Lagrangian & Hamiltonian mechanics

Rather than starting from the traditional beginnings of *Conservation of Energy, Conservation of momentum* or *Newton's Second Law*, the **Lagrangian** (and related **Hamiltonian**) formulations of dynamics offer a more systematic solution mechanism for mechanical systems, especially those with multiple variables.

In **Classical Mechanics**, the *Lagrangian* L is defined as:

$$L = T - V$$

Kinetic energy (T) — Potential energy (V)

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i}$$

The Euler-Lagrange equation yields the equations of motion

x_i Position or angular coordinate

$\dot{x}_i = \dfrac{dx_i}{dt}$ Rate of change of position or angle with time t

The idea is that the Euer-Lagrange equation extremizes* a quantity called the ***Action***

$$S = \int_{t_1}^{t_2} L\left(x_1, x_2, \ldots x_N, \dot{x}_1, \dot{x}_2, \ldots \dot{x}_N, t\right) dt$$

In many problems the Lagrangian is independent of time, so the Euler-Lagrange equation reduces to the **Beltrami Identity**. This is often the best starting point for problem solving in Lagrangian mechanics.

If $\dfrac{\partial L}{\partial t} = 0$ this means

$$L - \sum_i \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} = \text{constant}$$

Proof of the **Beltrami Identity**

$$\frac{dL}{dt} = \frac{\partial L}{\partial t} + \sum_i \left(\frac{\partial L}{\partial x_i}\dot{x}_i + \frac{\partial L}{\partial \dot{x}_i}\ddot{x}_i\right) \quad \text{Chain Rule}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i} \qquad \frac{d}{dt}L(x,\dot{x},t) = \frac{\partial L}{\partial x}\frac{dx}{dt} + \frac{\partial L}{\partial \dot{x}}\frac{d\dot{x}}{dt} + \frac{\partial L}{\partial t}\frac{dt}{dt}$$

$$\therefore \frac{dL}{dt} = \frac{\partial L}{\partial t} + \sum_i \left(\dot{x}_i \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) + \frac{\partial L}{\partial \dot{x}_i}\ddot{x}_i\right)$$

$$\therefore \frac{dL}{dt} = \frac{\partial L}{\partial t} + \sum_i \frac{d}{dt}\left(\dot{x}_i \frac{\partial L}{\partial \dot{x}_i}\right)$$

Product rule

$$\therefore \frac{d}{dt}\left(L - \sum_i \dot{x}_i \frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial t}$$

$$\therefore \frac{\partial L}{\partial t} = 0 \Rightarrow L - \sum_i \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} = \text{constant}$$

Conservation of Energy for a single particle

$$L = \tfrac{1}{2}mv^2 - V(x,y,z)$$

$$L = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(x,y,z)$$

$$L - \sum_i \dot{x}_i \frac{\partial L}{\partial \dot{x}_i} = k \quad \text{Beltrami Identity}$$

$$\therefore k = L - \left(\dot{x}\frac{\partial L}{\partial \dot{x}} + \dot{y}\frac{\partial L}{\partial \dot{y}} + \dot{z}\frac{\partial L}{\partial \dot{z}}\right)$$

$$k = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)^2 - V - \left(\dot{x}m\dot{x} + \dot{y}m\dot{y} + \dot{z}m\dot{z}\right)$$

$$k = -\tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)^2 - V$$

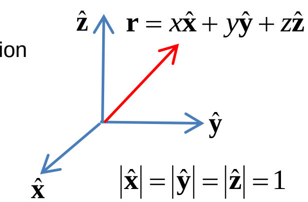
$$\therefore E = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right)^2 + V \qquad (E = -k)$$

i.e. we can associate energy with the (negative) of the constant in the Beltrami Identity

Newton's Second Law for a single particle

$$L = \tfrac{1}{2}m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V(x,y,z)$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial x_i} \quad \text{Euler-Lagrange Equation}$$



$$\therefore \frac{d}{dt}(m\dot{x}) = -\frac{\partial V}{\partial x_i} \Rightarrow m\ddot{x} = -\frac{\partial V}{\partial x}$$

$$\therefore \frac{d}{dt}(m\dot{y}) = -\frac{\partial V}{\partial y_i} \Rightarrow m\ddot{y} = -\frac{\partial V}{\partial y}$$

$$\therefore \frac{d}{dt}(m\dot{z}) = -\frac{\partial V}{\partial z_i} \Rightarrow m\ddot{z} = -\frac{\partial V}{\partial z}$$

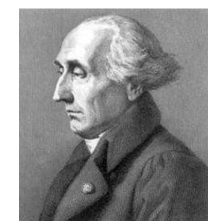
$$\therefore m\ddot{\mathbf{r}} = -\nabla V$$

i.e. using **Cartesian** basis vectors $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$

$$\nabla V = \hat{\mathbf{x}}\frac{\partial V}{\partial x} + \hat{\mathbf{y}}\frac{\partial V}{\partial y} + \hat{\mathbf{z}}\frac{\partial V}{\partial z}$$

Leonhard Euler 1707-1783

i.e. we associate a **force** with a gradient of potential energy, which is entirely consistent with any conservative field such as gravity or electromagnetism.



