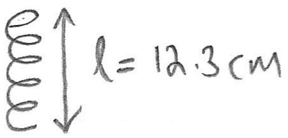
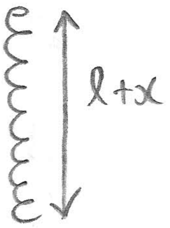
PHYSICAL PROPERTIES OF MATERIALS

1/

(i)





a) $x = 0.45\,l$

$x = 0.45 \times 12.3\text{ cm}$

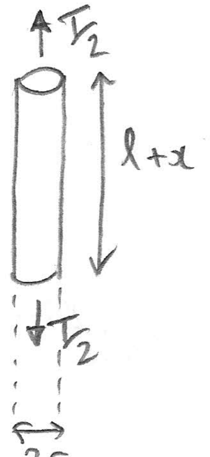
$\boxed{x = 5.5\text{ cm}}$ (5.535 cm)

b) $F = kx \quad \therefore k = F/x \quad \therefore k = \dfrac{6.7\text{N}}{5.535\times10^{-2}\text{m}} = \boxed{121\text{ N/m}}$

c) $E = \frac{1}{2}kx^2 = \frac{1}{2}\left(\dfrac{6.7}{5.535\times10^{-2}}\right)\left(5.535\times10^{-2}\right)^2$ (J)

$= \boxed{0.19\text{ J}}$

(ii)



$Y = 200\text{ GPa}$

$l = 10.000\text{ m}$

$x = 2\times10^{-3}\text{ m}$

$r = 42\times10^{-3}\text{ m}$

Stress $\sigma = \dfrac{T}{\pi r^2}$

$\dfrac{\sigma}{\varepsilon} = Y$ and $\varepsilon = \dfrac{x}{l}$

so $\dfrac{T}{\pi r^2} = \dfrac{Yx}{l}$

$\therefore \boxed{T = \dfrac{Yx\pi r^2}{l}}$

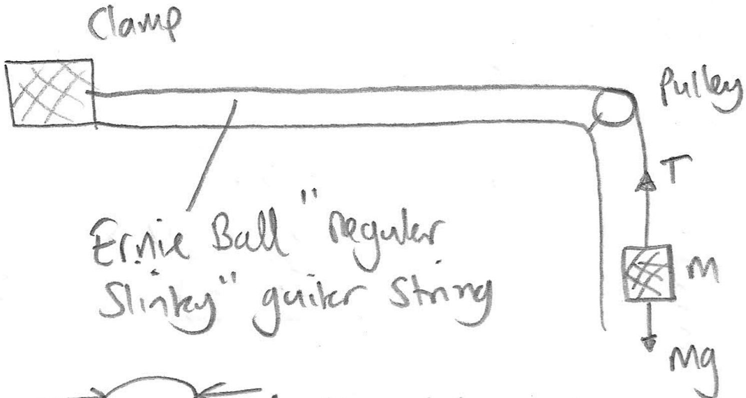
$\therefore T = \dfrac{200\times10^9 \times 2\times10^{-3} \times \pi \times (42\times10^{-3})^2}{10.00} = \boxed{2.22\times10^5\text{ N}}$

(ie a mass equivalent of 22.6 tonnes).

$\uparrow\ 22.6\times10^3 \times 9.81 = 2.22\times10^5$

If the $\boxed{\text{Poisson ratio}}$ (transverse to axial strain ratio) was not zero, the rod would probably 'neck' (ie $r\downarrow$) as it is stretched. Since $T = \dfrac{Yx\pi r^2}{l}$, for the same extension, this means a slightly lower tension T.

(iii)

