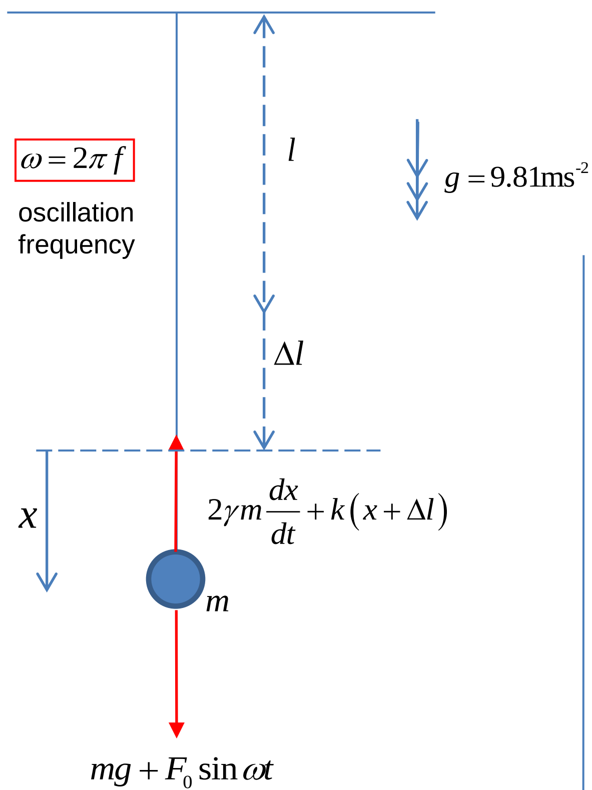
SHM equation from driven mechanical oscillations

Consider a particle of mass m suspended from a light elastic string from a fixed surface. The string has natural length l. Assume a *Hookean* law of elasticity i.e. restoring force is proportional to extension. The elastic constant in this case is k. Also assume mass is subject to air resistance which is proportional to velocity and mass m. The mass is also pulled 'driven' via an oscillatory force of magnitude F_0 and frequency $f = \omega/2\pi$.

In the absence of any driving force, the mass rests at string extension Δl. It is assumed at time $t = 0$ that extension from this equilibrium point, (x), is zero and the mass is at instantaneous rest.



By Newton's Second Law:

$$m\frac{d^2x}{dt^2} = mg + F_0\sin\omega t - 2\gamma m\frac{dx}{dt} - k(x+\Delta l)$$

At equilibrium $x = 0$, $F_0 = 0$

$$0 = mg - k\Delta l \quad \therefore \quad \Delta l = \frac{mg}{k}$$

Hence: $$m\frac{d^2x}{dt^2} + 2\gamma m\frac{dx}{dt} + kx = F_0\sin\omega t$$

$$\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \frac{k}{m}x = \frac{F_0}{m}\sin\omega t$$

This equation has a very similar form to the generic equation of **Simple Harmonic Motion (SHM)**

$$\boxed{\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = A_0\omega_0^2\sin\omega t}$$

$$\omega_0 = \frac{k}{m}, \quad A_0\omega_0^2 = \frac{F_0}{m}$$

The **general solution** to this *inhomogeneous, second order differential equation* has the form

$$x(t) = X(t) + Y(t)$$

where $$\frac{d^2X}{dt^2} + 2\gamma\frac{dX}{dt} + \omega_0^2 X = 0$$

and $$Y(t) = A\sin(\omega t - \phi)$$

i.e. a similar form (and frequency) to the driving force term. Experiments indicate that the latter, when damping is "low", corresponds to the steady state condition of the system. i.e. the $X(t)$ term decays with time and therefore is *transient*.*

It will be shown (on the next few pages) that oscillatory solutions are possible when the system is *under-damped*. This is when

$$\boxed{\omega_0 > \gamma}$$

Lets consider the steady state solutions, and substitute into the SHM equation

$$\boxed{x = A\sin(\omega t - \phi)}$$

$$-\omega^2 A\sin(\omega t-\phi) + 2\gamma\omega A\cos(\omega t-\phi) + \omega_0^2 A\sin(\omega t-\phi) = A_0\omega_0^2\sin\omega t$$

$$(\omega_0^2 - \omega^2)A\sin(\omega t-\phi) + 2\gamma\omega A\cos(\omega t-\phi) = A_0\omega_0^2\sin\omega t$$

$$\therefore (\omega_0^2-\omega^2)(\sin\omega t\cos\phi - \cos\omega t\sin\phi) + ...$$

$$2\gamma\omega(\cos\omega t\cos\phi + \sin\omega t\sin\phi) = \frac{A_0\omega_0^2}{A}\sin\omega t$$

$$\left((\omega_0^2-\omega^2)\cos\phi + 2\gamma\omega\sin\phi\right)\sin\omega t + ...$$

$$\left(-(\omega_0^2-\omega^2)\sin\phi + 2\gamma\omega\cos\phi\right)\cos\omega t = \frac{A_0\omega_0^2}{A}\sin\omega t$$

Comparing coefficients of $\sin\omega t$ and $\cos\omega t$

$$(\omega_0^2-\omega^2)\cos\phi + 2\gamma\omega\sin\phi = \frac{A_0\omega_0^2}{A}$$

$$-(\omega_0^2-\omega^2)\sin\phi + 2\gamma\omega\cos\phi = 0$$

See next page for algebra

$$\therefore \boxed{\tan\phi = \frac{2\gamma\omega}{\omega_0^2-\omega^2}} \quad \text{and} \quad \boxed{A = \frac{A_0\omega_0^2}{\sqrt{(\omega_0^2-\omega^2)^2 + 4\gamma^2\omega^2}}}$$