Joseph-Louis Lagrange 1736-1813

Hamiltonians & total energy

From the penultimate line of the proof of the Beltrami Identity:

$$\frac{d}{dt}\left(L - \sum_i \dot{x}_i \frac{\partial L}{\partial \dot{x}_i}\right) = \frac{\partial L}{\partial t}$$

Define the **Hamiltonian**

$$H = \sum_i p_i \dot{x}_i - L$$

'Canonical momentum'

$$p_i = \frac{\partial L}{\partial \dot{x}_i}$$

$$p_i = \frac{\partial L}{\partial \dot{x}_i} \quad \therefore \dot{x}_i = \frac{\partial H}{\partial p_i}$$

$$\therefore \frac{dH}{dt} = -\frac{\partial L}{\partial t}$$

So we can associate the Hamiltonian with the **total energy**. H is constant if L is time invariant.

Sir William Rowan Hamilton 1805–1865

*i.e. yields either a maximum or minimum value

Example #1: dynamics of a Hookean spring

$$L = \tfrac{1}{2}m\dot{x}^2 - \tfrac{1}{2}kx^2$$

x is the spring extension
k is the spring constant

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$

$$\therefore \boxed{m\ddot{x} = -kx}$$

i.e. Hooke's Law, SHM...

This is perhaps a bit contrived to *start* with the quadratic Elastic Potential Energy expression! .. But at least this shows *consistency* with the Newtonian method.

Example #2: Kepler's Laws for a planet orbiting a massive star (i.e. no movement of star due to gravitational attraction to planet, only vice versa)

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}} \quad \therefore v^2 = \dot{r}^2 + r^2\dot{\theta}^2$$

Plane polar coordinates

$$L = \tfrac{1}{2}m\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{GMm}{r}$$

Note sign! GPE is negative so $-V$ becomes positive

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$$

$$\therefore m\ddot{r} = mr\dot{\theta}^2 - \frac{GMm}{r^2}$$

$$\therefore \boxed{\ddot{r} - r\dot{\theta}^2 = -\frac{GM}{r^2}}$$

Confirm Newton II, with correct formula for centripetal acceleration

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$$

$$\therefore \frac{d}{dt}\left(mr^2\dot{\theta}\right) = 0$$

Conservation of angular momentum

$$\therefore \boxed{mr^2\dot{\theta} = J}$$

The last result is the first glimpse of the utility of the Lagrangian method. The conservation of angular momentum comes straight from the fact that the Lagrangian has no explicit dependence on angle, since KE doesn't, and Newton's Law of Gravitation acts purely radially.

From above:

$$r^2\dot{\theta} = \frac{J}{m} \quad \therefore \dot{\theta} = \frac{J}{mr^2} \quad \therefore r\dot{\theta}^2 = \frac{J^2}{m^2r^3}$$

$$\ddot{r} = r\dot{\theta}^2 - \frac{GM}{r^2}$$

$$\therefore \ddot{r} = \frac{J^2}{m^2r^3} - \frac{GM}{r^2}$$

ε is the *eccentricity* of the ellipse

$$\boxed{\varepsilon = \sqrt{1 - \frac{b^2}{a^2}}}$$

If orbits are *elliptical* (**Kepler's First Law**)

$$r = \frac{a\left(1-\varepsilon^2\right)}{1-\varepsilon\cos\theta} \quad \therefore \varepsilon\cos\theta = 1 - \frac{a\left(1-\varepsilon^2\right)}{r}$$

$$\therefore \dot{r} = \frac{-a\left(1-\varepsilon^2\right)\varepsilon\sin\theta}{\left(1-\varepsilon\cos\theta\right)^2}\dot{\theta} = -\frac{\varepsilon r^2\dot{\theta}\sin\theta}{a\left(1-\varepsilon^2\right)}$$

$$\therefore \dot{r} = -\frac{\varepsilon J\sin\theta}{ma\left(1-\varepsilon^2\right)}$$

$$\therefore \ddot{r} = -\frac{\varepsilon J\cos\theta}{ma\left(1-\varepsilon^2\right)}\dot{\theta} = -\left(1 - \frac{a\left(1-\varepsilon^2\right)}{r}\right)\frac{J}{ma\left(1-\varepsilon^2\right)}\frac{J}{mr^2}$$