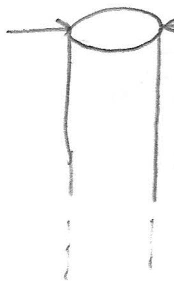


$g = 9.81$ N/kg

$l = 3.14$ m

$x = 2.7$ mm

$Y = 200$ GPa



$d = 2r = \frac{46}{1000} \times 25.4$ mm

$$Y = \frac{T/\pi r^2}{x/l}$$

$$T = Mg$$

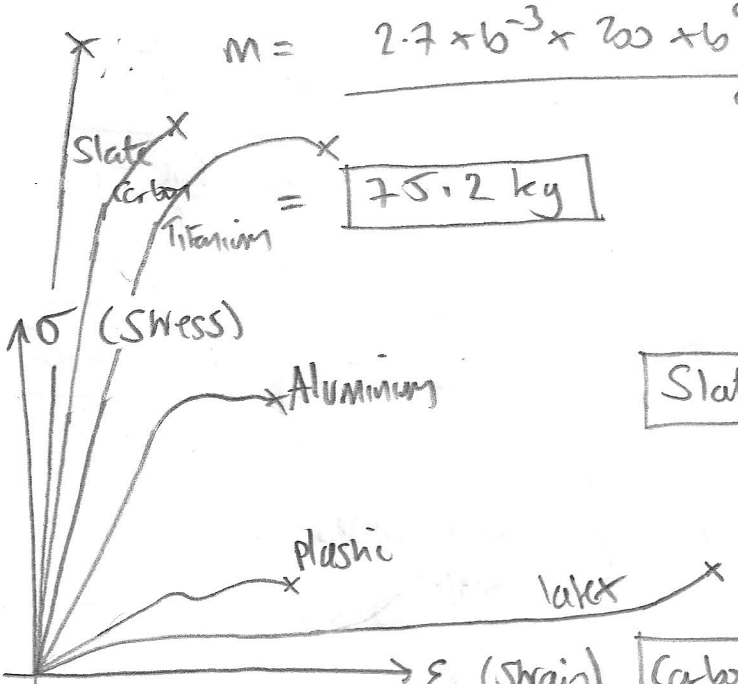
$$\therefore \boxed{\frac{x}{l} Y \pi r^2 / g = M}$$

$$\therefore M = \frac{2.7 \times 10^{-3} \times 200 \times 10^{9} \times \pi \times \left(\frac{46}{1000} \times 25.4 \times 10^{-3}\right)^2}{9.81}$$

$$= \boxed{75.2 \text{ kg}}$$

{ About the mass of Dr F! }

(iv)

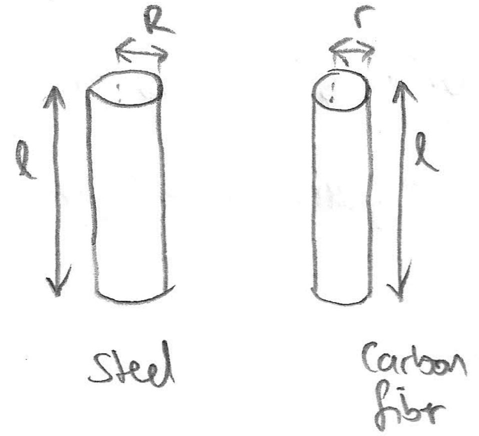


Slate: ceramic. High Y. Brittle, ie fractures before it yields. (at small strain)

Carbon Fiber: High Y (230 GPa is larger than steel), so very stiff. Weave and expoxy composite fabrication probably allows for some degree of yield, but expect to be less tough than titanium or aluminium.

Plastic should yield before fracture, (ie undergo plastic deformation first), and expect latex to do this in a more extreme way: ie lowest Y and largest strain.

v)



let tension T applied to steel rod be such that it is under maximum stress.

$$\frac{T}{\pi R^2} = 3\text{ GPa}$$

$$\therefore \quad T = 3\text{GPa} \times \pi R^2$$

If T is the same for the carbon fiber:

$$\frac{T}{\pi r^2} = Y_{cf}\, x/l \quad ①$$

Now extension x is the same as steel

$$\text{so:} \quad \frac{T}{\pi R^2} = Y_s\, x/l \quad ②$$

$$\Rightarrow \quad \left(\frac{r}{R}\right)^2 = Y_s/Y_{cf} \qquad ②/①$$

$$\Rightarrow \quad r = \sqrt{\frac{Y_s}{Y_{cf}}} \times 3.5\text{ mm}$$

$$r = \sqrt{\frac{200}{230}} \times 3.5\text{ mm}$$

$$\boxed{r = 3.26\text{mm}}$$

If T_{max} for steel is $3\text{GPa} \times \pi R^2$

and T_{max} for carbon fiber is $3\text{GPa} \times \pi r^2$

$\Rightarrow$ max force is $3\times10^9\text{ N/m}^2 \times \pi \times (3.26\times10^{-3})^2 = \boxed{1.00\times10^5\text{ N}}$

(10.2 tonne equivalent).

Fractional weight saving is $\dfrac{\pi R^2 \rho_s l - \pi r^2 \rho_{cf} l}{\pi R^2 \rho_s l} = 1 - \left(\dfrac{r}{R}\right)^2 \dfrac{\rho_{cf}}{\rho_s}$

$$= 1 - \left(\frac{3.26}{3.5}\right)^2 \left(\frac{2.00}{8.00}\right) = 0.783$$ so a $\boxed{78\%}$.

fractional weight saving.

