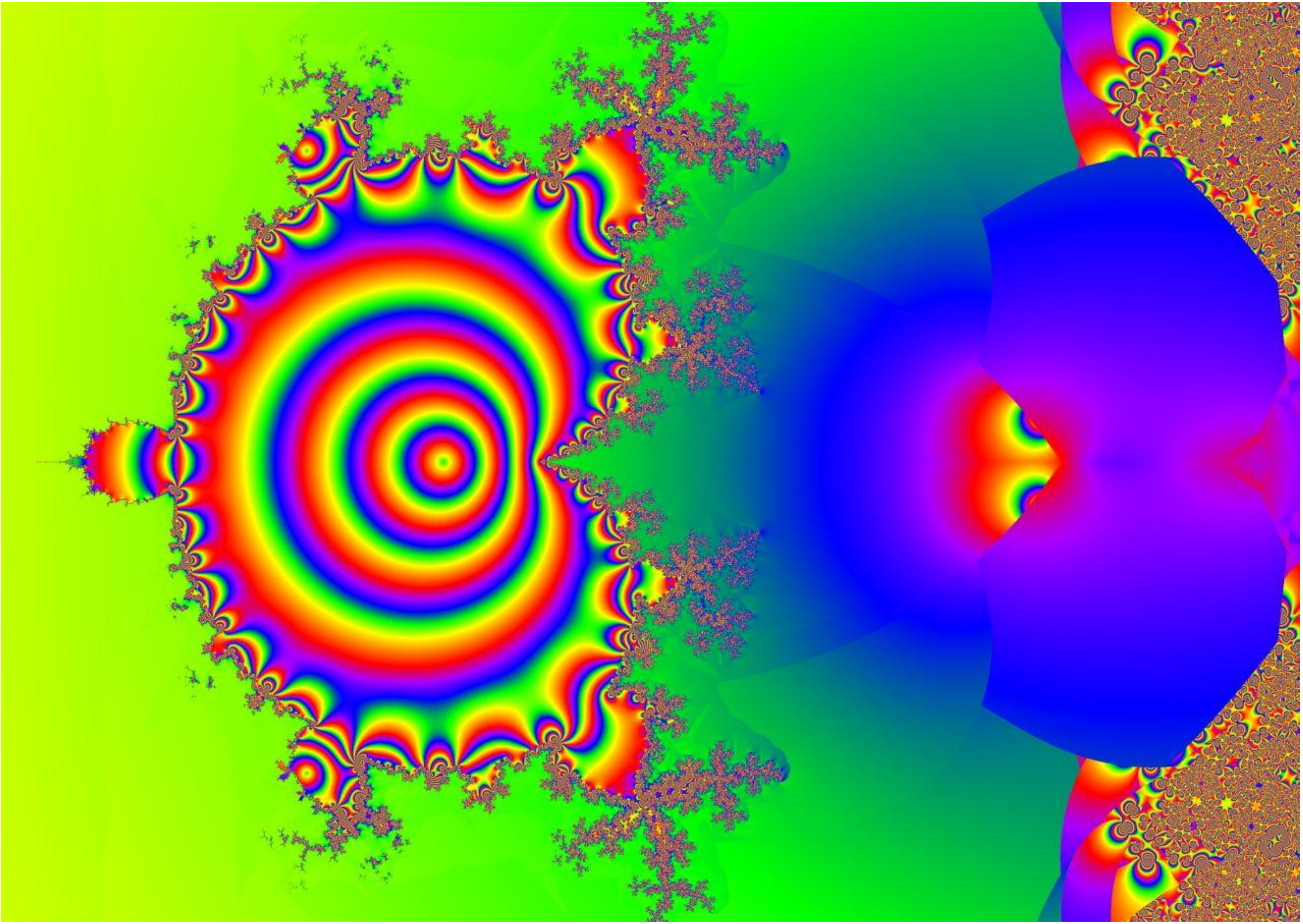
Sputnik interference



$$z \to z - \frac{\sin z - \cos z}{\sin z + \cos z}$$



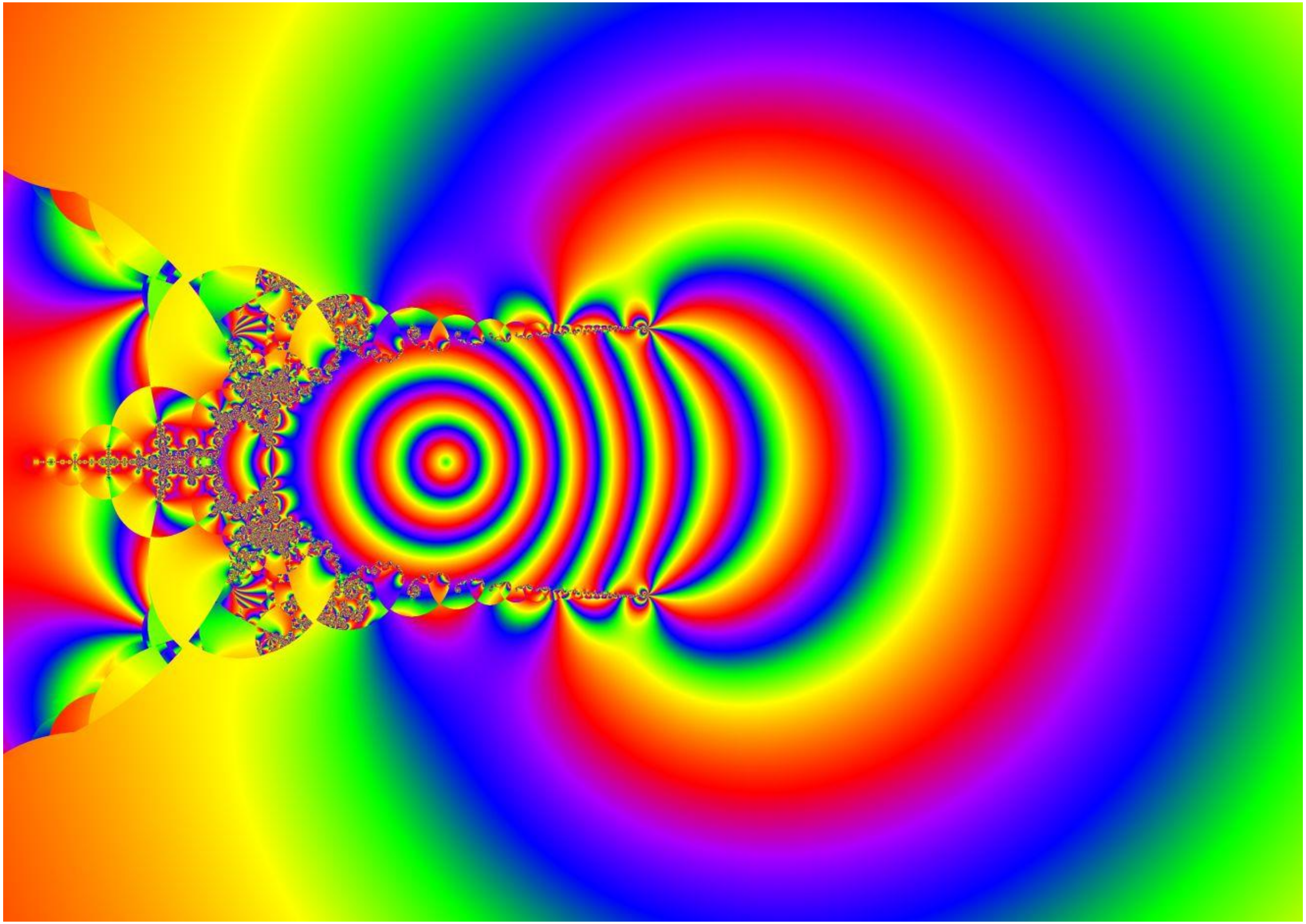
Monster of the deep!