*We'll look at $X(t)$ in a few pages! It is called the *Complimentary Function*, whereas $Y(t)$ is a *Particular Integral*

$$\left(\omega_0^2-\omega^2\right)\cos\phi+2\gamma\omega\sin\phi=\frac{A_0\omega_0^2}{A}$$

$$\left(\omega_0^2-\omega^2\right)+2\gamma\omega\tan\phi=\frac{A_0\omega_0^2}{A\cos\phi}$$

$$\left(\omega_0^2-\omega^2\right)+2\gamma\omega\tan\phi=\frac{A_0\omega_0^2}{A}\sqrt{1+\tan^2\phi}$$

$$\tan\phi=\frac{2\gamma\omega}{\omega_0^2-\omega^2}$$

$$\left(\omega_0^2-\omega^2\right)+2\gamma\omega\frac{2\gamma\omega}{\omega_0^2-\omega^2}=\frac{A_0\omega_0^2}{A}\sqrt{1+\frac{4\gamma^2\omega^2}{\left(\omega_0^2-\omega^2\right)^2}}$$

$$\frac{\left(\omega_0^2-\omega^2\right)^2+4\gamma^2\omega^2}{\omega_0^2-\omega^2}=\frac{A_0\omega_0^2}{A}\frac{1}{\omega_0^2-\omega^2}\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\gamma^2\omega^2}$$

$$\boxed{A=\frac{A_0\omega_0^2}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\gamma^2\omega^2}}}$$

Now A has a *maxima* when

$$\frac{dA}{d\omega}=0$$

$$\frac{dA}{d\omega}=-\tfrac{1}{2}\frac{A_0\omega_0^2\left(2\left(\omega_0^2-\omega^2\right)(-2\omega)+8\gamma^2\omega\right)}{\left(\left(\omega_0^2-\omega^2\right)^2+4\gamma^2\omega^2\right)^{\frac{3}{2}}}$$

$$\frac{dA}{d\omega}=\frac{2A_0\omega_0^2\omega\left(\omega_0^2-\omega^2-2\gamma^2\right)}{\left(\left(\omega_0^2-\omega^2\right)^2+4\gamma^2\omega^2\right)^{\frac{3}{2}}}$$

$$\frac{dA}{d\omega}=0\Rightarrow\omega_0^2-\omega^2-2\gamma^2=0$$

$$\therefore\boxed{\omega_{\max}=\sqrt{\omega_0^2-2\gamma^2}}$$

$$\omega=\sqrt{\omega_0^2-2\gamma^2}$$

$$\therefore\left(\omega_0^2-\omega^2\right)^2+4\gamma^2\omega^2$$

$$=\left(\omega_0^2-\omega_0^2-2\gamma^2\right)^2+4\gamma^2\left(\omega_0^2-2\gamma^2\right)$$

$$=4\gamma^4+4\gamma^2\omega_0^2-8\gamma^4$$

$$=4\gamma^2\omega_0^2-4\gamma^4$$

$$\therefore A_{\max}=\frac{A_0\omega_0^2}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\gamma^2\omega^2}}$$

$$A_{\max}=\frac{A_0\omega_0^2}{\sqrt{4\gamma^2\omega_0^2-4\gamma^4}}$$

$$\boxed{A_{\max}=\frac{A_0\omega_0^2}{2\gamma\sqrt{\omega_0^2-\gamma^2}}}$$

As the damping factor approaches the 'natural frequency' ω_0, the height of the **resonance** peak *increases* and the peak *sharpens*.

When the driving frequency approaches the natural frequency the phase of the steady state response approaches 90º. As the driving frequency increases, the steady state response will tend towards 180º out of phase with the driving oscillator.

$x=A\sin(\omega t-\phi)\rightarrow$

Steady-State SHM Summary

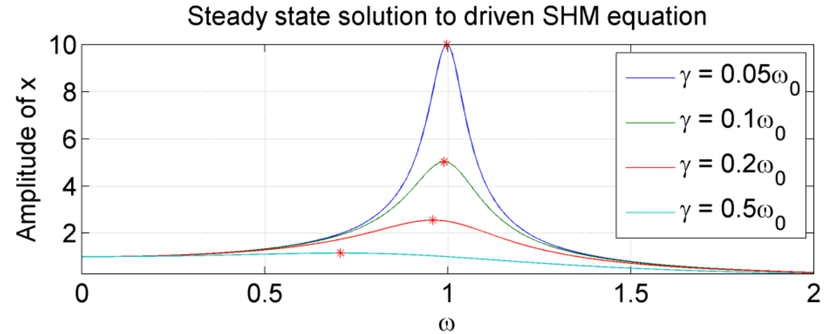
$$\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2x=A_0\omega_0^2\sin\omega t$$