(vi)

Zebedee from the Magic Roundabout.

During compression of spring by max amount x:

$F = kx \quad \therefore \quad k = F/x$

$F = 10\,N$ ...... But we don't know x!

$$\text{Now} \quad \underbrace{\tfrac{1}{2}kx^2}_{\text{(EPE)}} = \underbrace{mg(x+h)}_{\text{(GPE gain)}} \qquad h = 0.1\,m$$

height gain of unloaded Zebedee.

So using $x = F/k$:

$$\tfrac{1}{2}k\left(\frac{F}{k}\right)^2 = mg\left(\frac{F}{k} + h\right)$$

$$\tfrac{1}{2}k^2\, F^2/k^2 = mg(F + kh)$$

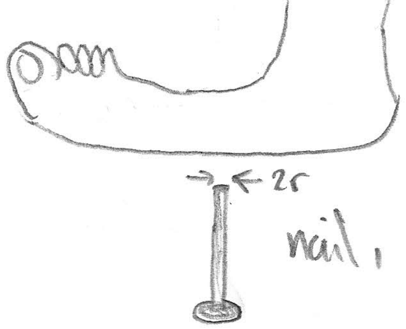
$$\tfrac{1}{2}F^2 = mgF + mgkh$$

$$\therefore \quad \boxed{\frac{\tfrac{1}{2}F^2 - mgF}{mgh} = k}$$

$F = 10\,N$
$m = 0.173\,kg$
$g = 9.81\,N/kg$
$h = 0.1\,m$

$$\Rightarrow \quad k = \frac{\tfrac{1}{2}(10)^2 - 0.173 \times 9.81 \times 10}{0.173 \times 9.81 \times 0.1}$$

$$\boxed{k = 195\,N/m}$$

(vii) 

Human foot (!)

$\rightarrow\ \leftarrow 2r$

nail, radius of point $r = \frac{0.5\,mm}{2}$

Not quite accurate as nail may have a tip

Puncture stress for human is $\sigma = \frac{1.2}{\pi r^2}$ (Pa)

$$= \frac{1.2\,N}{\pi \left(\frac{0.5 \times 10^{-3}}{2}\right)^2} = \boxed{6.11 \times 10^6\ Pa}$$

ie assume puncture area is cross section of nail.

Note: pressure of person on a bed of nails is:

$$mg/A = \frac{80\,kg \times 9.81\,N/kg}{1.8\,m \times 0.3\,m} = \boxed{1453\ N/m^2}$$

To avoid puncturing the person we need n nails / cm^2

such that $\underbrace{n \times 100^2}_{} \times 1.8 \times 0.3 \times 1.2\,N > 80\,kg \times 9.81$

ie person pressure < puncture force × nails/m^2

$n \times 100^2$ nails / m^2 ie force / nail < 1.2 N.

$$\Rightarrow \quad n > \frac{80 \times 9.81}{100^2 \times 1.8 \times 0.3 \times 1.2} \quad \text{nails}/cm^2$$

$$\boxed{n > 0.12\ \text{nails}/cm^2}$$

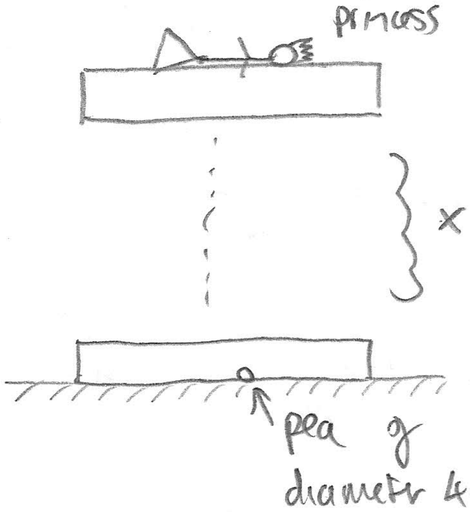
or if separation of nails is x cm

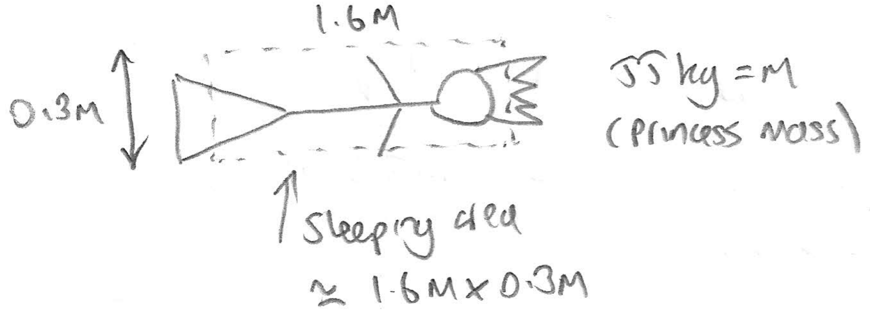
$$1/x^2 = n \Rightarrow \boxed{x = \frac{1}{\sqrt{n}}}$$

