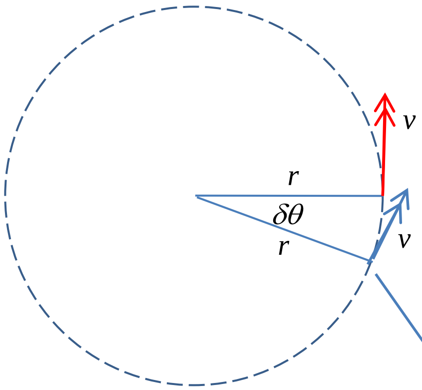
Circular motion. For simplicity let us consider *motion in a circle at a constant velocity*. Hence the *angular velocity* ω is constant and rotating period is T

$$\omega = \frac{2\pi}{T}$$

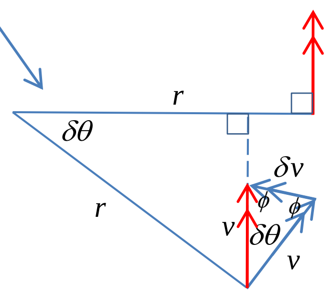
The rotation frequency is $f = \frac{1}{T}$ $\therefore$ $\omega = 2\pi f$

Note 1 revolutions per minute (RPM) = $\frac{2\pi}{60} \approx 0.105\text{rads}^{-1}$



Velocity $v = \frac{2\pi r}{T} = r\omega$

Consider an infinitesimally small sector of the circular path drawn downwards from the horizontal



$180^0 = 2\phi + \delta\theta$

$\phi = 90^0 - \frac{1}{2}\delta\theta$

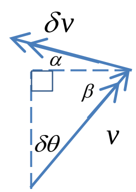
$\beta = 90^0 - \delta\theta$

$\alpha + \beta = \phi$

$\alpha = \phi - \beta$

$\alpha = 90^0 - \frac{1}{2}\delta\theta - 90^0 + \delta\theta$

$\alpha = \frac{1}{2}\delta\theta$



To execute a circular motion, the velocity vector must continuously change in direction. Although its magnitude is not changing, the directional part is, so therefore a body undergoing circular motion must be accelerating since ***acceleration is the rate of change of velocity***, and both are vector quantities.

$$\mathbf{a} = \frac{d\mathbf{v}}{dt}$$

The change of velocity vector δv points at an angle of $\frac{1}{2}\delta\theta$ from the horizontal
Hence the **acceleration must point towards the centre of the circle**

Now since the velocity vectors either side of the sector form an isosceles triangle

$\delta v = 2v\sin\frac{1}{2}\delta\theta$

The time taken to traverse the arc is $\delta t = \frac{r\delta\theta}{v}$

Hence the magnitude of the acceleration is $a = \frac{\delta v}{\delta t} = \frac{2v^2\sin\frac{1}{2}\delta\theta}{r\delta\theta}$

In the limit when $\delta\theta$ is small $\sin\frac{1}{2}\delta\theta \approx \frac{1}{2}\delta\theta$

Hence: $a = \frac{v^2}{r}$

This is known as '**centripetal acceleration**'

In summary, for circular motion of *constant angular frequency and fixed radius*:

$\omega = \frac{2\pi}{T}$, $\omega = 2\pi f$

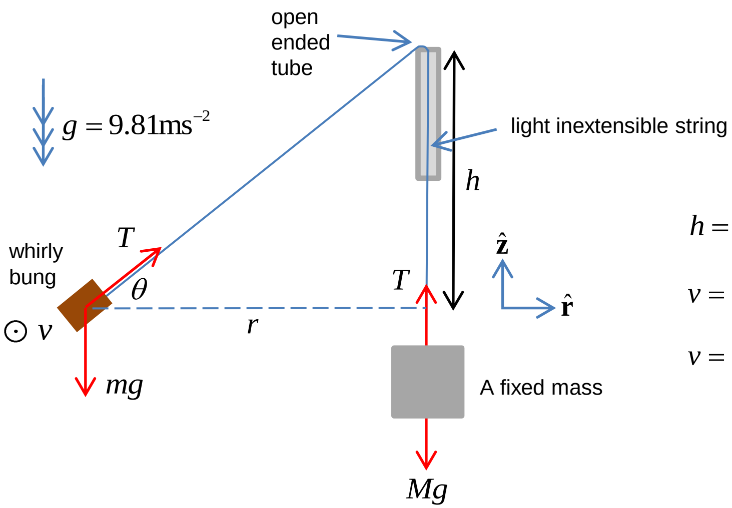
$v = r\omega$

$a = \frac{v^2}{r}$, $a = r\omega^2$

$1\text{RPM} = \frac{\pi}{30} \approx 0.105\text{rads}^{-1}$

Whirly Bungs experiment

This simple experiment is an example of horizontal circular motion. A rubber bung, attached to a fixed mass via a light inextensible string threaded through an open ended tube, is whirled in a horizontal circle until 'dynamic equilibrium' is obtained. i.e. the radius of circular motion is *constant*, as is the angle of inclination.