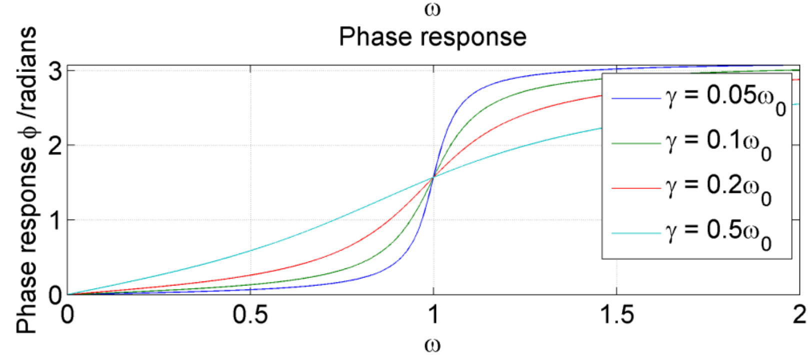
$$x=\frac{A_0\omega_0^2}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\gamma^2\omega^2}}\sin\left(\omega t-\phi\right)$$

$$\phi=\tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2-\omega^2}\right)$$

$$|x|_{\max}\ \text{when}\quad\omega=\sqrt{\omega_0^2-2\gamma^2}$$

$$|x|_{\max}=\frac{A_0\omega_0^2}{2\gamma\sqrt{\omega_0^2-\gamma^2}}$$





Solving the **SHM equation** (steady state) using **complex variables.** i.e. we make use of **De-Moivre's Theorem**

$$\boxed{\frac{d^2z}{dt^2}+2\gamma\frac{dz}{dt}+\omega_0^2 z = A_0\omega_0^2 e^{i\omega t}}$$

$$z = Ae^{i(\omega t-\phi)}$$

$$\left(-\omega^2+2i\gamma\omega+\omega_0^2\right)Ae^{i(\omega t-\phi)} = A_0\omega_0^2 e^{i\omega t}$$

$$\left(-\omega^2+2i\gamma\omega+\omega_0^2\right)Ae^{-i\phi} = A_0\omega_0^2$$

Compute the modulus (complex number magnitude)

$$\left|-\omega^2+2i\gamma\omega+\omega_0^2\right|A = A_0\omega_0^2\left|e^{i\phi}\right| = A_0\omega_0^2$$

$$A = \frac{A_0\omega_0^2}{\left|-\omega^2+2i\gamma\omega+\omega_0^2\right|}$$

$$\therefore \boxed{A = \frac{A_0\omega_0^2}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\gamma^2\omega^2}}}$$

Compute the modulus (complex number magnitude)

$$\left(-\omega^2+2i\gamma\omega+\omega_0^2\right)A = A_0\omega_0^2 e^{i\phi}$$

$$A\left|-\omega^2+2i\gamma\omega+\omega_0^2\right|e^{i\arg\left(-\omega^2+2i\gamma\omega+\omega_0^2\right)} = A_0\omega_0^2 e^{i\phi}$$

$$\left|-\omega^2+2i\gamma\omega+\omega_0^2\right|A = A_0\omega_0^2$$

$$\therefore \phi = \arg\left(-\omega^2+2i\gamma\omega+\omega_0^2\right)$$

$$\therefore \boxed{\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2-\omega^2}\right)}$$

i.e. the steady state solution $x = A\sin(\omega t-\phi)$ is the *imaginary* part of z

$$\mathrm{Im}\left(A_0e^{i\omega t}\right) = \mathrm{Im}\left(A_0\cos\omega t + i\sin\omega t\right) = A_0\sin\omega t$$

$$\mathrm{Im}\left(Ae^{i(\omega t-\phi)}\right) = \mathrm{Im}\left(A\cos\omega t + iA\sin\omega t\right) = A\sin\left(\omega t-\phi\right)$$

Steady-State SHM Summary

$$\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2 x = A_0\omega_0^2\sin\omega t$$

$$x = \frac{A_0\omega_0^2}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\gamma^2\omega^2}}\sin\left(\omega t-\phi\right)$$

$$\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2-\omega^2}\right)$$

$$|x|_{\max} \text{ when } \omega = \sqrt{\omega_0^2-2\gamma^2}$$

$$|x|_{\max} = \frac{A_0\omega_0^2}{2\gamma\sqrt{\omega_0^2-\gamma^2}}$$

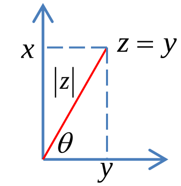
As you can see, this is a far more efficient method of arriving at the steady state solution.

We will therefore adopt a complex number from now on.