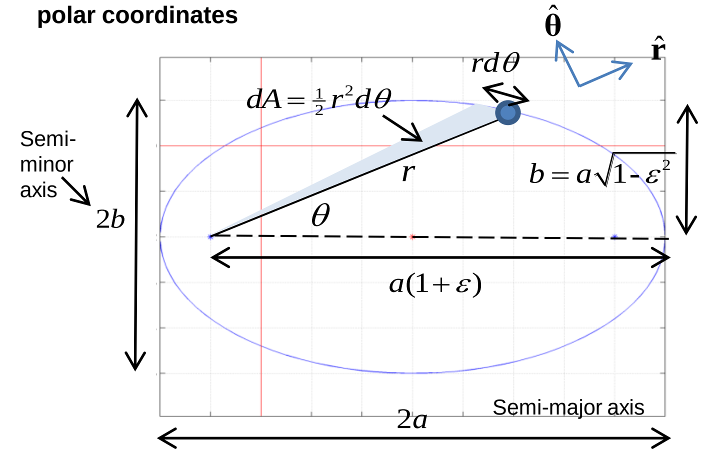
$$\frac{J^2}{m^2r^3} - \frac{GM}{r^2} = \left(-1 + \frac{a\left(1-\varepsilon^2\right)}{r}\right)\frac{J^2}{m^2a\left(1-\varepsilon^2\right)r^2}$$

$$\therefore \frac{J^2}{m^2r^3} - \frac{GM}{r^2} = \frac{J^2}{m^2r^3} - \frac{J^2}{m^2a\left(1-\varepsilon^2\right)r^2}$$

$$\Rightarrow \boxed{J = \sqrt{GMm^2a\left(1-\varepsilon^2\right)}}$$

So elliptical orbits satisfy the laws of motion, and yield an expression for total angular momentum J, which is constant.

An ellipse defined in plane polar coordinates



$$\boxed{\frac{dA}{dt} = \tfrac{1}{2}r^2\dot{\theta} = \tfrac{1}{2}\frac{J}{m} = \tfrac{1}{2}\sqrt{G\left(1-\varepsilon^2\right)a}}$$

So equal areas are swept out in equal times, which is **Kepler's Second Law.**

Since equal areas are swept out in equal times, the orbital period is the area of the ellipse divided by the rate of area sweep. This proves **Kepler's Third Law.**

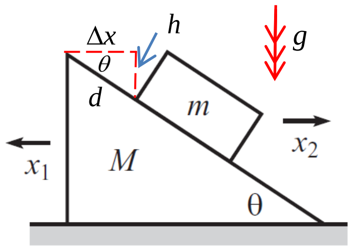
$$P = \frac{\pi ab}{\frac{dA}{dt}} \Rightarrow P = \frac{\pi a^2\sqrt{1-\varepsilon^2}}{\frac{1}{2}\sqrt{GM\left(1-\varepsilon^2\right)a}}$$

Ellipse area is πab

$$\boxed{P^2 = \frac{4\pi^2}{GM}a^3}$$

Kepler's Third Law: The **square** of the orbital **period** of a planet is directly proportional to the **cube** of the **semi-major axis** of its orbit.

Example #3: Moving plane*



Consider a rectangular block of mass m sliding frictionlessly from the top of a planar wedge of mass M. This also slides frictionlessly on a horizontal surface.

Assuming the block has not reached the horizontal surface, after t seconds the horizontal separation is:

$$\Delta x = x_1 + x_2$$

$$\therefore h = (x_1 + x_2)\tan\theta \qquad \text{vertical drop of block}$$

The Lagrangian for the system is:

$$L = T - V$$

$$T = \underbrace{\tfrac{1}{2}M\dot{x}_1^2 + \tfrac{1}{2}m\dot{x}_2^2 + \tfrac{1}{2}m\dot{h}^2}_{\text{Kinetic energy}}$$

$$V = \underbrace{-mgh}_{\text{Gravitational potential energy}}$$

$$\therefore \boxed{L = \tfrac{1}{2}M\dot{x}_1^2 + \tfrac{1}{2}m\dot{x}_2^2 + \tfrac{1}{2}m(\dot{x}_1 + \dot{x}_2)^2\tan^2\theta + mg(x_1 + x_2)\tan\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_1}\right) = \frac{\partial L}{\partial x_1}$$

Applying the Euler-Lagrange equation for mass M

$$\therefore \frac{d}{dt}\left(M\dot{x}_1 + m(\dot{x}_1 + \dot{x}_2)\tan^2\theta\right) = mg\tan\theta$$

$$\therefore \boxed{M\ddot{x}_1 + m(\ddot{x}_1 + \ddot{x}_2)\tan^2\theta = mg\tan\theta}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}_2}\right) = \frac{\partial L}{\partial x_2}$$

Applying the Euler-Lagrange equation for mass m

$$\therefore \frac{d}{dt}\left(m\dot{x}_2 + m(\dot{x}_1 + \dot{x}_2)\tan^2\theta\right) = mg\tan\theta$$

$$\therefore \boxed{m\ddot{x}_2 + m(\ddot{x}_1 + \ddot{x}_2)\tan^2\theta = mg\tan\theta}$$

Hence: $M\ddot{x}_1 - m\ddot{x}_2 = 0 \;\longrightarrow\; \therefore \frac{d}{dt}(M\dot{x}_1 - m\dot{x}_2) = 0$

