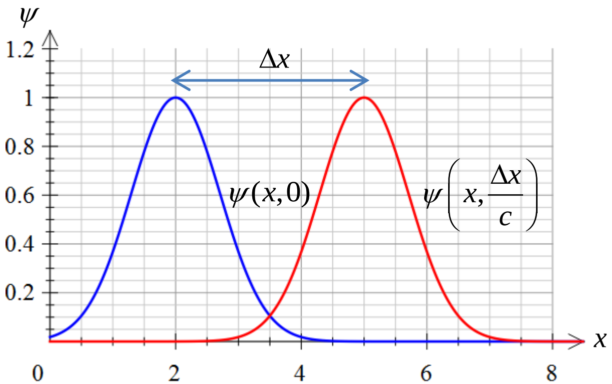
A Mathematical Anatomy of Waves

A *wave* is essentially a *disturbance* that propagates at a fixed velocity through space. The disturbance could be in gas pressure, movement of a string under tension, ground or water movement, or indeed fluctuations in electromagnetic fields which constitute light, radio waves, X-rays etc. Waves are a very general phenomena in Physics, and this handout describes characteristics which are germane to all waves.

Consider a disturbance of **amplitude** $\psi(x,t)$ moving at **speed** c in the x direction



The key feature of a wave is that is a *spatial translation* of a disturbance *f(x)* as time progresses. *There may also be a decay (or growth) of amplitude with time, but this process shall be modelled separately.*

$$\boxed{\psi(x,t)=f(x-ct)}$$

We can differentiate this to form the **Wave Equation**

$$\psi(x,t)=f(x-ct)$$

$$z=x-ct \quad \therefore \psi=f(z)$$

$$\frac{\partial\psi}{\partial t}=\frac{df}{dz}\frac{\partial z}{\partial t}=-c\frac{df}{dz}$$

i.e. using the Chain Rule

$$\frac{\partial^2\psi}{\partial t^2}=\frac{d}{dz}\left(\frac{\partial\psi}{\partial t}\right)\frac{\partial z}{\partial t}$$

$$\frac{\partial^2\psi}{\partial t^2}=\frac{d}{dz}\left(-c\frac{df}{dz}\right)(-c)=c^2\frac{d^2f}{dz^2}$$

$$\frac{\partial\psi}{\partial x}=\frac{df}{dz}\frac{dz}{dx}=\frac{df}{dz}\times 1=\frac{df}{dz}$$

$$\frac{\partial^2\psi}{\partial x^2}=\frac{d}{dz}\left(\frac{\partial\psi}{\partial x}\right)\frac{\partial z}{\partial x}$$

$$\frac{\partial^2\psi}{\partial x^2}=\frac{d}{dz}\left(\frac{df}{dz}\right)\times 1=\frac{d^2f}{dz^2}$$

$$\therefore \frac{d^2f}{dz^2}=\boxed{\frac{\partial^2\psi}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}}$$

Also: $\boxed{\left(\frac{\partial\psi}{\partial x}\right)^2=\frac{1}{c^2}\left(\frac{\partial\psi}{\partial t}\right)^2}$

All *periodic* disturbances can be constructed from a summation (or 'superposition') of sine or cosine functions or different *amplitude*, *phase shift* and *frequency*. This is called a Fourier series.

Note argument , or *phase*, of cosine function is in *radians*

$$\boxed{\psi(x,t)=A\cos(kx-\omega t-\phi_0)}$$

$$f(x)=A\cos\left(2\pi\frac{x}{\lambda}-\phi_0\right)$$

$$f(x-ct)=A\cos\left(2\pi\frac{x-ct}{\lambda}-\phi_0\right)$$

$$f(x-ct)=A\cos(kx-\omega t-\phi_0)$$

$\phi=kx-\omega t-\phi_0$

Phase of waveform

A **Amplitude**

ϕ_0 Phase when x and t = 0

λ **Wavelength** $\quad k=\frac{2\pi}{\lambda}$ **Wavenumber**

T **Period** of the wave is the time taken for the wave to move one wavelength

$f=\frac{1}{T}$ **Frequency** is the number of periods (or oscillations) per second

$c=f\lambda$ The wave moves one wavelength per period which gives this formula for **wave speed**