$$z = y + ix$$

$$|z| = \sqrt{x^2+y^2}$$

$$\arg z = \tan^{-1}\tfrac{x}{y}$$

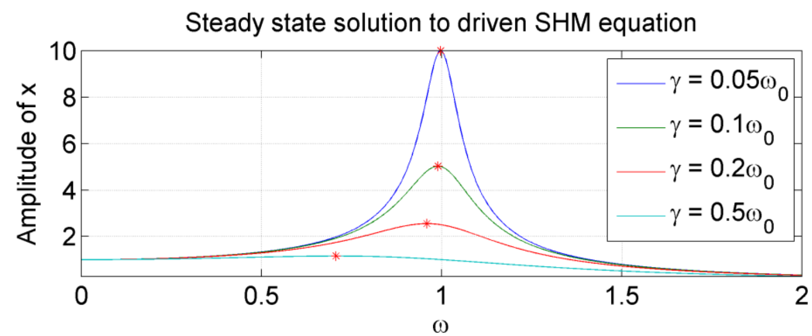


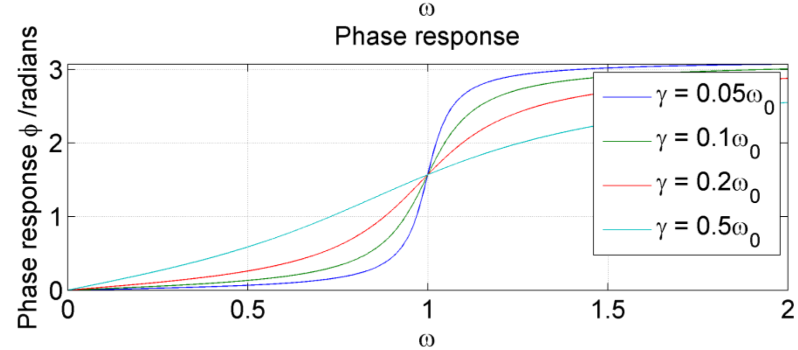
$$e^{i\theta} = \cos\theta + i\sin\theta$$

$$z = |z|e^{i\theta}$$

$$\arg z = \theta$$

Yes I know x,y appear to be the wrong way round, but the use of sin in our forcing frequency means x associates with the imaginary part. It is merely for convenience!





General solution of the Forced SHM equation i.e. including the *transient* term

The equation of forced SHM

$$\boxed{\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2 x = A_0\omega_0^2\sin\omega t}$$

General solution

We shall exploit the *linearity* of the SHM equation by expressing the general solution $x(t)$ in terms of a *Complementary Function* X (which is the solution when $A_0 = 0$) plus a *Particular Integral* B, which will have the same functional form as the 'forcing term'.

$$x = X + Y$$

$$X = Ae^{qt}$$ i.e. *guess* solutions of this form

$$Y = B\sin\left(\omega t-\phi\right)$$

$$\frac{d^2X}{dt^2}+2\gamma\frac{dX}{dt}+\omega_0^2 X = 0$$

$$q^2Ae^{qt}+2\gamma qAe^{qt}+\omega_0^2Ae^{qt} = 0$$

$$q^2+2\gamma q+\omega_0^2 = 0$$

$$q = \frac{-2\gamma\pm\sqrt{4\gamma^2-4\omega_0^2}}{2}$$

$$q = -\gamma\pm\sqrt{\gamma^2-\omega_0^2}$$

$$q = -\gamma\pm i\sqrt{\omega_0^2-\gamma^2}$$ alternative!

$X(t)$ must be a superposition of *two* solutions in general, since the SHM equation is second order. i.e. two integrations to solve and therefore two integration constants

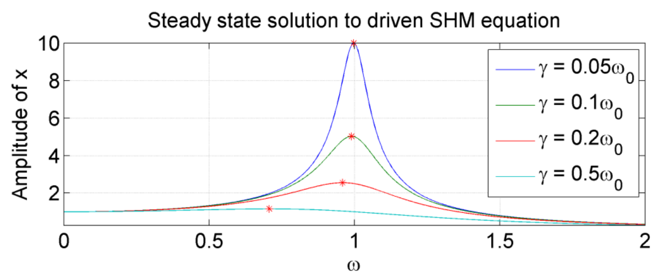
$$\boxed{X(t)=\begin{cases}\text{'Critically damped'} & \\ e^{-\gamma t}\left(A_1+A_2t\right) & \gamma=\omega_0 \\ \text{'Overdamped'} & \\ e^{-\gamma t}\left(A_1e^{t\sqrt{\gamma^2-\omega_0^2}}+A_2e^{-t\sqrt{\gamma^2-\omega_0^2}}\right) & \gamma>\omega_0 \\ \text{'Underdamped – i.e. oscillatory' *} & \\ Ae^{-\gamma t}\cos\left(t\sqrt{\omega_0^2-\gamma^2}-\Phi\right) & \gamma<\omega_0\end{cases}}$$

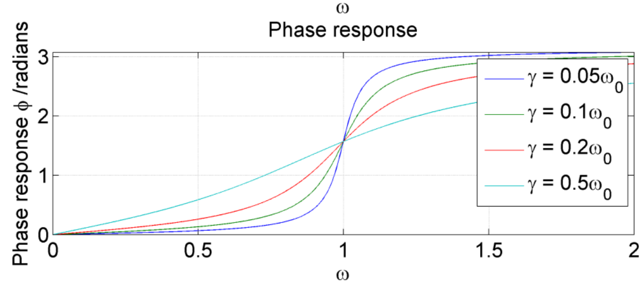
We already know:

$$Y(t)=B\sin\left(\omega t-\phi\right)\qquad B=\frac{A_0\omega_0^2}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\gamma^2\omega^2}}$$

$$\phi=\tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2-\omega^2}\right)\qquad B_{\max}\text{ when }\omega=\sqrt{\omega_0^2-2\gamma^2}$$

$$B_{\max}=\frac{A_0\omega_0^2}{2\gamma\sqrt{\omega_0^2-\gamma^2}}$$





If we consider only the *oscillatory* solutions:

$$\boxed{\begin{aligned}&\frac{d^2x}{dt^2}+2\gamma\frac{dx}{dt}+\omega_0^2 x = A_0\omega_0^2\sin\omega t\\&\gamma<\omega_0\\&x(t)=Ae^{-\gamma t}\cos\left(t\sqrt{\omega_0^2-\gamma^2}-\Phi\right)+B\sin\left(\omega t-\phi\right)\\&B=\frac{A_0\omega_0^2}{\sqrt{\left(\omega_0^2-\omega^2\right)^2+4\gamma^2\omega^2}}\\&\phi=\tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2-\omega^2}\right)\end{aligned}}$$