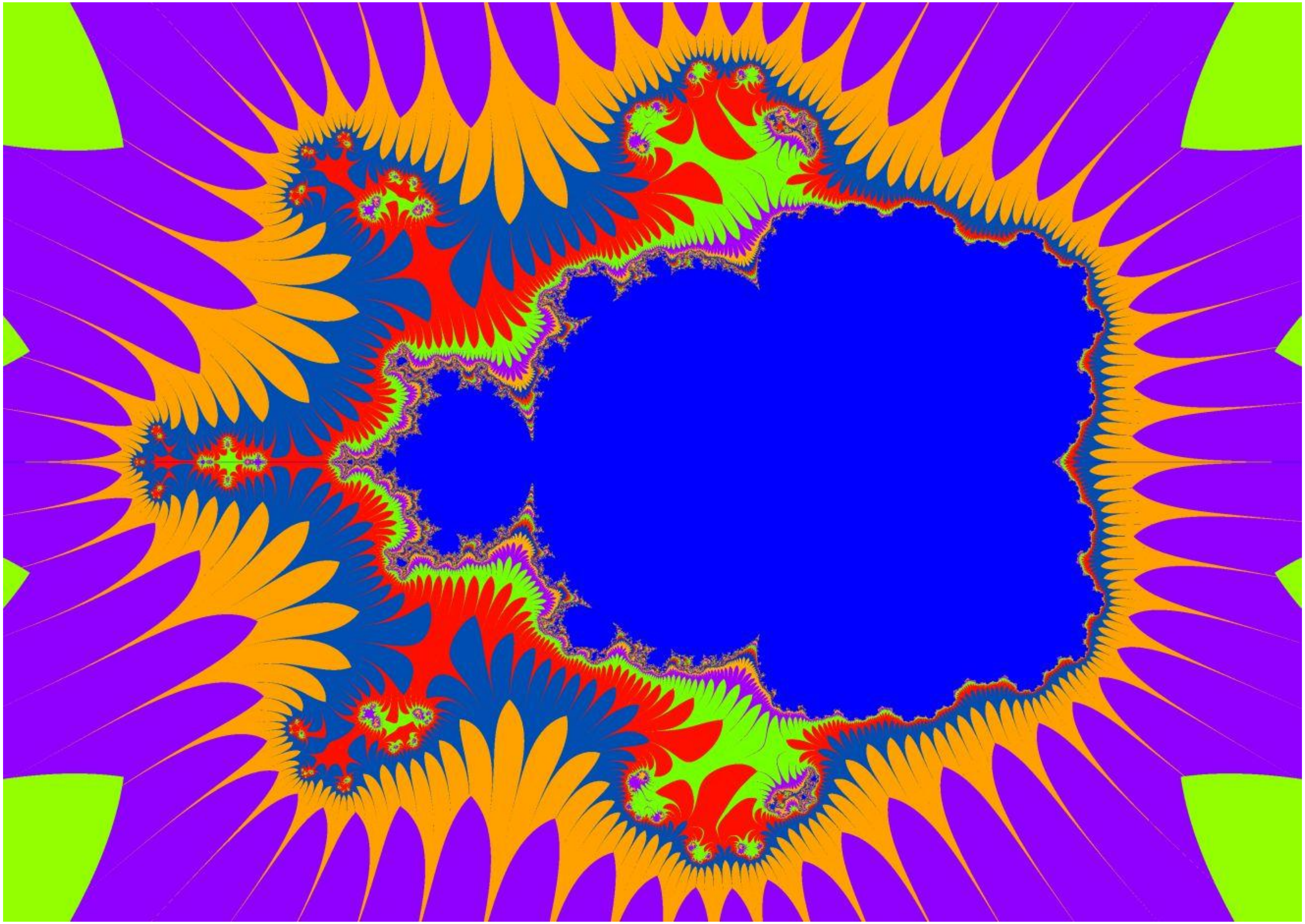
The Mandlerocket!