$\omega=2\pi f$ To simplify we can remove the 2 x π factors by defining an **'angular speed'** (this is the same formula for angular speed in *circular motion*, which has similar mathematics)

$\omega=ck$ Alternative **wave speed equation**

It is often convenient to use *complex numbers* to represent wave phenomena. Using *De-Moivre's Theorem:*

$$\boxed{\begin{aligned}&A\cos\phi=\mathrm{Re}\left(Ae^{i\phi}\right)\\&\phi=kx-\omega t-\phi_0\end{aligned}}$$

A wave might be therefore written as:

$\boxed{\psi(x,t)=ae^{i(kx-\omega t)}}$ The constant *a* might also be complex to incorporate the initial phase shift

$$\psi=\cos(kx-\omega t-\phi_0)$$

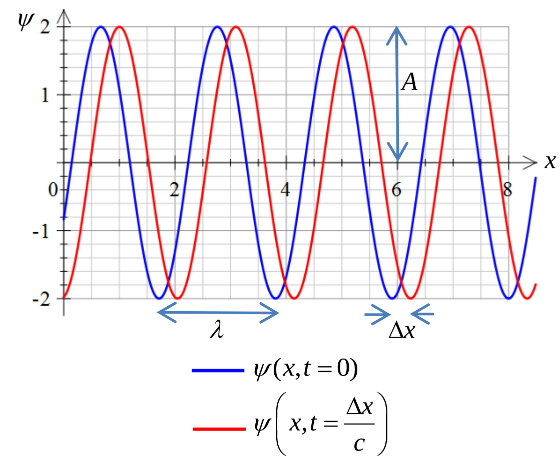
$$\frac{\partial^2\psi}{\partial t^2}=-\omega^2\cos(kx-\omega t-\phi_0)=-\omega^2\psi$$

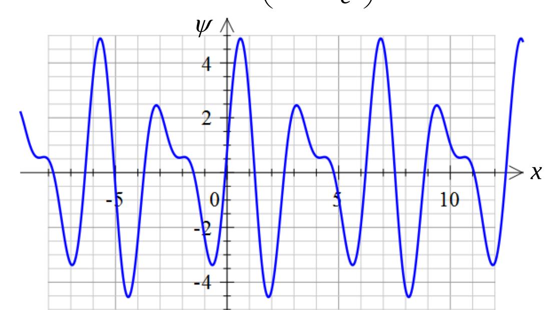
$$\frac{\partial^2\psi}{\partial x^2}=-k^2\cos(kx-\omega t-\phi_0)=-k^2\psi$$

$$\therefore \psi=\frac{1}{k^2}\frac{\partial^2\psi}{\partial x^2}=\frac{1}{\omega^2}\frac{\partial^2\psi}{\partial t^2}$$

$$\omega=ck \Rightarrow \boxed{\frac{\partial^2\psi}{\partial x^2}=\frac{1}{c^2}\frac{\partial^2\psi}{\partial t^2}}$$

Satisfies the **wave equation**

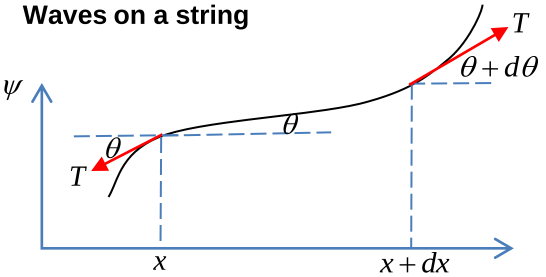




$$\psi=2\cos(3x-2)+3\cos(2x-1)$$

A *superposition* of two cosine functions of different amplitude, frequency and initial phase shift can be used to create any *periodic* (i.e. time repeating) signal

Waves on a string



Consider an infinitesimally small section of a string of mass per unit length μ under constant tension T

$\theta \ll 1$ so string length $\approx dx$ and $\theta \approx \sin\theta \approx \tan\theta$