Let us constrain this further by considering initial conditions

$$x(0)=x_0,\quad \left.\frac{dx}{dt}\right|_{t=0}=\dot{x}_0$$

$$x_0 = A\cos\left(-\Phi\right)+B\sin\left(-\phi\right)$$

$$x_0 = A\cos\Phi - B\sin\left(\phi\right)$$

$$x_0 + B\sin\phi = A\cos\Phi$$

Noting even and odd properties of sine and cosine

$$\frac{dx}{dt} = -\gamma Ae^{-\gamma t}\cos\left(t\sqrt{\omega_0^2-\gamma^2}-\Phi\right)+...$$

$$-Ae^{-\gamma t}\sqrt{\omega_0^2-\gamma^2}\sin\left(t\sqrt{\omega_0^2-\gamma^2}-\Phi\right)+B\omega\cos\left(\omega t-\phi\right)$$

$$\left.\frac{dx}{dt}\right|_{t=0} = -\gamma A\cos\left(-\Phi\right)+...$$

$$-A\sqrt{\omega_0^2-\gamma^2}\sin\left(-\Phi\right)+B\omega\cos\left(-\phi\right)$$

$$\dot{x}_0 = -\gamma A\cos\Phi + A\sqrt{\omega_0^2-\gamma^2}\sin\Phi + B\omega\cos\phi$$

*Note $\cos\theta=\frac{1}{2}\left(e^{i\theta}+e^{-i\theta}\right),\ \sin\theta=\frac{1}{2i}\left(e^{i\theta}-e^{-i\theta}\right)$

$$\dot{x}_0 = -\gamma\left(x_0 + B\sin\phi\right) + \frac{x_0 + B\sin\phi}{\cos\Phi}\sqrt{\omega_0^2 - \gamma^2}\,\sin\Phi + B\omega\cos\phi$$

$$\dot{x}_0 = -\gamma\left(x_0 + B\sin\phi\right) + \left(x_0 + B\sin\phi\right)\sqrt{\omega_0^2 - \gamma^2}\,\tan\Phi + B\omega\cos\phi$$

$$\boxed{\frac{\dot{x}_0 + \gamma\left(x_0 + B\sin\phi\right) - B\omega\cos\phi}{\left(x_0 + B\sin\phi\right)\sqrt{\omega_0^2 - \gamma^2}} = \tan\Phi \qquad \frac{x_0 + B\sin\phi}{\cos\Phi} = A}$$

Now let us consider similar initial conditions for ***critical damping***:

$$\boxed{\begin{aligned} &\gamma = \omega_0 \\ &x(t) = e^{-\gamma t}\left(A_1 + A_2 t\right) + B\sin\left(\omega t - \phi\right) \\ &B = \frac{A_0\omega_0^2}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2}} \\ &\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2 - \omega^2}\right) \end{aligned}}$$

Notice the linear dependence on t is necessary to make this work

$$x(0) = x_0, \quad \left.\frac{dx}{dt}\right|_{t=0} = \dot{x}_0$$

$$x_0 = A_1 - B\sin\phi$$

$$\frac{dx}{dt} = e^{-\gamma t}\left(A_2\right) - \gamma e^{-\gamma t}\left(A_1 + A_2 t\right) + B\omega\cos\left(\omega t - \phi\right)$$

$$\dot{x}_0 = A_2 - \gamma A_1 + B\omega\cos\phi$$

$$\boxed{A_1 = x_0 + B\sin\phi}$$

$$A_2 = \dot{x}_0 + \gamma A_1 - B\omega\cos\phi$$

$$\boxed{A_2 = \dot{x}_0 + \gamma\left(x_0 + B\sin\phi\right) - B\omega\cos\phi}$$

Finally, let us consider the ***over-damped*** solutions:

$$\boxed{\begin{aligned} &\gamma > \omega_0 \\ &x(t) = e^{-\gamma t}\left(A_1 e^{t\sqrt{\gamma^2 - \omega_0^2}} + A_2 e^{-t\sqrt{\gamma^2 - \omega_0^2}}\right) + B\sin\left(\omega t - \phi\right) \\ &B = \frac{A_0\omega_0^2}{\sqrt{\left(\omega_0^2 - \omega^2\right)^2 + 4\gamma^2\omega^2}} \\ &\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2 - \omega^2}\right) \end{aligned}}$$

$$x(0) = x_0, \quad \left.\frac{dx}{dt}\right|_{t=0} = \dot{x}_0$$

$$x_0 = A_1 + A_2 - B\sin\phi \quad \therefore A_1 + A_2 = x_0 + B\sin\phi$$

$$\frac{dx}{dt} = e^{-\gamma t}\sqrt{\gamma^2 - \omega_0^2}\left(A_1 e^{t\sqrt{\gamma^2 - \omega_0^2}} - A_2 e^{-t\sqrt{\gamma^2 - \omega_0^2}}\right) + \ldots$$

$$-\gamma e^{-\gamma t}\left(A_1 e^{t\sqrt{\gamma^2 - \omega_0^2}} + A_2 e^{-t\sqrt{\gamma^2 - \omega_0^2}}\right) + B\omega\cos\left(\omega t - \phi\right)$$