Which is a statement of **conservation of linear momentum in the horizontal direction** i.e. where there is *no net external force* (there is gravity, but this acts downwards)

$$\therefore \boxed{\ddot{x}_2 = \frac{M}{m}\ddot{x}_1}$$

$$\therefore M\ddot{x}_1 + m\ddot{x}_1\left(1 + \frac{M}{m}\right)\tan^2\theta = mg\tan\theta$$

$$\therefore M\ddot{x}_1 + m\ddot{x}_1\tan^2\theta + M\ddot{x}_1\tan^2\theta = mg\tan\theta$$

$$\therefore \ddot{x}_1\left(M(1 + \tan^2\theta) + m\tan^2\theta\right) = mg\tan\theta$$

$$\therefore \ddot{x}_1\left(\frac{M}{\cos^2\theta} + m\frac{\sin^2\theta}{\cos^2\theta}\right) = mg\frac{\sin\theta}{\cos\theta} \;\longleftarrow\; 1 + \tan^2\theta = \frac{1}{\cos^2\theta}, \quad \tan\theta = \frac{\sin\theta}{\cos\theta}$$

$$\therefore \ddot{x}_1\left(M + m\sin^2\theta\right) = mg\sin\theta\cos\theta$$

$$\boxed{\begin{aligned} \therefore \ddot{x}_1 &= \frac{mg\sin\theta\cos\theta}{M + m\sin^2\theta} = \frac{g\sin\theta\cos\theta}{\frac{M}{m} + \sin^2\theta} \\ \therefore \ddot{x}_2 &= \frac{Mg\sin\theta\cos\theta}{M + m\sin^2\theta} = \frac{g\sin\theta\cos\theta}{1 + \frac{m}{M}\sin^2\theta} \end{aligned}}$$

i.e. **constant acceleration motion** for both block and wedge. If system starts from rest:

$$\boxed{\begin{aligned} \dot{x}_1 &= \frac{g\sin\theta\cos\theta}{\frac{M}{m} + \sin^2\theta}t \\ x_1 &= \tfrac{1}{2}\frac{g\sin\theta\cos\theta}{\frac{M}{m} + \sin^2\theta}t^2 \end{aligned}} \qquad \boxed{\begin{aligned} \dot{x}_2 &= \frac{g\sin\theta\cos\theta}{1 + \frac{m}{M}\sin^2\theta}t \\ x_2 &= \tfrac{1}{2}\frac{g\sin\theta\cos\theta}{1 + \frac{m}{M}\sin^2\theta}t^2 \end{aligned}}$$

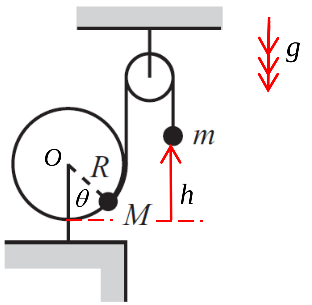
Limiting case when $M \gg m$

$$\boxed{\begin{aligned} \dot{x}_1 &\to 0 \\ x_1 &\to 0 \end{aligned}} \qquad \boxed{\begin{aligned} \dot{x}_2 &= gt\sin\theta\cos\theta \\ x_2 &= \tfrac{1}{2}gt^2\sin\theta\cos\theta \end{aligned}}$$

.. which is consistent with no horizontal movement when the angle tends to 90°.
i.e. in this case the block simply falls vertically!

* Morin, *Classical Mechanics* problem 6.1

Example #4: Hoop & Pulley*



Mass M is fixed to a circular hoop of negligible mass, which can freely rotate about fixed origin O. Between M and m is a light inextensible string, which runs over a light and frictionless pulley. When the system is disturbed by a small amount from equilibrium it undergoes small oscillations of frequency f.

The speed of mass M (which is tangential to the hoop) is:

$$v = R\dot{\theta}$$

Since the string is inextensible, this must also be the speed of the vertically hanging mass m.

The total kinetic energy is therefore:

$$T = \tfrac{1}{2}MR^2\dot{\theta}^2 + \tfrac{1}{2}mR^2\dot{\theta}^2$$

The total (gravitational) potential energy of the system is:

$$V = Mg\left(R - R\cos\theta\right) + mgh$$

Now since the string is inextensible:

$$\dot{h} = -R\dot{\theta} \quad \therefore h = h_0 - R\theta$$
$$\therefore V = MgR - MgR\cos\theta + mgh_0 - mgR\theta$$

The Lagrangian for the system is therefore:

$$L = T - V$$
$$L = \tfrac{1}{2}\left(m+M\right)R^2\dot{\theta}^2 + MgR\cos\theta - MgR - mgh_0 + mgR\theta$$

Applying the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$$
$$\tfrac{1}{2}\left(M+m\right)R^2\dot{\theta}^2 + MgR\cos\theta + mgR\theta - MgR + mgh_0$$
$$\therefore \frac{d}{dt}\left(\left(M+m\right)R^2\dot{\theta}\right) = -MgR\sin\theta + mgR$$
$$\therefore \left(M+m\right)R^2\ddot{\theta} = -MgR\sin\theta + mgR$$
$$\therefore \quad \ddot{\theta} = \frac{g}{R}\left(\frac{m - M\sin\theta}{m+M}\right)$$

Equilibrium occurs when:

$$\ddot{\theta} = 0$$
$$\therefore m - M\sin\theta = 0$$
$$\therefore \theta = \sin^{-1}\left(\tfrac{m}{M}\right)$$

So as M becomes much larger than m, mass M will tend to the bottom of the hoop i.e. $\theta = 0$ which makes sense!