0.12 nails/cm^2

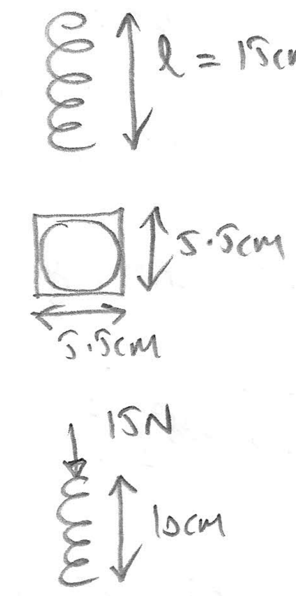
$\therefore \boxed{x = 2.87\,cm}$

2/





Pocket spring:



# of springs compressed in parallel are:

$$N = \frac{1.6 \times 0.3}{(5.5 \times 10^{-2})^2} = \boxed{159} \quad \text{(rounding up)}$$

so if $k = \frac{15N}{5cm} = 3N/cm$

Compression x is: $Nkx = mg$

for one mattress

$$x = \frac{mg}{Nk} = \frac{55 \times 9.81}{159 \times 3}$$

$$= \boxed{1.13 \text{ cm}}$$

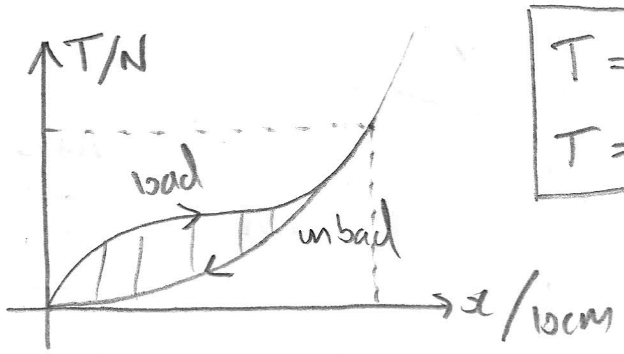
Now since we have 20 mattresses stacked (i.e. springs in series), the total change in height is $20 \times 1.13cm = \boxed{22.6 \text{ cm}}$

The pea diameter is $\times$ $\frac{4.0 \times 10^{-1} cm}{22.6 cm} \simeq \boxed{1.8\%}$ of this. This sounds like it could be detectable ... So what is the flaw here?

... The answer is the pea will make a very tiny difference to the compression of one spring, and the resulting upward force/unit area on the princess won't change much. The local pressure on her is what she 'measures'/feels, not changes to local displacement.

↳ However! Since all mattresses compress by 1.13 cm under the princesses weight, $\frac{0.4\text{cm}}{1.13\text{cm}} = \boxed{35\%}$ which is quite a difference, and independent of the # of mattresses! So perhaps she could feel the pea after all.....

3/



$$\boxed{\begin{aligned} T &= 4(x-2)^3 + 32 \\ T &= x^3 \end{aligned}}$$

load

unload

They intersect when $\boxed{x = 4, 0}$

$[\ 4(4-2)^3 + 32 = 64\ ,\quad 4^3 = 64\ ]$

[or via algebra:

$$4\left(x^3 + 3(-2)x^2 + 3(-2)^2 x + (-2)^3\right) + 32 = x^3$$

$$3x^3 - 24x^2 + 48x - 4\times 8 + 32 = 0$$

$$3x^3 - 24x^2 + 48x = 0$$

$$x^3 - 8x^2 + 16x = 0$$

$$x(x^2 - 8x + 16) = 0$$

$$x(x-4)^2 = 0 \quad \text{so} \quad \boxed{x = 0, 4}\ .\ ]$$

Hysteresis loss is: $\Delta E = \oint T\,dx = \int_0^4 \left(4(x-2)^3 + 32\right)dx - \int_0^4 x^3\,dx$