$h = r\tan\theta$

$v = r\omega$

$v = \dfrac{2\pi r}{P}$

Time for one rotation i.e. 'period'

Newton II for bung $\quad // \hat{\mathbf{r}}: \ \dfrac{mv^2}{r} = T\cos\theta$

$// \hat{\mathbf{z}}: \ 0 = T\sin\theta - mg$

Newton II for mass $\quad // \hat{\mathbf{z}}: \ 0 = T - Mg$

Hence:

$T\sin\theta = mg$

$T = Mg$

$\therefore \sin\theta = \dfrac{m}{M}$

$\therefore \theta = \sin^{-1}\left(\frac{m}{M}\right)$

$T\cos\theta = \dfrac{mv^2}{r}$

$T\sin\theta = mg$

$\therefore \tan\theta = \dfrac{gr}{v^2} = \dfrac{grP^2}{4\pi^2 r^2}$

$\therefore P = 2\pi\sqrt{\dfrac{r\tan\theta}{g}}$

$\therefore P = 2\pi\sqrt{\dfrac{h}{g}}$

To measure *r*, mark string in cm spacings and hence measure bung to tube length

$l = \sqrt{r^2 + h^2}$

$\therefore r = \sqrt{l^2 - h^2}$

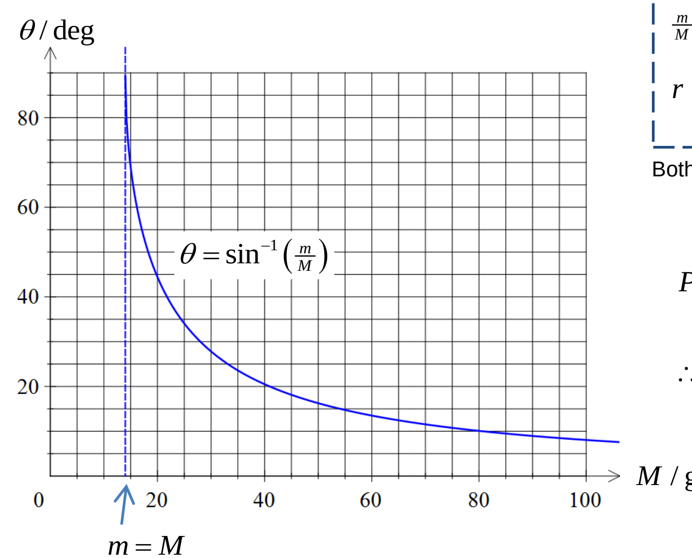
Use **video capture** to measure *h*, *calibrated* by direct measurement of l

The natural *variables* for the whirly bungs experiment are the mass of the fixed mass M, and the period of rotation P.

Time ten rotations and then average

Predicted quantities are the angle θ and the radius *r*

For a typical bung $\quad m = 14\text{g}$



For **analysis** plot:

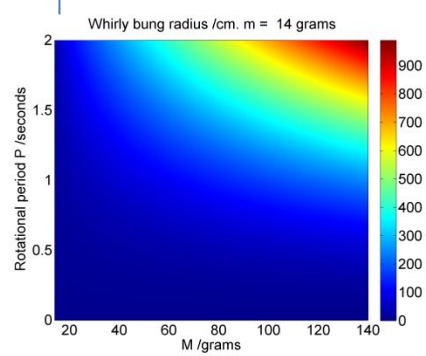
$\frac{m}{M}$ vs $\sin\theta$

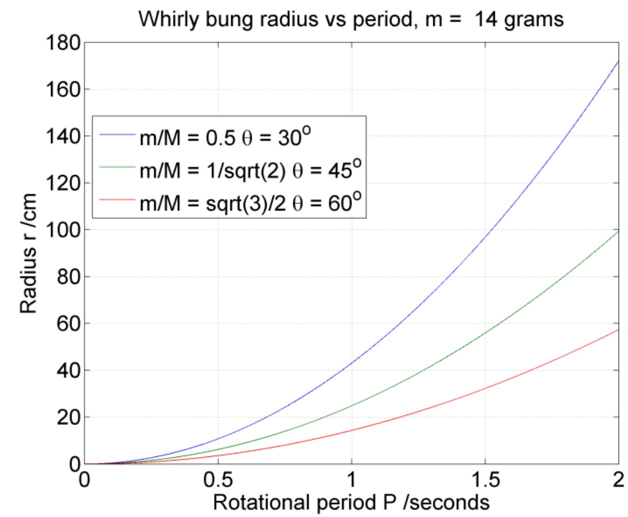
r vs $\dfrac{P^2}{\tan\theta}$

Both should be straight lines

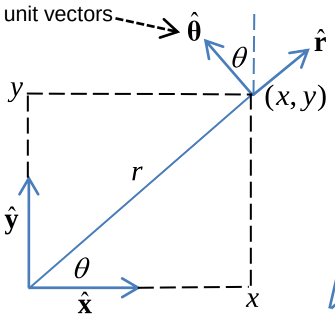
$P = 2\pi\sqrt{\dfrac{r\tan\theta}{g}}$

$\therefore r = \dfrac{gP^2}{4\pi^2\tan\theta}$





Circular motion. For simplicity we shall initially consider motion described by **plane polar coordinates** r, θ rather than general 3D motion described in spherical polars.



Position vector $\mathbf{r}$ can be described in Cartesians or plane polars

$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}}$$
$$\mathbf{r} = r\hat{\mathbf{r}}$$

The conversion is:

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$\mathbf{r} = r\cos\theta\hat{\mathbf{x}} + r\sin\theta\hat{\mathbf{y}}$$

Plane polar unit vectors are

$$\hat{\mathbf{r}} = \frac{\partial \mathbf{r}}{\partial r} \bigg/ \left|\frac{\partial \mathbf{r}}{\partial r}\right| = \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}$$

$$\hat{\boldsymbol{\theta}} = \frac{\partial \mathbf{r}}{\partial \theta} \bigg/ \left|\frac{\partial \mathbf{r}}{\partial \theta}\right| = -\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}$$

Use $\mathbf{r} = r\cos\theta\hat{\mathbf{x}} + r\sin\theta\hat{\mathbf{y}}$ to work this out

Unlike Cartesian unit vectors, time derivatives of plane polar unit vectors are *not* constant

$$\frac{d\hat{\mathbf{r}}}{dt} = \dot{\theta}\hat{\boldsymbol{\theta}}$$
$$\frac{d\hat{\boldsymbol{\theta}}}{dt} = -\dot{\theta}\hat{\mathbf{r}}$$

Hence velocity and acceleration can be written in terms of plane polar coordinate and associated unit vectors

$$\mathbf{v} = \dot{\mathbf{r}} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} \quad \leftarrow \quad \mathbf{v} = \frac{d}{dt}(r\hat{\mathbf{r}}) = \dot{r}\hat{\mathbf{r}} + r\frac{d\hat{\mathbf{r}}}{dt} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$$

$$\mathbf{a} = \ddot{\mathbf{r}} = \left(\ddot{r} - r\dot{\theta}^2\right)\hat{\mathbf{r}} + \left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)\hat{\boldsymbol{\theta}}$$

Define **angular velocity**

$$\omega = \dot{\theta} = \frac{d\theta}{dt}$$
$$\dot{\omega} = \ddot{\theta} = \frac{d^2\theta}{dt^2}$$

Hence:

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\omega\hat{\boldsymbol{\theta}}$$
$$\mathbf{a} = \left(\ddot{r} - r\omega^2\right)\hat{\mathbf{r}} + \left(2\dot{r}\omega + r\dot{\omega}\right)\hat{\boldsymbol{\theta}}$$

$$v^2 = \mathbf{v}\cdot\mathbf{v} = \dot{r}^2 + r^2\omega^2$$

$$\therefore \text{ if } \dot{r} = 0$$

$$\frac{v^2}{r} = r\omega^2$$
$$\mathbf{a} = -\frac{v^2}{r}\hat{\mathbf{r}} + r\dot{\omega}\hat{\boldsymbol{\theta}}$$

