Matrices and matrix transformations test Name: Dr French

Date: 23/01/2012

Write all questions on this paper. Calculator allowed but probably not really necessary.

Time: 35 minutes. TOTAL MARKS = 50

1. Multiply out the following matrices [2 marks]

(i) $\begin{pmatrix} -1 & 3 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} 7 & 2 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} -7 & -5 \\ 35 & 8 \end{pmatrix}$

(ii) $\begin{pmatrix} 5 & -2 & 1 \\ 0 & 3 & -4 \\ 3 & 2 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 2 \\ 4 & -1 & 0 \\ -2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} -5 & 5 & 11 \\ 20 & -15 & -4 \\ 13 & -5 & 5 \end{pmatrix}$ [3 marks]

(iii) $\begin{pmatrix} -3 & 2 \\ 1 & -4 \end{pmatrix}^{-1} = \frac{1}{12-2}\begin{pmatrix} -4 & -2 \\ -1 & -3 \end{pmatrix} = \frac{-1}{10}\begin{pmatrix} 4 & 2 \\ 1 & 3 \end{pmatrix}$ [3 marks]

$= \begin{pmatrix} -0.4 & -0.2 \\ -0.1 & -0.3 \end{pmatrix}$

(iv) $\begin{pmatrix} -4 & 2 \\ 7 & 3 \end{pmatrix}^3 = \begin{pmatrix} -4 & 2 \\ 7 & 3 \end{pmatrix}\begin{pmatrix} 30 & -2 \\ -7 & 23 \end{pmatrix} = \begin{pmatrix} -134 & 54 \\ 189 & 55 \end{pmatrix}$ [3 marks]

Since $\begin{pmatrix} -4 & 2 \\ 7 & 3 \end{pmatrix}\begin{pmatrix} -4 & 2 \\ 7 & 3 \end{pmatrix} = \begin{pmatrix} 30 & -2 \\ -7 & 23 \end{pmatrix}$

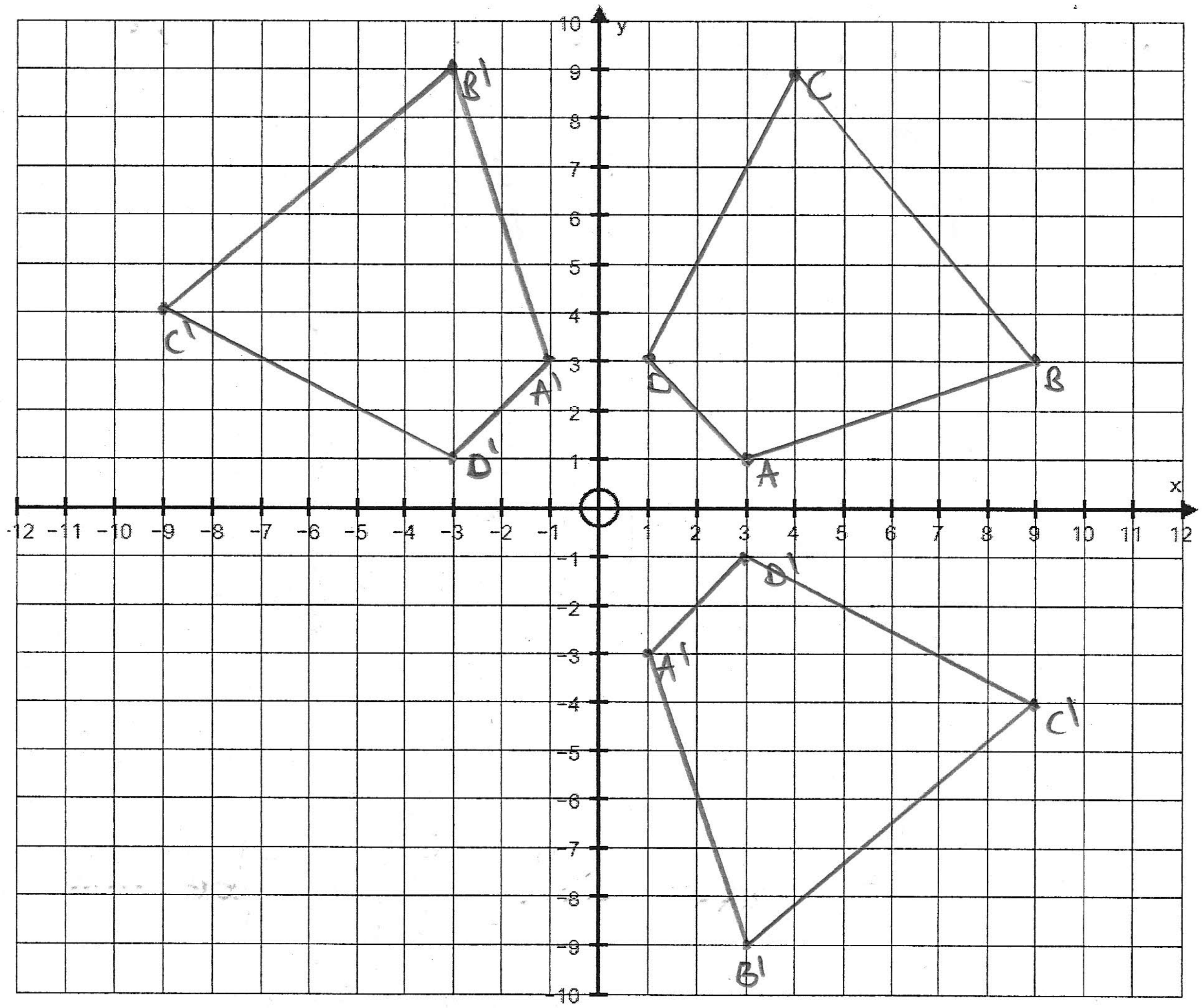
(v) $\begin{pmatrix} -3 & 2 \\ 1 & -4 \end{pmatrix}^{-1}\begin{pmatrix} -4 & 2 \\ 7 & 3 \end{pmatrix}^{-1}\begin{pmatrix} -4 & 2 \\ 7 & 3 \end{pmatrix}\begin{pmatrix} -3 & 2 \\ 1 & -4 \end{pmatrix}$ [2 marks]