$$\dot{x}_0 = \sqrt{\gamma^2 - \omega_0^2}\left(A_1 - A_2\right) - \gamma\left(A_1 + A_2\right) + B\omega\cos\phi$$

$$\frac{\dot{x}_0 + \gamma\left(x_0 + B\sin\phi\right) - B\omega\cos\phi}{\sqrt{\gamma^2 - \omega_0^2}} = A_1 - A_2$$

$$\therefore\ 2A_1 = x_0 + B\sin\phi + \frac{\dot{x}_0 + \gamma\left(x_0 + B\sin\phi\right) - B\omega\cos\phi}{\sqrt{\gamma^2 - \omega_0^2}}$$

$$\boxed{\begin{aligned} &A_1 = \tfrac{1}{2}\left(x_0 + B\sin\phi\right) + \frac{\dot{x}_0 + \gamma\left(x_0 + B\sin\phi\right) - B\omega\cos\phi}{2\sqrt{\gamma^2 - \omega_0^2}} \\ &A_2 = \tfrac{1}{2}\left(x_0 + B\sin\phi\right) - \frac{\dot{x}_0 + \gamma\left(x_0 + B\sin\phi\right) - B\omega\cos\phi}{2\sqrt{\gamma^2 - \omega_0^2}} \end{aligned}}$$

SHM in summary

$$\boxed{\frac{d^2x}{dt^2} + 2\gamma\frac{dx}{dt} + \omega_0^2 x = A_0\omega_0^2 \sin\omega t} \qquad x(0) = x_0, \quad \left.\frac{dx}{dt}\right|_{t=0} = \dot{x}_0$$

'Transient'

Under-damped - oscillatory

$$\gamma < \omega_0$$

$$x(t) = Ae^{-\gamma t}\cos\left(t\sqrt{\omega_0^2-\gamma^2} - \Phi\right) + B\sin(\omega t - \phi)$$

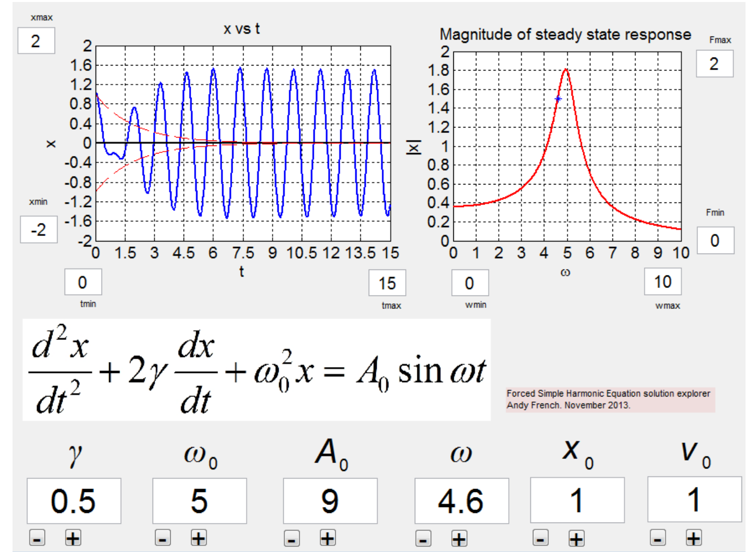
$$\Phi = \tan^{-1}\left(\frac{\dot{x}_0 + \gamma(x_0 + B\sin\phi) - B\omega\cos\phi}{(x_0 + B\sin\phi)\sqrt{\omega_0^2-\gamma^2}}\right)$$

$$A = \frac{x_0 + B\sin\phi}{\cos\Phi}$$

$$B = \frac{A_0\omega_0^2}{\sqrt{\left(\omega_0^2-\omega^2\right)^2 + 4\gamma^2\omega^2}}$$

$$\phi = \tan^{-1}\left(\frac{2\gamma\omega}{\omega_0^2-\omega^2}\right)$$

'Steady state' solution amplitude and phase



Critically damped

$$\gamma = \omega_0$$

$$x(t) = e^{-\gamma t}\left(A_1 + A_2 t\right) + B\sin(\omega t - \phi)$$

$$A_1 = x_0 + B\sin\phi$$

$$A_2 = \dot{x}_0 + \gamma(x_0 + B\sin\phi) - B\omega\cos\phi$$

$$B_{\max} \text{ when } \quad \omega = \sqrt{\omega_0^2 - 2\gamma^2}$$

$$B_{\max} = \frac{A_0\omega_0^2}{2\gamma\sqrt{\omega_0^2-\gamma^2}}$$

Note this *resonance frequency* will always be less than or equal to the *natural frequency*

Over-damped

$$\gamma > \omega_0$$

$$x(t) = e^{-\gamma t}\left(A_1 e^{t\sqrt{\gamma^2-\omega_0^2}} + A_2 e^{-t\sqrt{\gamma^2-\omega_0^2}}\right) + B\sin(\omega t - \phi)$$

$$A_1 = \tfrac{1}{2}(x_0 + B\sin\phi) + \frac{\dot{x}_0 + \gamma(x_0 + B\sin\phi) - B\omega\cos\phi}{2\sqrt{\gamma^2-\omega_0^2}}$$