$$\int (x-2)^3\,dx = \frac{1}{4}(x-2)^4 + C$$

$$\text{So} \quad \Delta E = \left[(x-2)^4 + 32x - \tfrac{1}{4}x^4 \right]_0^4$$

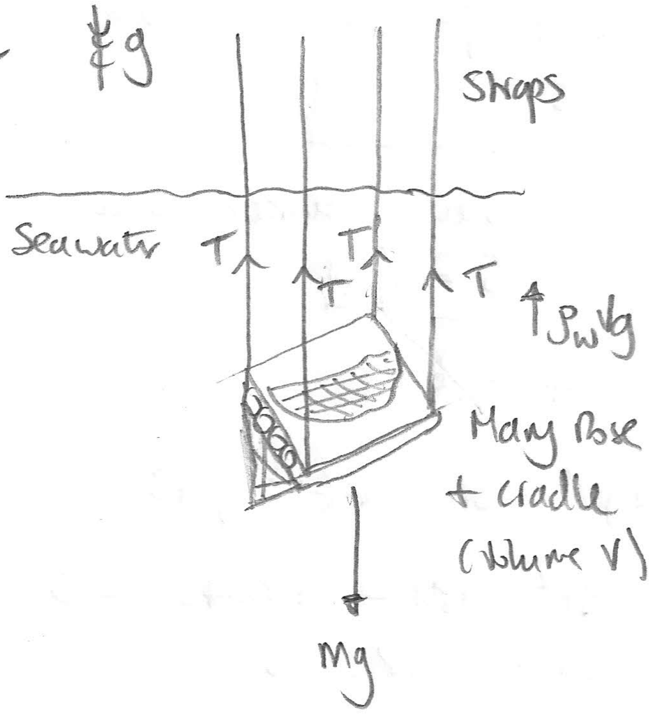
$$\Delta E = 2^4 + 32 \times 4 - \tfrac{1}{4} \times 4^4 - 2^4$$

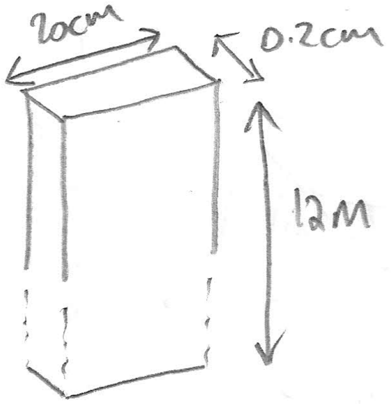
$$= 2^4 + 2^7 - 2^6 - 2^4$$

$$= 2^6(2-1) = \boxed{64}$$

Now x units are $0.1\,\text{m}$ so $\therefore \Delta E = \boxed{6.4\,\text{J}}$

4/ $\downarrow g$





Straps

$\boxed{Y = 196\,\text{GPa}}$

(i.e. probably woven steel?)

Assume Mary Rose lifted at constant speed i.e. in equilibrium!

when submerged, $4T + \underbrace{\rho_w V g}_{\text{upthrust}} = Mg$

Now $\rho = \frac{M}{V}$ and $\rho = \frac{3}{2}\rho_w$

so $V = \frac{M}{\rho} \Rightarrow V = \frac{M}{\frac{3}{2}\rho_w} \Rightarrow V = \frac{2}{3}\frac{M}{\rho_w}$

$$\therefore 4T + \rho_w \frac{2}{3}\frac{M}{\rho_w} g = Mg$$

$$\Rightarrow 4T = Mg\left(1 - \tfrac{2}{3}\right) \quad \therefore \boxed{T = \frac{Mg}{12}}$$

Now when not submerged: $\boxed{4T' = mg}$

let extensions be x (submerged) and x' (not submerged)