$z \rightarrow \mathrm{asin}(z^2 + z_0)$

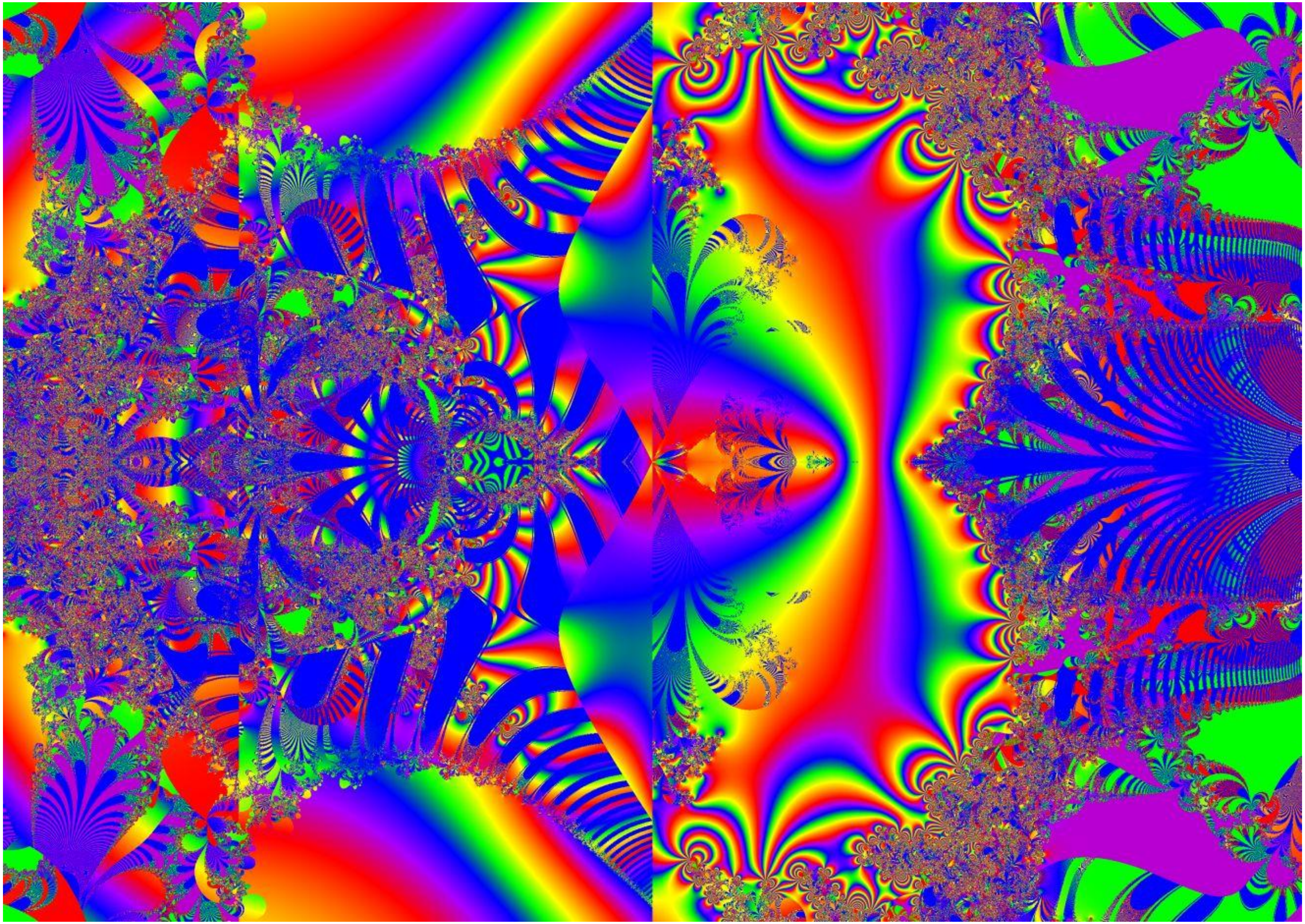
7 steps to enlightenment $z \rightarrow \mathrm{atan}(z^2 + z_0)$