θ in radians

By *Newton's Second Law:*

mass x acceleration — Sum of forces in vertical direction

$$\mu dx \frac{\partial^2 \psi}{\partial t^2} \approx T\sin(\theta + d\theta) - T\sin\theta$$

$$\therefore \frac{\mu}{T}\frac{\partial^2 \psi}{\partial t^2} \approx \frac{\tan(\theta + d\theta) - \tan\theta}{dx}$$ i.e. $\sin\theta \approx \tan\theta$

$$\tan\theta = \frac{\partial \psi}{\partial x}$$

$$\frac{\tan(\theta + d\theta) - \tan\theta}{dx} = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial x}\right) = \frac{\partial^2 \psi}{\partial x^2}$$

Hence: $$\frac{\partial^2 \psi}{\partial x^2} = \frac{\mu}{T}\frac{\partial^2 \psi}{\partial t^2}$$

Comparing with the Wave Equation: $$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}$$

$$\boxed{c = \sqrt{\frac{T}{\mu}}}$$ Speed of waves on a string

Guitar strings

The fundamental frequencies associated with musical notes can be modelled using the following equation:

$$\boxed{f(n) = 110 \times 2^{\frac{1}{12}n}}$$

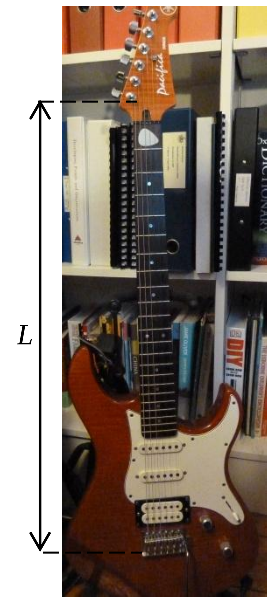
This is a good approximation of the Pythagorean 'harmonious proportions' e.g. a perfect 5th interval like A to E, is a frequency ratio of 3/2. 2^(7/2) = 1.4983.

where f is the frequency in Hz and n is the number of *semitones* above the A note, set to be 110Hz.

	A	A# or Bb	B	C	C# or Db	D	D# or Eb	E	F	F# or Gb	G	G# or Ab	A (next octave)
n	0	1	2	3	4	5	6	7	8	9	10	11	12
Hz	110	116.5	123.5	130.8	138.6	146.8	155.6	164.8	174.6	185.0	196	207.7	220

String tensions can be calculated if one knows the geometry of each string, the material density and the desired note frequency. The following results correspond to D'Addario EXL110 electric guitar strings.
(Density data from https://courses.physics.illinois.edu/phys406/Student_Projects/Fall00/DAchilles/Guitar_String_Tension_Experiment.pdf)

Note	n	Frequency /Hz	Diameter (inches)	String length /m	String density /kgm^{-3}	Tension /N
E	19	329.63	0.010 plain	0.75	7690	95.3
B	14	246.94	0.013 plain	0.73	7950	88.5
G	10	196	0.017 plain	0.71	8220	93.2
D	5	146.83	0.026 wound	0.69	6930	97.5
A	0	110	0.036 wound	0.67	6610	94.3
E	-5	82.41	0.046 wound	0.66	6540	83.0



Assume *fundamental* mode of string vibration i.e. two nodes at either end of the string

$$c = f\lambda = \sqrt{\frac{T}{\mu}}$$

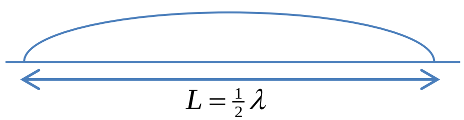
$$\therefore T = 4f^2L^2\mu$$

$$\rho = \frac{\mu L}{L\pi\left(\frac{1}{2}d\right)^2} = \frac{4\mu}{\pi d^2}$$

$$\therefore \mu = \tfrac{1}{4}\rho\pi d^2$$

$$\therefore T = 4f^2L^2\tfrac{1}{4}\rho\pi d^2$$

$$\boxed{T = \pi\rho\left(fLd\right)^2}$$



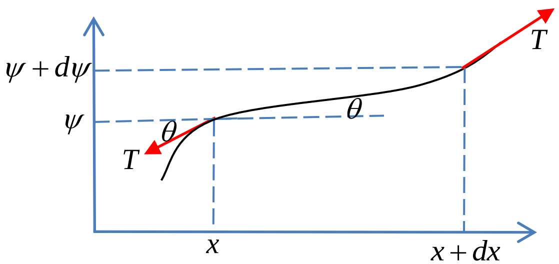
A wave which has fixed nodes, i.e. does not propagate in the x direction, is called a **standing wave.**
Standing waves can be written as $\psi(x) = \Phi(x)\tau(t)$ i.e. the time dependent part is separated.