Let us consider small angle deviations from this point:

$$\theta = \theta_{eq} + \delta, \quad \delta \ll \theta_{eq}, \quad \delta \ll 1$$
$$\sin\theta_{eq} = \tfrac{m}{M}$$

$$\therefore \sin\theta = \sin\left(\theta_{eq} + \delta\right)$$
$$\therefore \sin\theta = \sin\theta_{eq}\cos\delta + \cos\theta_{eq}\sin\delta$$
$$\therefore \sin\theta \approx \tfrac{m}{M} + \cos\theta_{eq}\delta$$
$$\therefore \sin\theta \approx \tfrac{m}{M} + \delta\sqrt{1 - \frac{m^2}{M^2}}$$
$$\therefore \sin\theta \approx \tfrac{m}{M} + \delta\tfrac{1}{M}\sqrt{M^2 - m^2}$$
$$\therefore \sin\theta \approx \tfrac{m}{M} + \delta\tfrac{1}{M}\sqrt{(M+m)(M-m)}$$

$$\cos^2\theta + \sin^2\theta = 1$$
$$\therefore \cos\theta = \sqrt{1 - \sin^2\theta}$$

Positive root in range of θ germane to this system

Hence:

$$\ddot{\theta} = \frac{g}{R}\left(\frac{m - M\sin\theta}{m+M}\right)$$
$$\therefore \ddot{\delta} \approx \frac{g}{R}\left(\frac{m - M\left\{\tfrac{m}{M} + \delta\tfrac{1}{M}\sqrt{(M+m)(M-m)}\right\}}{m+M}\right)$$
$$\therefore \ddot{\delta} \approx \frac{g}{R}\left(\frac{m - m - \delta\sqrt{(M+m)(M-m)}}{m+M}\right)$$
$$\therefore \ddot{\delta} \approx -\frac{g}{R}\sqrt{\frac{M-m}{M+m}}\delta$$

This is the equation of **simple harmonic motion** $\ddot{\delta} \approx -\omega^2\delta$ where frequency:

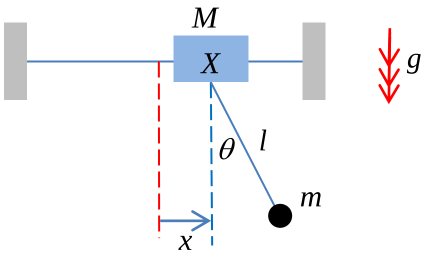
$$f = \frac{\omega}{2\pi}$$

Hence frequency of small oscillations is:

$$f = \tfrac{1}{2\pi}\sqrt{\frac{g}{R}}\left(\frac{M-m}{M+m}\right)^{\frac{1}{4}}$$

* Morin, *Classical Mechanics* problem 6.10

Example #5: Pendulum with free support*



A mass m swings from the central underside X of rectangular block of mass M. The mass m is connected to X via a light inextensible string of length l.

Block M is free to slide horizontally without friction.

It is assumed oscillations of the system are small and the amplitude is such that M does not collide with the side walls.

Let x be the displacement of block M from the centre point between the side walls. The horizontal position of the pendulum is therefore:

$$x_m = x + l\sin\theta$$
$$\therefore \dot{x}_m = \dot{x} + l\cos\theta\dot{\theta}$$

The vertical displacement of the pendulum is:

$$y_m = l\cos\theta$$
$$\therefore \dot{y}_m = -l\sin\theta\dot{\theta}$$

The total kinetic energy of the system is therefore:

$$T = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\left(\dot{x}_m^2 + \dot{y}_m^2\right)$$
$$T = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\left(\dot{x}^2 + 2\dot{x}l\cos\theta\dot{\theta} + l^2\cos^2\theta\dot{\theta}^2 + l^2\sin^2\theta\dot{\theta}^2\right)$$
$$\boxed{T = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\left(\dot{x}^2 + 2\dot{x}l\cos\theta\dot{\theta} + l^2\dot{\theta}^2\right)}$$

The potential energy of the system (taking zero to be at the block height) is:

$$\boxed{V = -mgl\cos\theta}$$

Hence the Lagrangian for the system is:

$$L = T - V$$
$$\boxed{L = \tfrac{1}{2}M\dot{x}^2 + \tfrac{1}{2}m\left(\dot{x}^2 + 2\dot{x}l\cos\theta\dot{\theta} + l^2\dot{\theta}^2\right) + mgl\cos\theta}$$