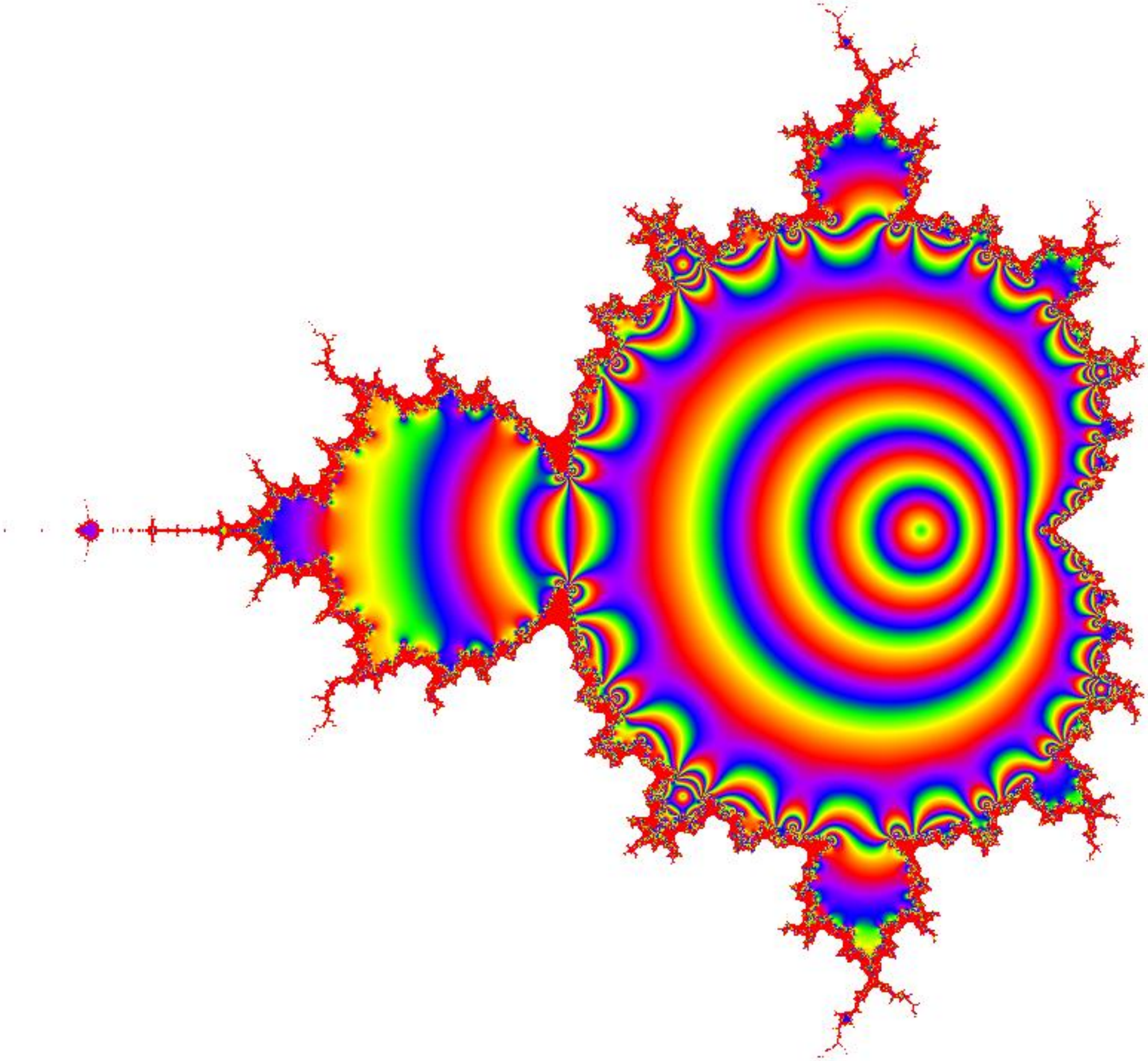
The light bulb $z \rightarrow \log(z^2 + z_0)$



Micro mandlebeast $z \rightarrow (z^2 + z_0)^2$



The profusion of power $z \rightarrow (z^2 + z_0)^z$



Day of Julia

Summary

- Introduction
  - Computers and mathematics - a harmonious relationship!
- Software demonstrations
  - Harmonograph
  - Spherium
  - Julia
- **Rene Descartes' theory of the rainbow**

(separate presentation)