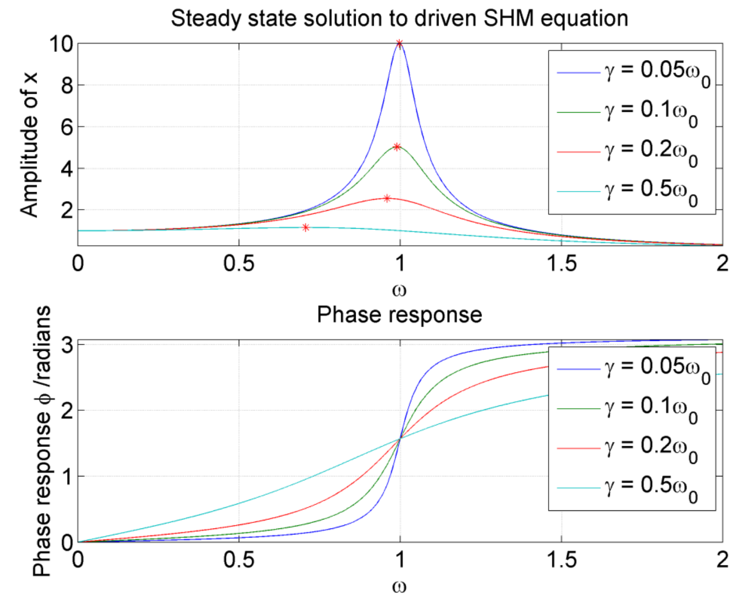
$$A_2 = \tfrac{1}{2}(x_0 + B\sin\phi) - \frac{\dot{x}_0 + \gamma(x_0 + B\sin\phi) - B\omega\cos\phi}{2\sqrt{\gamma^2-\omega_0^2}}$$

Note we only get a **resonance peak** when

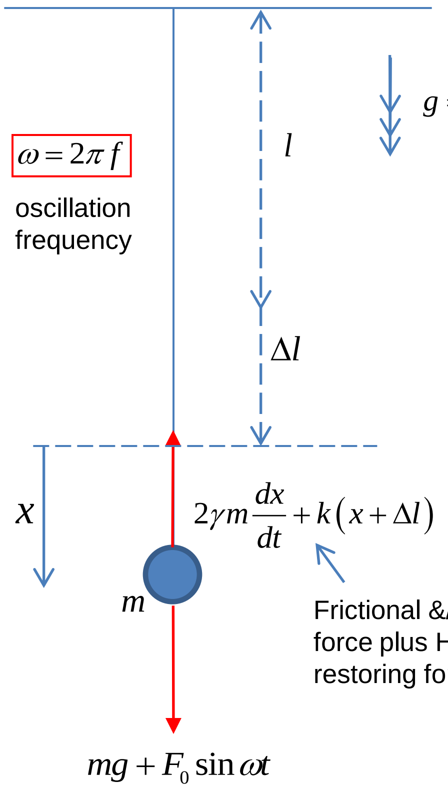
$$\omega_0^2 - 2\gamma^2 \geq 0$$

$$\gamma \leq \frac{\omega_0}{\sqrt{2}}$$

$$\gamma \leq 0.7011\omega_0$$



Example system #1: Mass, spring system



$\boxed{\omega = 2\pi f}$

oscillation frequency

By Newton's Second Law:

$$m\frac{d^2x}{dt^2} = mg + F_0 \sin \omega t - 2\gamma m \frac{dx}{dt} - k(x + \Delta l)$$

At equilibrium $x = 0$, $F_0 = 0$

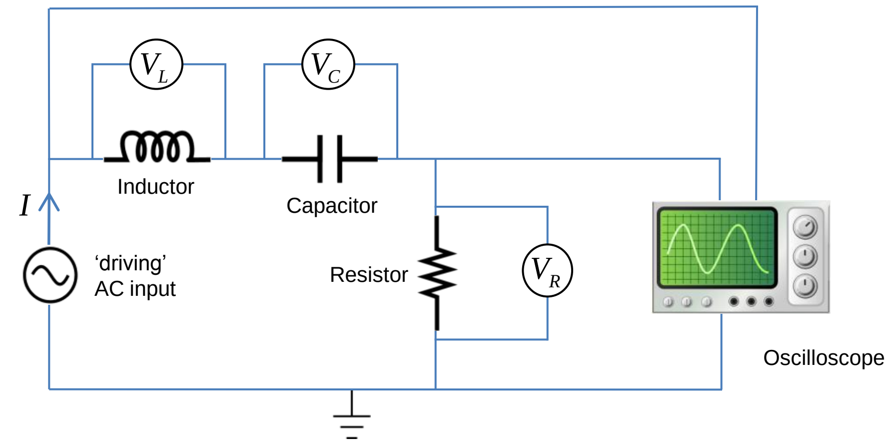
$$0 = mg - k\Delta l \quad \therefore \quad \Delta l = \frac{mg}{k}$$

Hence: $$m\frac{d^2x}{dt^2} + 2\gamma m \frac{dx}{dt} + kx = F_0 \sin \omega t$$

$$\boxed{\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = A_0 \omega_0^2 \sin \omega t}$$

$$\boxed{\omega_0 = \frac{k}{m}, \quad A_0 \omega_0^2 = \frac{F_0}{m}}$$

Example system #2: R,L,C electrical circuit



Let current I flow through the circuit above. The net EMF V - V_L must equal the sum of the potential drops across each electrical component.