$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

2. Vertices A,B,C,D of a shape in (x,y) coordinates are collected together in a 2 x 4 matrix

$$\mathbf{V} = \begin{pmatrix} 3 & 9 & 4 & 1 \\ 1 & 3 & 9 & 3 \end{pmatrix}$$

(a) Plot the shape on the axes below and label all vertices [2 marks]



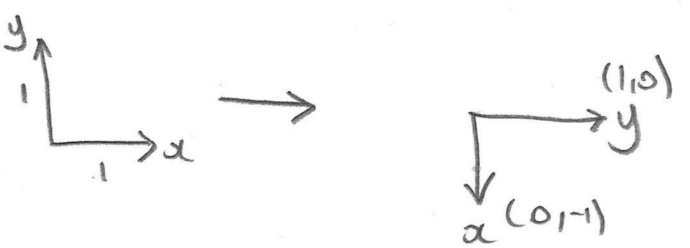
(b) A matrix transformation has form

$$\mathbf{T} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

(i) Work out what $\mathbf{T}^2$ is as a 2 x 2 matrix. By considering where the basis vectors ("red") (1,0) and ("blue") (0,1) go under this transformation, describe the transformation. You are expected to draw where the "red" and "blue" basis vectors go as part of your answer.

$$\underline{\underline{T}}^2 = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

[5 marks]



$\boxed{\underline{\underline{T}}^2 \text{ is a: Rotation of } 90^\circ \text{ clockwise about } (0,0)}$

$$\underline{\underline{T}}^2 V = \begin{pmatrix}0&1\\-1&0\end{pmatrix}\begin{pmatrix}3&9&4&1\\1&3&9&3\end{pmatrix} = \begin{pmatrix}1&3&9&3\\-3&-9&-4&-1\end{pmatrix}$$

(ii) Plot $\mathbf{T^2V}$ on the graph above. Does this agree with your answer to part (i) ? Yes! [2 marks]

(iii) Find $(\mathbf{T}^2)^{-1}\mathbf{V}$ [3 marks]

$$(\underline{\underline{T}}^2)^{-1} = \frac{1}{1}\begin{pmatrix}0&-1\\1&0\end{pmatrix} = \begin{pmatrix}0&-1\\1&0\end{pmatrix}$$

$$\therefore (\underline{\underline{T}}^2)^{-1}\underline{V} = \begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}3&9&4&1\\1&3&9&3\end{pmatrix} = \begin{pmatrix}-1&-3&-9&-3\\3&9&4&1\end{pmatrix}$$

(iv) Plot $(\mathbf{T}^2)^{-1}\mathbf{V}$ on the plot above and therefore describe this transformation [3 marks]

Rotation of 90° anticlockwise about (0,0)

(v) Without evaluating the matrices, describe the transformations

(i) $\mathbf{T}$ [2 marks]

45° rotation clockwise about (0,0)