Applying the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) = \frac{\partial L}{\partial x}$$
$$\frac{d}{dt}\left(M\dot{x} + m\dot{x} + ml\cos\theta\dot{\theta}\right) = 0$$
$$\boxed{(M+m)\ddot{x} - ml\dot{\theta}^2\sin\theta + ml\ddot{\theta}\cos\theta = 0}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$$
$$\frac{d}{dt}\left(m\dot{x}l\cos\theta + ml^2\dot{\theta}\right) = -m\dot{x}l\dot{\theta}\sin\theta - mgl\sin\theta$$
$$m\ddot{x}l\cos\theta - m\dot{x}l\sin\theta\dot{\theta} + ml^2\ddot{\theta} = -m\dot{x}l\dot{\theta}\sin\theta - mgl\sin\theta$$
$$\therefore \boxed{l\ddot{\theta} + \ddot{x}\cos\theta + g\sin\theta = 0}$$

Now consider small angle approximations:

$$\sin\theta \approx \theta, \quad \cos\theta \approx 1 - \tfrac{1}{2}\theta^2$$

$$(M+m)\ddot{x} - ml\dot{\theta}^2\sin\theta + ml\ddot{\theta}\cos\theta = 0$$
$$\therefore (M+m)\ddot{x} - ml\dot{\theta}^2\theta + ml\ddot{\theta}\left(1 - \tfrac{1}{2}\theta^2\right) \approx 0$$
$$\therefore (M+m)\ddot{x} + ml\ddot{\theta} + \theta\left(-ml\dot{\theta}^2\right) + \theta^2\left(-\tfrac{1}{2}ml\ddot{\theta}\right) \approx 0$$
$$\therefore (M+m)\ddot{x} + ml\ddot{\theta} \approx 0$$

$$l\ddot{\theta} + \ddot{x}\cos\theta + g\sin\theta = 0$$
$$\therefore l\ddot{\theta} + \ddot{x} + g\theta \approx 0$$
$$\therefore \ddot{x} \approx -g\theta - l\ddot{\theta}$$

Hence:

$$(M+m)\left(-g\theta - l\ddot{\theta}\right) + ml\ddot{\theta} \approx 0$$
$$\therefore l\ddot{\theta}(-M - m + m) = g\theta(M+m)$$
$$\therefore \ddot{\theta} = -\frac{g}{l}\left(1 + \tfrac{m}{M}\right)\theta$$

SHM: $\ddot{\theta} \approx -\omega^2\theta \qquad f = \dfrac{\omega}{2\pi} \qquad \theta = \theta_0\cos(\omega t - \phi)$

$$\boxed{\therefore f = \tfrac{1}{2\pi}\sqrt{\frac{g}{l}}\left(1 + \tfrac{m}{M}\right)^{\frac{1}{2}}} \qquad \boxed{\omega = \sqrt{\frac{g}{l}}\left(1 + \tfrac{m}{M}\right)^{\frac{1}{2}}}$$

$$\ddot{x} \approx -g\theta - l\ddot{\theta}$$
$$\therefore \ddot{x} = -g\theta + g\left(1 + \tfrac{m}{M}\right)\theta$$
$$\therefore \ddot{x} = \tfrac{m}{M}g\theta_0\cos(\omega t - \phi)$$

General solution is CF+ PI
CF is solution to:

$$\ddot{x} = 0 \quad \therefore x_{CF} = At + B$$

PI is of the form:

$$x_{PI} = C\cos(\omega t - \phi)$$
$$\therefore -\omega^2 C\cos(\omega t - \phi) = \tfrac{m}{M}\theta_0 g\cos(\omega t - \phi)$$
$$\therefore C = -\frac{\tfrac{m}{M}\theta_0 g}{\omega^2} = -\tfrac{m}{M}\theta_0 g\frac{l}{g}\left(1 + \tfrac{m}{M}\right)^{-1} = -\frac{ml\theta_0}{m+M}$$

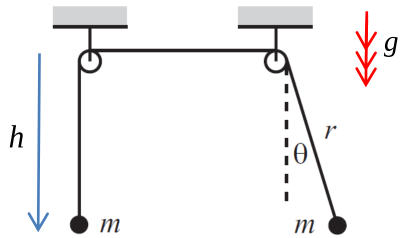
$$\boxed{\begin{aligned}&\therefore x(t) = At + B - \frac{ml\theta_0}{m+M}\cos(\omega t - \phi)\\ &\therefore \theta(t) = \theta_0\cos(\omega t - \phi)\end{aligned}}$$

Sensible initial conditions

$$\boxed{\begin{aligned}&\phi = 0, \quad B = 0\\ &\therefore \dot{x}(0) = A\end{aligned}}$$

* Morin, *Classical Mechanics* problem 6.14

Example #6: Two masses, one swinging*



Two masses are connected via light and inextensible string which run over light and frictionless pulleys.

Since in general the left mass may move, we must assume the most general expression (in polar coordinates r, θ) for the speed of the right mass.