$$V_R + V_C = V - V_L$$

$$V_R = IR \qquad Q = CV$$

$$V_C = \frac{1}{C}\int I dt \longleftarrow I = \frac{dQ}{dt} \therefore CV = \int I dt$$

$$V_L = L\frac{dI}{dt}$$

V_L is the 'back EMF' due to induction in the inductor coil

Consider an oscillatory driving voltage $V = V_0 \cos \omega t$

$$\frac{dV_R}{dt} + \frac{dV_C}{dt} + \frac{dV_L}{dt} = \frac{dV}{dt}$$

$$L\frac{d^2I}{dt^2} + R\frac{dI}{dt} + \frac{1}{C}I = -\omega V_0 \sin \omega t$$

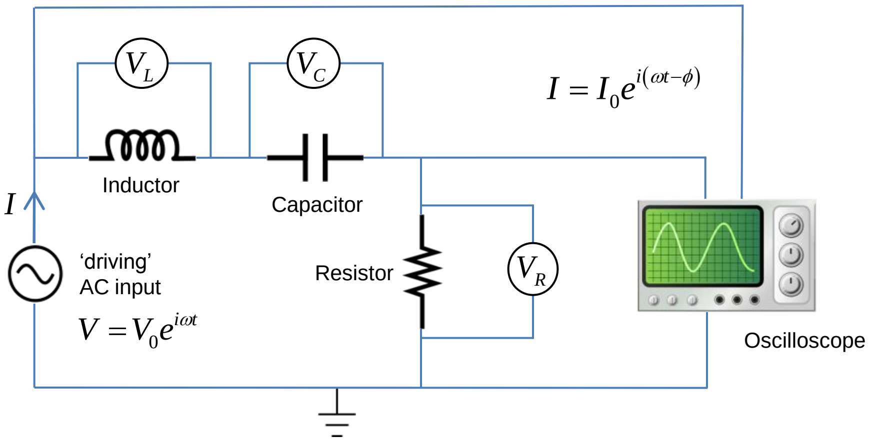
$$\frac{d^2I}{dt^2} + \frac{R}{L}\frac{dI}{dt} + \frac{1}{LC}I = \frac{-\omega V_0}{L} \sin \omega t$$

i.e. equation of forced SHM in *I(t)*

$$\boxed{\gamma = \frac{R}{2L}, \quad \omega_0^2 = \frac{1}{LC}, \quad A_0 \omega_0^2 = \frac{-\omega V_0}{L}, \quad x = I}$$

$$\boxed{\frac{d^2x}{dt^2} + 2\gamma \frac{dx}{dt} + \omega_0^2 x = A_0 \omega_0^2 \sin \omega t}$$

Example system #3: R,L,C electrical circuit – using complex impedances

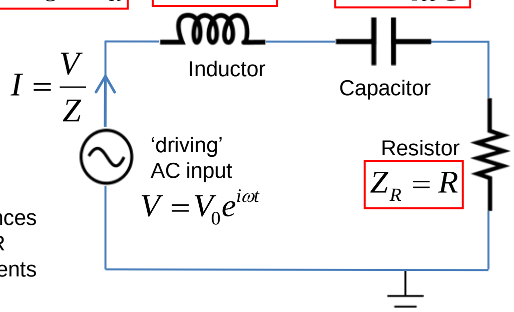


$$V_R + V_C = V - V_L$$
$$V_R = IR$$
$$V_C = \frac{1}{C}\int I dt \quad \longleftarrow \quad Q = CV$$
$$I = \frac{dQ}{dt} \therefore CV = \int I dt$$
$$V_L = L\frac{dI}{dt}$$
$$IR + \frac{1}{C}\int I dt + L\frac{dI}{dt} = V_0 e^{i\omega t}$$

$$IR + \frac{1}{C}\int I dt + L\frac{dI}{dt} = V_0 e^{i\omega t}$$
$$I_0\, \mathrm{e}^{i(\omega t - \phi)}\left(R + \frac{1}{i\omega C} + i\omega L\right) = V_0 e^{i\omega t}$$
$$Z = R + \frac{1}{i\omega C} + i\omega L$$
$$\therefore V = I\left|Z\right| e^{i\arg(Z)}$$
$$\phi = \arg(Z)$$
$$\therefore V = IZ$$

Ohm's Law, generalized for AC

$$\boxed{Z = Z_L + Z_C + Z_R} \quad \boxed{Z_L = i\omega L} \quad \boxed{Z_C = \frac{1}{i\omega C}}$$



Impedances for L,C,R components

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{Z_R}{Z_L + Z_C + Z_R}$$
$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{R}{R + i\left(\omega L - \frac{1}{\omega C}\right)}$$
$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{RC\omega}{RC\omega + i\left(\omega^2 LC - 1\right)}$$

$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{2\pi f \tau}{2\pi f \tau + i\left(\frac{4\pi^2 f^2}{4\pi^2 f_0^2} - 1\right)}$$
$$\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{2\pi x f_0 \tau}{2\pi x f_0 \tau + i\left(x^2 - 1\right)}$$
$$\boxed{\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{2\pi x \alpha}{2\pi x \alpha + i\left(x^2 - 1\right)}}$$

Note is you want a voltage gain in resonance swap over the resistor and the capacitor in the circuit above

$$x = \frac{f}{f_0} \quad f_0 = \tfrac{1}{2\pi}\sqrt{\frac{1}{LC}}$$
$$\tau = RC \quad \alpha = f_0 \tau$$

Characteristic time and frequency parameters for the RCL circuit