(ii) $\mathbf{T}^3$ [2 marks]

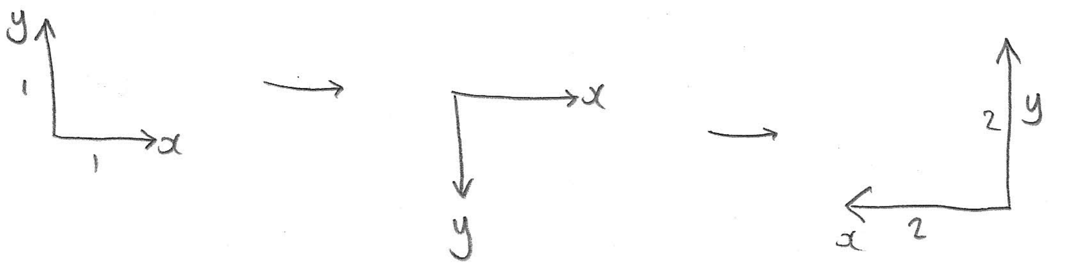
45° × 3 = 135° clockwise rotation about (0,0)

(iii) $\mathbf{T}^{-0.128}$ [3 marks]

45° × 0.128 = 5.76° anticlockwise rotation about (0,0)

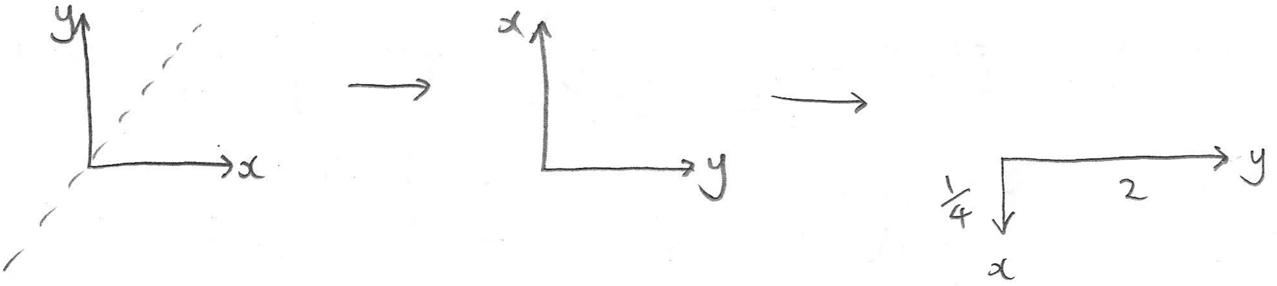
3) Using 'red and blue basis vector diagrams' derive matrices which represent the following transformations

(i) Reflection in the x-axis, then enlargement scale factor -2 about the origin [3 marks]



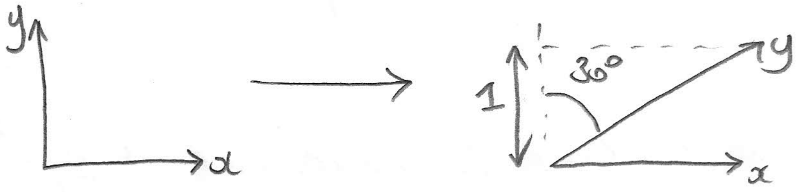
$$\begin{pmatrix}-2&0\\0&2\end{pmatrix}$$

(ii) Reflection in the y=x, then a stretch of scale factor 2 parallel to the x axis and -0.25 parallel to the y axis. [3 marks]



$$\begin{pmatrix} 0 & 2 \\ \frac{1}{4} & 0 \end{pmatrix}$$

(iii) A shear parallel to the x axis of angle 30 degrees from the y axis. Bonus mark if the answer is written in surd form. [3 marks]



$$\begin{pmatrix} 1 & \tan 30^\circ \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{\sqrt{3}} \\ 0 & 1 \end{pmatrix}$$

4. A graphic design company has asked you to write a computer program which will automatically rotate digital photographs and line-art.

For artistic purposes they would like to rotate about *any central point* rather than the origin.

YOUR TASK!