Since the string is inextensible, the rate of increase of r is also the upward velocity of the left mass.

$$\therefore \dot{h} = -\dot{r}$$
$$\therefore h(t) = h_0 - r$$

Hence total kinetic energy is:

$$T = \tfrac{1}{2} m\dot{r}^2 + \tfrac{1}{2} m\left(\dot{r}^2 + r^2\dot{\theta}^2\right)$$

Total potential energy is (taking zero to be at the level of the pulley)

$$V = -mgh - mgr\cos\theta$$
$$V = -mg\left(h_0 - r + r\cos\theta\right)$$

Hence the Lagrangian for the system is:

$$L = T - V$$
$$L = \tfrac{1}{2} m\left(2\dot{r}^2 + r^2\dot{\theta}^2\right) + mg\left(h_0 - r + r\cos\theta\right)$$

Applying the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{r}}\right) = \frac{\partial L}{\partial r}$$
$$\therefore \frac{d}{dt}\left(2m\dot{r}\right) = mr\dot{\theta}^2 - mg + mg\cos\theta$$
$$\therefore \boxed{\ddot{r} = \tfrac{1}{2} r\dot{\theta}^2 + \tfrac{1}{2} g\left(\cos\theta - 1\right)}$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$$
$$\therefore \frac{d}{dt}\left(mr^2\dot{\theta}\right) = -mgr\sin\theta$$
$$\therefore \boxed{\ddot{\theta} = -\frac{g}{r}\sin\theta}$$

Let us assume a small angle approximation ** :

$$\boxed{\begin{aligned}&\ddot{\theta} \approx -\frac{g}{r}\theta\\ &\ddot{r} \approx \tfrac{1}{2} r\dot{\theta}^2 + \tfrac{1}{2} g\left(1 - \tfrac{1}{2}\theta^2 - 1\right) = \tfrac{1}{2}\dot{\theta}^2 - \tfrac{1}{4} g\theta^2\end{aligned}}$$

Let us also assume r changes slowly compared to the pendulum speed. The angular equation is therefore SHM (i.e. r is essentially time invariant compared to θ).

$$\boxed{\begin{aligned}&\theta = \varepsilon\cos\left(\omega t - \phi\right)\\ &\omega = \sqrt{\frac{g}{r}}\end{aligned}}$$

ε is the angular amplitude of the SHM

** $\sin\theta \approx \theta, \ \ \cos\theta \approx 1 - \tfrac{1}{2}\theta^2$

Hence:

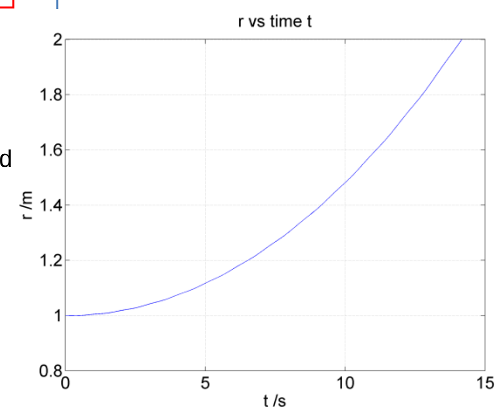
$$\theta = \varepsilon\cos\left(\omega t - \phi\right)$$
$$\dot{\theta} = -\varepsilon\omega\sin\left(\omega t - \phi\right)$$

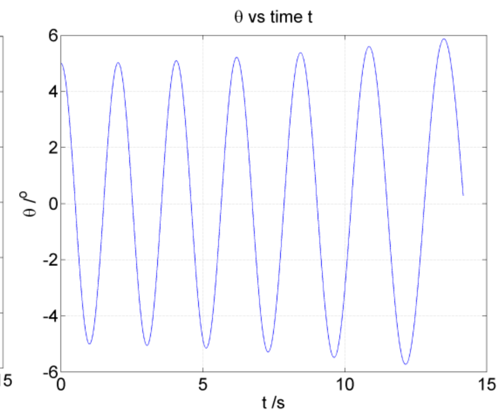
$$\ddot{r} \approx \tfrac{1}{2} r\dot{\theta}^2 - \tfrac{1}{4} g\theta^2$$
$$\therefore \ddot{r} \approx \tfrac{1}{2} r\varepsilon^2\omega^2\sin^2\left(\omega t - \phi\right) - \tfrac{1}{4} g\varepsilon^2\cos^2\left(\omega t - \phi\right)$$
$$\therefore \ddot{r} \approx \tfrac{1}{2} r\varepsilon^2\frac{g}{r}\sin^2\left(\omega t - \phi\right) - \tfrac{1}{4} g\varepsilon^2\cos^2\left(\omega t - \phi\right)$$
$$\therefore \ddot{r} \approx \tfrac{1}{2} g\varepsilon^2\left\{\sin^2\left(\omega t - \phi\right) - \tfrac{1}{2}\cos^2\left(\omega t - \phi\right)\right\}$$
$$\therefore \ddot{r} \approx \tfrac{1}{2} g\varepsilon^2\left\{1 - \cos^2\left(\omega t - \phi\right) - \tfrac{1}{2}\cos^2\left(\omega t - \phi\right)\right\}$$
$$\therefore \ddot{r} \approx \tfrac{1}{2} g\varepsilon^2\left\{1 - \tfrac{3}{2}\cos^2\left(\omega t - \phi\right)\right\}$$

The average value of the harmonic term is: $\overline{\cos^2\left(\omega t - \phi\right)} = \tfrac{1}{2}$

Hence: $\boxed{\ddot{r} \approx \tfrac{1}{2} g\varepsilon^2\left\{1 - \tfrac{3}{2}\tfrac{1}{2}\right\} \approx \tfrac{1}{8} g\varepsilon^2}$

So we might expect the amplitude of the pendulum swing to slowly increase and therefore the left mass to slowly rise.