Newton's Second Law expressed in plane polar coordinates looks like this:

$$m\mathbf{a} = \sum_i \mathbf{F}_i$$

$$\left(m\ddot{r} - mr\omega^2\right)\hat{\mathbf{r}} + \left(2m\dot{r}\omega + mr\dot{\omega}\right)\hat{\boldsymbol{\theta}} = \sum_i \mathbf{F}_i$$

$$m\ddot{r}\hat{\mathbf{r}} = \sum_i \mathbf{F}_i + mr\omega^2\hat{\mathbf{r}} - \left(2m\dot{r}\omega + mr\dot{\omega}\right)\hat{\boldsymbol{\theta}}$$

Centrifugal 'force' ($mr\omega^2\hat{\mathbf{r}}$)

Coriolis and **tangential** 'forces' ($-\left(2m\dot{r}\omega + mr\dot{\omega}\right)\hat{\boldsymbol{\theta}}$)

So if a body is in a rotating frame of reference, i.e. rotates at angular speed ω, it can *appear* that there are *additional* forces acting.

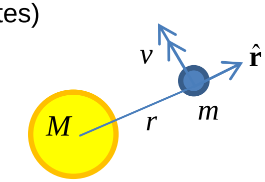
These 'forces' are a consequence that the *frame itself is accelerating*.

Example: What is the orbital speed of the Earth about the Sun, assuming a circular orbit? How does orbital radius and period vary?

If a circular orbit $\dot{r} = 0$ Also since gravity is a central force, there can be no tangential acceleration so $\dot{\omega} = 0$. Newton's Second Law is therefore (expressed in plane polar coordinates)

$$m\mathbf{a} = -\frac{mv^2}{r}\hat{\mathbf{r}} = -\frac{GMm}{r^2}\hat{\mathbf{r}} \quad \therefore v = \sqrt{\frac{GM}{r}}$$

Let the Earth be mass m and the Sun mass M



$$\frac{v^2}{r} = r\omega^2 = \frac{4\pi^2}{T^2}r$$

Hence:

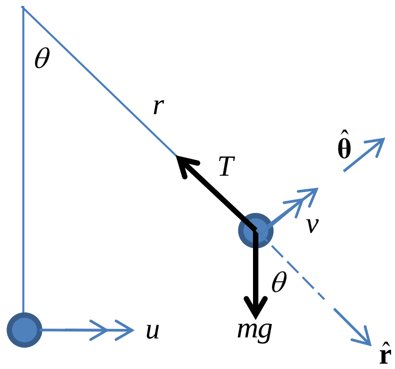
$$\frac{v^2}{r} = \frac{GM}{r^2}$$
$$\frac{4\pi^2}{T^2}r = \frac{GM}{r^2}$$
$$T^2 = \frac{4\pi^2}{GM}r^3$$

This is *Kepler's Third Law*

M = 2 x 10^{30} kg
G = 6.67 x 10^{-11} $\text{m}^3\,\text{kg}^{-1}\text{s}^{-2}$
T = 365 days = 3.154 x10^7s

r = 150 million km
v = 29.8 kms^{-1}

Example: A mass on a light inextensible string or length r is projected with a velocity u from the horizontal with the string taught. What angle does this pendulum swing before the string become slack? What is the minimum velocity such that the string will swing all the way round? Ignore air resistance.



Using plane polar coordinates, noting $\dot{r}=0$

acceleration **a** is $\mathbf{a}=-\frac{v^2}{r}\hat{\mathbf{r}}+r\ddot{\theta}\hat{\boldsymbol{\theta}}$

Newton's Second Law is therefore

$$-\frac{mv^2}{r}\hat{\mathbf{r}}+mr\ddot{\theta}\hat{\boldsymbol{\theta}}=-T\hat{\mathbf{r}}+mg\cos\theta\hat{\mathbf{r}}-mg\sin\theta\hat{\boldsymbol{\theta}}$$

Hence: $\hat{\mathbf{r}}:\ -\frac{mv^2}{r}=-T+mg\cos\theta$

By conservation of energy $\frac{1}{2}mu^2=\frac{1}{2}mv^2+mgr(1-\cos\theta)$

Hence: $mv^2=mu^2-2mgr(1-\cos\theta)$

Substituting into the radial component of Newton II:

$$\frac{2mgr(1-\cos\theta)-mu^2}{r}=-T+mg\cos\theta$$

$$\boxed{T=mg\left(3\cos\theta-2\right)+\frac{mu^2}{r}}$$

The string is taught for $T>0\quad \therefore mg\left(3\cos\theta-2\right)+\frac{mu^2}{r}>0\Rightarrow\cos\theta>\frac{2}{3}-\frac{u^2}{3gr}$

The minimum value of $\cos\theta$ is -1.