A shape with coordinates stored as column vectors in a matrix $\mathbf{V}$ needs to be rotated anticlockwise by ninety degrees about centre (2,3). The rotated shape coordinates are stored in matrix $\mathbf{V}'$. **(SEE FIGURE 1)**

Use an appropriate method to find out the following matrices to achieve this process via the following equation

$$\mathrm{V}' = \mathrm{A}(\mathrm{V}-\mathrm{B})+\mathrm{C}$$

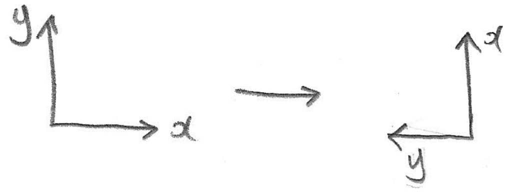
The matrices must apply to the following test coordinates

$$\mathbf{V} = \begin{pmatrix} 2 & 3 & 2.6 & 1.4 & 1.1 \\ 4 & 3.1 & 2.2 & 2.2 & 3.3 \end{pmatrix}$$

A good start is to find out what matrix rotates coordinates anticlockwise by ninety degrees, *about the origin.* What could we do to effectively change our origin? What other sort of transformation do we need?

Write workings and ideas here: [6 marks]

Anticlockwise rotation of $90°$



$\therefore \underline{A} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

$$\underline{B} = \underline{C} = \begin{pmatrix} 2 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 3 & 3 \end{pmatrix}$$

If $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$ was the origin we could rotate by $90°$ about it via multiplication of $\underline{V}$ by $\underline{A}$ $(\underline{A}\,\underline{V})$

To make this the case, shift by $-\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. After transformation is applied, shift back by $\begin{pmatrix} 2 \\ 3 \end{pmatrix}$. To apply this shift to every vector in $\underline{V}$ we need five copies in $\underline{B} = \underline{C}$

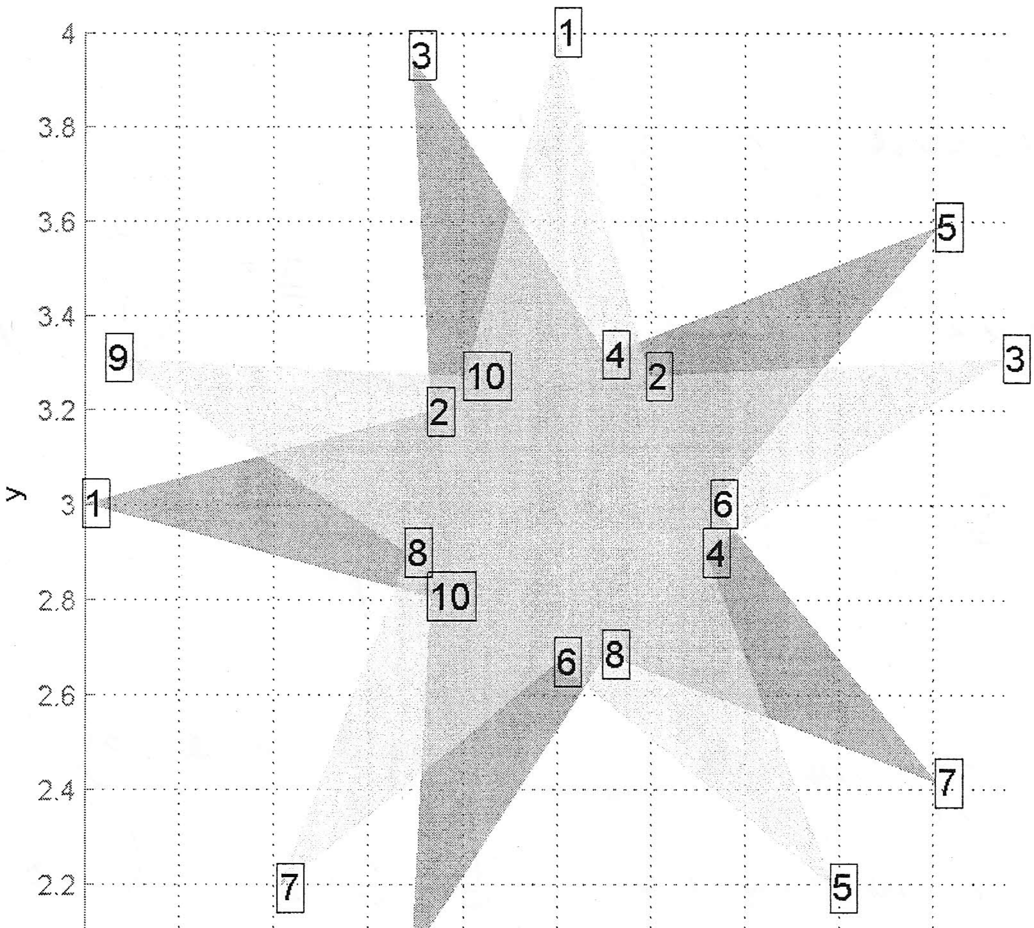


Figure 1: Rotation about (2,3) anticlockwise by 90 degrees