$$\frac{T'}{A} \Big/ \frac{x'}{l} = Y$$

where: $A = 20 \times 10^{-2} \text{m} \times 0.2 \times 10^{-2} \text{m}$

$\boxed{A = 4.0 \times 10^{-4} \text{ m}^2}$

$l = 12 \text{m}$

$Y = 196 \text{ GPa}$

so $\frac{T'}{A} = \frac{Yx'}{l}$ $\therefore$ $\boxed{x' = \frac{T'l}{AY}}$

Similarly: $\boxed{x = \frac{Tl}{AY}}$

Now $\Delta x = x' - x = \boxed{1.0 \times 10^{-3} \text{ m}}$

$$\Delta x = (T' - T)\, l/AY$$

$$\Delta x = \left(\tfrac{1}{4}mg - \tfrac{1}{12}mg\right) l/AY$$

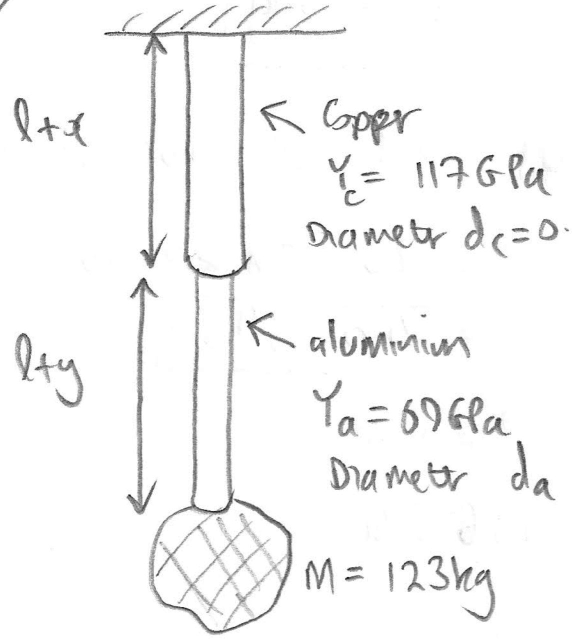
$$\Delta x = \frac{mg}{12}(3-1)\, l/AY$$

$$\Delta x = \tfrac{1}{6} mg\, l/AY \qquad \therefore \quad \boxed{m = \frac{6AY\Delta x}{gl}}$$

$$\therefore \quad m = \frac{6 \times 4.0 \times 10^{-4} \times 196 \times 10^{9} \times 10^{-3} \times 10}{9.81 \times 12}$$

$\boxed{m = 40 \text{ tonnes}}$ (i.e. about 6.7% of the fully laden warship of 600 tonnes).

5/



$g = 9.81$ N/kg

$l = 0.666$ m

$x + y = 0.1$ m (total stretch)

Tension in both wires is assumed to be the same.

so $T = mg$

$$\frac{T}{\pi\left(\frac{d_c}{2}\right)^2} = Y_c \frac{x}{l}$$

$$\frac{T}{\pi\left(\frac{d_a}{2}\right)^2} = Y_a \frac{y}{l}$$

$$\therefore \frac{4mg}{\pi}\left(\frac{1}{Y_c d_c^2} + \frac{1}{Y_a d_a^2}\right) = \frac{x+y}{l}$$

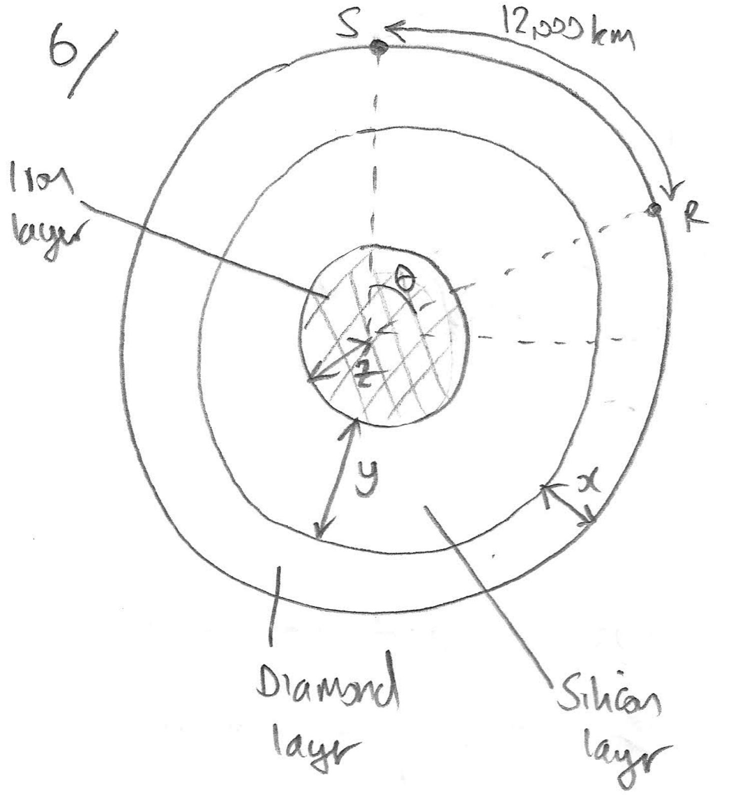
$$\therefore \frac{(x+y)\pi}{4mgl} - \frac{1}{Y_c d_c^2} = \frac{1}{Y_a d_a^2}$$

$$\therefore \boxed{d_a = \left(\frac{Y_a (x+y)\pi}{4mgl} - \frac{Y_a}{Y_c}\frac{1}{d_c^2}\right)^{-\frac{1}{2}}}$$

$$\therefore d_a = \left(\frac{69 \times 10^9 (0.1)\pi}{4 \times 123 \times 9.81 \times 0.666} - \frac{69}{117}\frac{1}{(0.4 \times 10^{-3})^2}\right)^{-\frac{1}{2}}$$

$$= \boxed{0.572 \text{ mm}}$$

6/



S Source of S waves
R receiver of S waves

Speed of S waves is

$$c = \sqrt{G/\rho}$$

In diamond:

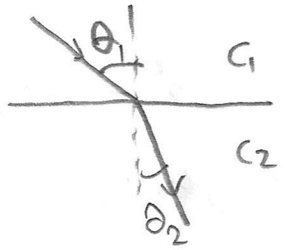
$$c_D = \sqrt{\frac{478 \times 10^9}{3510}} = \boxed{1.17 \times 10^4 \text{ m/s}}$$

In silicon:

$$c_S = \sqrt{\frac{65 \times 10^9}{2380}} = \boxed{5.23 \times 10^3 \text{ m/s}}$$

$x = 3000$ km
$y = 4000$ km
$z = 3000$ km
} ∴ radius $r = 10{,}000$ km ($\simeq 60\%$ more than Earth)

Snell's law is:



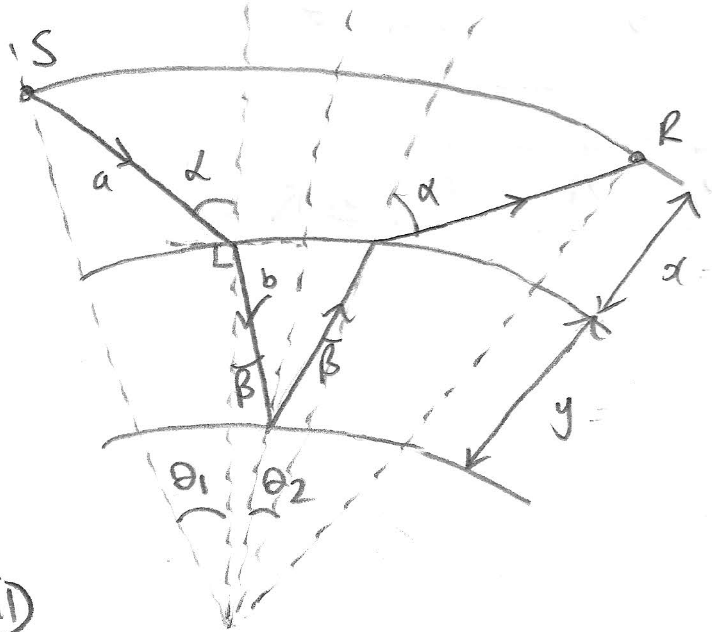
$$\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}$$

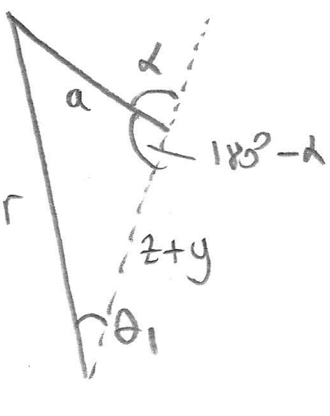
so if $c_2 < c_1$ as in our case $\Rightarrow$ $\theta_2 = \sin^{-1}\left(\frac{c_2}{c_1}\sin\theta_1\right)$ is smaller than θ_1

Travel time $S \to R$ is

$$\boxed{\Delta t = \frac{2a}{c_D} + \frac{2b}{c_S}}$$

Need to find a, b

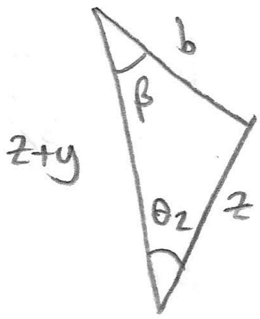




(11)

Sine rule: $\dfrac{a}{\sin\theta_1} = \dfrac{r}{\sin(180^\circ-\alpha)}$ (1)

Similarly:



$\dfrac{b}{\sin\theta_2} = \dfrac{z}{\sin\beta}$ (2)