So for the mass to move all the way round the circle $\frac{2}{3}-\frac{u^2}{3gr}<-1$

$$\frac{5}{3}<\frac{u^2}{3gr}$$

$$\boxed{u>\sqrt{5gr}}$$

Alternative derivation of the conservation of energy equation via direct integration of the tangential component of Newton's Second law

$\hat{\boldsymbol{\theta}}:\ r\ddot{\theta}=-g\sin\theta$ Now using the chain rule $\boxed{\ddot{\theta}=\frac{d\dot{\theta}}{dt}=\dot{\theta}\frac{d\dot{\theta}}{d\theta}}$

Hence $\dot{\theta}\frac{d\dot{\theta}}{d\theta}=-\frac{g}{r}\sin\theta$

$$\dot{\theta}d\dot{\theta}=-\frac{g}{r}\sin\theta d\theta$$

Now in our situation the velocity of the mass (which is tangential in direction)

$v=r\dot{\theta}$

$\therefore\dot{\theta}=\frac{v}{r}\ \ \&\ \ d\dot{\theta}=\frac{dv}{r}$

$\Rightarrow\dot{\theta}d\dot{\theta}=\frac{vdv}{r^2}$

Hence:

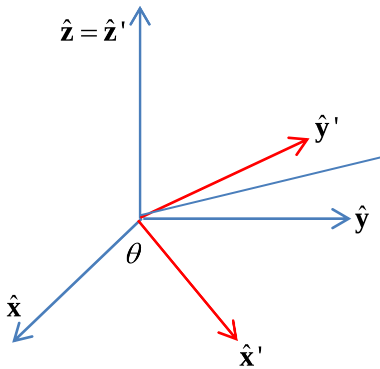
$$\frac{vdv}{r^2}=-\frac{g}{r}\sin\theta d\theta$$

$$\int_u^v vdv=-gr\int_0^\theta\sin\theta d\theta$$

$$\tfrac{1}{2}v^2-\tfrac{1}{2}u^2=gr\left[\cos\theta\right]_0^\theta$$

$$\boxed{v^2=u^2-2gr(1-\cos\theta)}$$

Rather than using plane polar coordinates, we can derive a useful expression for velocity and acceleration within a **rotating Cartesian frame of reference.** This also will yield extra 'forces' of the *Centrifugal* and *Coriolis* variety.

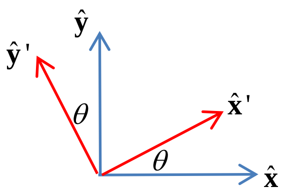


$$\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$
$$\mathbf{r} = x'\hat{\mathbf{x}}' + y'\hat{\mathbf{y}}' + z\hat{\mathbf{z}}$$

Define frame angular velocity vector

$$\Omega = \dot{\theta}$$
$$\boldsymbol{\Omega} = \Omega\hat{\mathbf{z}}$$

Assume $\dot{\Omega} = 0$



$$\hat{\mathbf{x}}' = \cos\theta\hat{\mathbf{x}} + \sin\theta\hat{\mathbf{y}}$$
$$\hat{\mathbf{y}}' = -\sin\theta\hat{\mathbf{x}} + \cos\theta\hat{\mathbf{y}}$$

The time derivatives of the rotated x and y basis vectors are *not* constant

$$\frac{d\hat{\mathbf{x}}'}{dt} = -\Omega\sin\theta\hat{\mathbf{x}} + \Omega\cos\theta\hat{\mathbf{y}} = \Omega\hat{\mathbf{y}}'$$
$$\frac{d\hat{\mathbf{y}}'}{dt} = -\Omega\cos\theta\hat{\mathbf{x}} - \Omega\sin\theta\hat{\mathbf{y}} = -\Omega\hat{\mathbf{x}}'$$