For the guitar string fundamental mode: $\boxed{\psi(x,t) = \sin(\pi x / L)\sin(2\pi f t)}$

Plucking a real guitar string activates many more *harmonics*, which gives it its distinctive sound.

1 inch = 0.0254m

Wave energy and power



If a wave on a string under constant tension T propagates from x to $x + dx$

Work done is:

$$dW = T\left(\sqrt{(d\psi)^2 + (dx)^2} - dx\right)$$ i.e. force x extension

$$dW = Tdx\left(\sqrt{1+\left(\frac{\partial \psi}{\partial x}\right)^2} - 1\right)$$ Assume a small amplitude disturbance $\frac{\partial \psi}{\partial x} \ll 1$

$$dW \approx Tdx\left(1+\tfrac{1}{2}\left(\frac{\partial \psi}{\partial x}\right)^2 - 1\right)$$ First term Binomial expansion

Hence **potential energy per unit length** of string is:

$$\frac{dW}{dx} = \boxed{E_P = \tfrac{1}{2}T\left(\frac{\partial \psi}{\partial x}\right)^2}$$

Kinetic energy per unit length is:

$$\boxed{E_K = \tfrac{1}{2}\mu\left(\frac{\partial \psi}{\partial t}\right)^2}$$ μ mass per unit length of string

Total energy per unit length is therefore

$$E = E_P + E_K$$

$$E = \tfrac{1}{2}\left(T\left(\frac{\partial \psi}{\partial x}\right)^2 + \mu\left(\frac{\partial \psi}{\partial t}\right)^2\right)$$

Now from the Wave Equation

$$\boxed{\left(\frac{\partial \psi}{\partial x}\right)^2 = \frac{\mu}{T}\left(\frac{\partial \psi}{\partial t}\right)^2} \longleftarrow c = \sqrt{\frac{T}{\mu}}$$

Hence:

$$E = \tfrac{1}{2}\left(T\frac{\mu}{T}\left(\frac{\partial \psi}{\partial t}\right)^2 + \mu\left(\frac{\partial \psi}{\partial t}\right)^2\right)$$

$$\boxed{E = \mu\left(\frac{\partial \psi}{\partial t}\right)^2}$$

If we consider a *harmonic* wave
$\psi = A\cos(kx - \omega t - \phi_0)$

Average energy per unit length is

$$\bar{E} = \tfrac{1}{T}\int_0^T \mu\left(\frac{\partial \psi}{\partial t}\right)^2 dt$$

$$\bar{E} = \frac{\mu A^2\omega^2}{T}\int_0^T \sin^2(kx - \omega t - \phi_0)dt$$

$$\boxed{\bar{E} = \tfrac{1}{2}\mu A^2\omega^2}$$

After time *dt* an extra *cdt* is oscillating.
Hence average **input power** to wave is:

$$P = \frac{\bar{E}cdt}{dt} = \boxed{\tfrac{1}{2}\mu A^2\omega^2 c}$$

Wave **impedance** can be defined as

$$\boxed{Z = \left|\frac{\text{driving force}}{\text{velocity}}\right|}$$

For the wave on a string

$$Z = \left|\frac{T\sin\theta}{\frac{\partial \psi}{\partial t}}\right|$$

$$\theta \ll 1 \therefore \theta \approx \sin\theta \approx \tan\theta = \frac{\partial \psi}{\partial x}$$ i.e. assume small oscillations

$$\therefore Z^2 = T^2\frac{\left(\frac{\partial \psi}{\partial x}\right)^2}{\left(\frac{\partial \psi}{\partial t}\right)^2} = T^2\frac{\frac{\mu}{T}\left(\frac{\partial \psi}{\partial t}\right)^2}{\left(\frac{\partial \psi}{\partial t}\right)^2} = \mu T$$

$$\boxed{Z = \sqrt{\mu T}}$$

Impedance of a wave on a string

$$T = \mu c^2 \longleftarrow c = \sqrt{\frac{T}{\mu}}$$

$$\therefore Z^2 = \mu^2 c^2$$

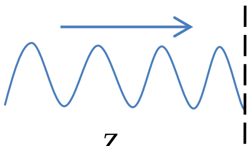
$$\boxed{Z = \mu c}$$

Hence wave **power** is:

$$\boxed{P = \tfrac{1}{2}ZA^2\omega^2}$$

This is a *general* result for wave phenomena

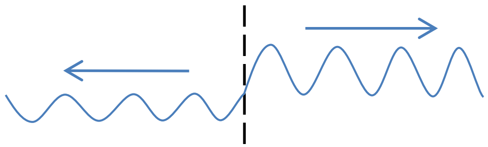
Reflection and transmission of waves on boundaries



Z_1 $\qquad$ Z_2

Incident wave

$\psi_I = Ie^{i(k_1 x - \omega t)}$



Reflected wave

$\psi_R = Re^{i(-k_1 x - \omega t)}$

Z_1

Transmitted wave

$\psi_T = Te^{i(k_2 x - \omega t)}$

Z_2

Note the number of wave per second must be the *same* on either side of the boundary

$\omega_1 = \omega_2 = \omega$

However, the wave speed (and hence wave impedance) changes, which will modify the wavelength and hence the wavenumber.

$Z = \sqrt{\mu T}$ Impedance of a wave on a string

$\frac{Z_1}{Z_2} = \sqrt{\frac{\mu_1}{\mu_2}}$ Expect tension to be continuous across the boundary (i.e. same on both sides)

$c = \sqrt{\frac{T}{\mu}}$ Wave speed of string under tension T with mass per unit length μ

$\therefore \frac{c_1}{c_2} = \sqrt{\frac{\mu_2}{\mu_1}} = \frac{Z_2}{Z_1}$ General result relating **wave speed** and **impedance** $\boxed{\frac{c_1}{c_2} = \frac{Z_2}{Z_1}}$

$k = \frac{2\pi}{\lambda}$

$\omega = 2\pi f$

$c = f\lambda \Rightarrow \omega = ck$

$\therefore k = \frac{\omega}{c}$

$\therefore \frac{k_1}{k_2} = \frac{c_2}{c_1} = \frac{Z_1}{Z_2} \leftarrow \frac{c_1}{c_2} = \sqrt{\frac{\mu_2}{\mu_1}} = \frac{Z_2}{Z_1}$

Note: This analysis has been done for a wave on a tensioned string, but the end results are quite general

Let us assume the wave amplitude, and its gradient $\frac{\partial \psi}{\partial x}$ are *continuous* across the boundary

$I = R + T$

$k_1 I = -k_1 R + k_2 T$

$\frac{k_1}{k_2} I = -\frac{k_1}{k_2} R + T$

$T = I - R$

$\therefore \frac{k_1}{k_2} I = -\frac{k_1}{k_2} R + I - R$

$R\left(\frac{Z_1}{Z_2} + 1\right) = I\left(\frac{Z_1}{Z_2} - 1\right)$

$\boxed{\frac{R}{I} = \frac{Z_1 - Z_2}{Z_2 + Z_1}}$ **Reflection coefficient**

$\therefore T = \left(1 - \frac{Z_1 - Z_2}{Z_2 + Z_1}\right) I$

$\boxed{\frac{T}{I} = \frac{2Z_1}{Z_2 + Z_1}}$ **Transmission coefficient**

Since wave power is proportional to the *square* of wave amplitude

$\boxed{P = \tfrac{1}{2} Z A^2 \omega^2}$

Reflected power / incident power

$\boxed{\frac{P_R}{P_I} = \left|\frac{Z_1 - Z_2}{Z_1 + Z_2}\right|^2}$

Writing the ratios in this way enables the formulae to be extensible to complex impedances, which result in oscillating electrical circuits

Transmitted power / incident power

$\boxed{\frac{P_T}{P_I} = 1 - \left|\frac{Z_1 - Z_2}{Z_1 + Z_2}\right|^2}$

Limiting cases

$Z_2 \gg Z_1$

$R = -I$

$T = 0$

i.e. the reflected wave is *inverted*

$Z_2 = Z_1$

$R = 0$

$T = I$

i.e. we don't expect any reflections unless we have an impedance change! This explains why electrical connectors are ideally *impedance matched.*