Now $2(\theta_1+\theta_2) = \theta$ (3) where $r\theta = 12{,}000$ km

ie $\theta = \dfrac{12}{10}$ radians

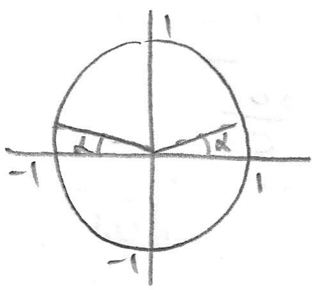
$\approx \boxed{68.8^\circ}$

From Snell: $\dfrac{\sin\alpha}{c_0} = \dfrac{\sin\beta}{c_s}$ (4)

Also using the cosine rule: $b^2 = (z+y)^2 + z^2 - 2z(z+y)\cos\theta_2$

and $a^2 = r^2 + (z+y)^2 - 2r(z+y)\cos\theta_1$

$\sin(180-\alpha) = \sin\alpha$ so (1) is: $\dfrac{a}{\sin\theta_1} = \dfrac{r}{\sin\alpha}$



(1)/(2) $\dfrac{a}{\sin\theta_1}\dfrac{\sin\theta_2}{b} = \dfrac{r}{\sin\alpha}\dfrac{\sin\beta}{z}$

$\therefore \boxed{\dfrac{a}{b}\dfrac{\sin\theta_2}{\sin\theta_1} = \dfrac{r}{z}\dfrac{c_s}{c_0}}$ (4)

So $\dfrac{a^2}{b^2}\dfrac{(1-\cos^2\theta_2)}{1-\cos^2\theta_1} = \dfrac{r^2}{z^2}\left(\dfrac{c_s}{c_0}\right)^2$

$\therefore$ $$\frac{r^2+(z+y)^2-2r(z+y)\cos\theta_1}{(z+y)^2+z^2-2z(z+y)\cos\theta_2}\;\frac{1-\cos^2\theta_2}{1-\cos^2\theta_1} = \frac{r^2}{z^2}\left(\frac{c_s}{c_0}\right)^2 \quad (*)$$

Now $\boxed{\theta_1 = \theta/2-\theta_2}$, so we need to solve $(*)$ for θ_2.
Then we can find θ_1 and hence a, b (and α, β too if we wish).

The resulting cubic equation in $\cos\theta_2$ is going to be unwieldy in general, so let us substitute the (nice) r, z, y, x values.

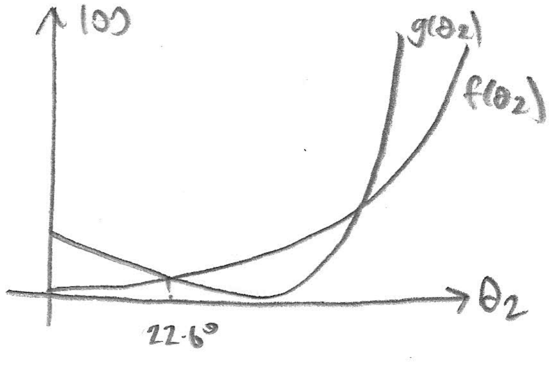
$$\frac{10^2+(3+4)^2-2(10)(7)\cos\left(\frac{\theta}{2}-\theta_2\right)}{(3+4)^2+3^2-2(3)(3+4)\cos\theta_2}\;\frac{1-\cos^2\theta_2}{1-\cos^2\left(\frac{\theta}{2}-\theta_2\right)} = \frac{100}{9}\left(\frac{c_s}{c_0}\right)^2$$

let $\dfrac{100}{9}\left(\dfrac{c_s}{c_0}\right)^2 = k.$ $\quad (k \simeq 2.22)$

$$\frac{100+49-140\cos\left(\theta/2-\theta_2\right)}{7^2+3^2-42\cos\theta_2}\;\frac{1-\cos^2\theta_2}{1-\cos^2\left(\theta/2-\theta_2\right)} = k$$

$$\underbrace{\left(149-140\cos\left(\tfrac{\theta}{2}-\theta_2\right)\right)\left(1-\cos^2\theta_2\right)}_{f(\theta_2)} = \underbrace{k\left(58-42\cos\theta_2\right)\left(1-\cos^2\left(\tfrac{\theta}{2}-\theta_2\right)\right)}_{g(\theta_2)}$$

Solve numerically $f(\theta_2) = g(\theta_2)$.



$\Rightarrow \boxed{\theta_2 \simeq 22.6°}$

$\Rightarrow \boxed{\begin{array}{l} a = 3453\ \text{km} \\ b = 4386\ \text{km}\end{array}} \Rightarrow \boxed{\Delta t = 2.27\,\text{s}}$