What we want is an expression for velocity and acceleration in terms of rotating frame coordinates x', y', z' so we can use Newton's Second law and solve Mechanics problems. Note by our definition of the rotating frame $z' = z$

$$\mathbf{v} = \frac{d}{dt}\left(x'\hat{\mathbf{x}}' + y'\hat{\mathbf{y}}' + z'\hat{\mathbf{z}}\right)$$
$$\mathbf{v} = \frac{dx'}{dt}\hat{\mathbf{x}}' + x'\frac{d\hat{\mathbf{x}}'}{dt} + \frac{dy'}{dt}\hat{\mathbf{y}}' + y'\frac{d\hat{\mathbf{y}}'}{dt} + \frac{dz'}{dt}\hat{\mathbf{z}}$$
$$\mathbf{v} = \mathbf{v}' + x'\frac{d\hat{\mathbf{x}}'}{dt} + y'\frac{d\hat{\mathbf{y}}'}{dt}$$
$$\mathbf{v} = \mathbf{v}' + x'\Omega\hat{\mathbf{y}}' - y'\Omega\hat{\mathbf{x}}'$$

where $\mathbf{v}' = \frac{dx'}{dt}\hat{\mathbf{x}}' + \frac{dy'}{dt}\hat{\mathbf{y}}' + \frac{dz'}{dt}\hat{\mathbf{z}}$

$$\boldsymbol{\Omega}\times\mathbf{r} = \Omega\hat{\mathbf{z}}\times\left(x'\hat{\mathbf{x}}' + y'\hat{\mathbf{y}}' + z\hat{\mathbf{z}}\right)$$
$$\boldsymbol{\Omega}\times\mathbf{r} = \Omega x'\hat{\mathbf{y}}' - \Omega y'\hat{\mathbf{x}}'$$

$$\hat{\mathbf{x}}\times\hat{\mathbf{y}} = \hat{\mathbf{z}}$$
$$\hat{\mathbf{x}}\times\hat{\mathbf{z}} = -\hat{\mathbf{y}}$$
$$\hat{\mathbf{y}}\times\hat{\mathbf{z}} = \hat{\mathbf{x}}$$

Cartesian basis vectors form a **right-handed set**

Hence $\mathbf{v} = \mathbf{v}' + \boldsymbol{\Omega}\times\mathbf{r}$

The time derivatives $\frac{d\hat{\mathbf{x}}'}{dt} = \Omega\hat{\mathbf{y}}'$ $\frac{d\hat{\mathbf{y}}'}{dt} = -\Omega\hat{\mathbf{x}}'$

can be used to determine an expression for acceleration.

$$\mathbf{a} = \frac{d}{dt}\left(\frac{dx'}{dt}\hat{\mathbf{x}}' + \frac{dy'}{dt}\hat{\mathbf{y}}' + \frac{dz'}{dt}\hat{\mathbf{z}} + x'\Omega\hat{\mathbf{y}}' - y'\Omega\hat{\mathbf{x}}'\right)$$

The algebra can be simplified by defining a 'rotational frame derivative'

$$\mathbf{a} = \frac{d\mathbf{v}}{dt} = \left(\frac{d}{dt} + \boldsymbol{\Omega}\times\right)\left(\mathbf{v}' + \boldsymbol{\Omega}\times\mathbf{r}\right)$$

$$\mathbf{a} = \mathbf{a}' + 2\boldsymbol{\Omega}\times\mathbf{v}' + \boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\mathbf{r}$$

where $\mathbf{a}' = \frac{d^2x'}{dt^2}\hat{\mathbf{x}}' + \frac{d^2y'}{dt^2}\hat{\mathbf{y}}' + \frac{d^2z'}{dt^2}\hat{\mathbf{z}}$

Newton's Second Law expressed in these coordinates looks like:

$$m\mathbf{a} = \sum_i \mathbf{F}_i$$
$$m\mathbf{a}' + 2m\boldsymbol{\Omega}\times\mathbf{v}' + m\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\mathbf{r} = \sum_i \mathbf{F}_i$$
$$m\mathbf{a}' = \sum_i \mathbf{F}_i - 2m\boldsymbol{\Omega}\times\mathbf{v}' + m\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\mathbf{r}$$

So in the rotating frame, we have two additional 'forces' to add to the sum of external forces

$$\mathbf{f}_{\text{centrifugal}} = m\boldsymbol{\Omega}\times\boldsymbol{\Omega}\times\left(x'\hat{\mathbf{x}}' + y'\hat{\mathbf{y}}' + z\hat{\mathbf{z}}\right)$$
$$\mathbf{f}_{\text{coriolis}} = -2m\boldsymbol{\Omega}\times\left(\frac{dx'}{dt}\hat{\mathbf{x}}' + \frac{dy'}{dt}\hat{\mathbf{y}}' + \frac{dz'}{dt}\hat{\mathbf{z}}\right)$$

$$m\mathbf{a}' = \sum_i \mathbf{F}_i + \mathbf{f}_{\text{coriolis}} + \mathbf{f}_{\text{centrifugal}}$$