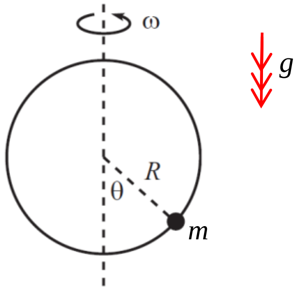


$r_0 = 1\text{m}, \ g = 9.81\text{ms}^{-2}, \ \theta_0 = 5^\circ, \ \dot{\theta}(0) = 0, \ \Delta t = 0.01\text{s}$

Verlet solver
i.e. constant acceleration motion between fixed time steps

* Morin, *Classical Mechanics* problem 6.4

Example #7: Bead on a rotating hoop*



A bead of mass m is free to slide frictionlessly on a circular hoop of radius R. The hoop itself rotates at constant angular speed ω about a vertical axis.

The kinetic energy of the bead is:
(noting motion is constrained to the circle in the plane, and also circular into the plane)

$$T = \tfrac{1}{2}m\left(R\dot{\theta}\right)^2 + \tfrac{1}{2}m\left(\omega R\sin\theta\right)^2$$

$$T = \tfrac{1}{2}mR^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right)$$

Taking zero GPE to be the lowest point on the hoop, the potential energy of the bead is:

$$V = mgR(1-\cos\theta)$$

Hence the Lagrangian for the system is:

$$L = T - V$$

$$L = \tfrac{1}{2}mR^2\left(\dot{\theta}^2 + \omega^2\sin^2\theta\right) + mgR\cos\theta - mgR$$

Applying the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) = \frac{\partial L}{\partial \theta}$$

$$\therefore \frac{d}{dt}\left(mR^2\dot{\theta}\right) = mR^2\omega^2\sin\theta\cos\theta - mgR\sin\theta$$

$$\therefore \ddot{\theta} = -\left(\frac{g}{R} - \omega^2\cos\theta\right)\sin\theta$$

So equilibrium when:

$$\frac{g}{R} - \omega^2\cos\theta_0 = 0$$

$$\Rightarrow \theta_0 = \cos^{-1}\left(\frac{g}{R\omega^2}\right)$$

$$\sin\theta_0 = 0$$

$$\Rightarrow \theta_0 = 0, \pi$$

The 'bead at the top of the hoop' is clearly an *unstable* equilibrium.

Using a small angle perturbation from the bottom of the hoop:

$$\ddot{\theta} \approx -\left(\frac{g}{R} - \omega^2\right)\theta$$

which is SHM of frequency

$$f = \tfrac{1}{2\pi}\sqrt{\frac{g}{R} - \omega^2}$$

Let us also consider small oscillations about: $\theta_0 = \cos^{-1}\left(\frac{g}{R\omega^2}\right)$

$$\theta = \theta_0 + \delta, \quad \delta \ll 1, \qquad \cos\theta_0 = \frac{g}{R\omega^2}$$

$$\sin\theta = \sin\left(\theta_0 + \delta\right) = \sin\theta_0\cos\delta + \cos\theta_0\sin\delta \approx \sin\theta_0 + \delta\cos\theta_0$$

$$\cos\theta = \cos\left(\theta_0 + \delta\right) = \cos\theta_0\cos\delta - \sin\theta_0\sin\delta \approx \cos\theta_0 - \delta\sin\theta_0$$

$$\therefore \sin\theta\cos\theta \approx \left(\sin\theta_0 + \delta\cos\theta_0\right)\left(\cos\theta_0 - \delta\sin\theta_0\right)$$

$$\therefore \sin\theta\cos\theta \approx \sin\theta_0\cos\theta_0 + \delta\cos^2\theta_0 - \delta\sin^2\theta_0 - \delta^2\sin\theta_0\cos\theta_0$$

$$\therefore \sin\theta\cos\theta \approx \sin\theta_0\cos\theta_0 + \delta\left(1 - 2\sin^2\theta_0\right)$$

$$\ddot{\theta} = -\left(\frac{g}{R} - \omega^2\cos\theta\right)\sin\theta$$

$$\therefore \ddot{\delta} \approx -\left(\frac{g}{R}\left(\sin\theta_0 + \cos\theta_0\delta\right) - \omega^2\sin\theta_0\cos\theta_0 - \omega^2\delta\left(1 - 2\sin^2\theta_0\right)\right)$$

$$\therefore \ddot{\delta} \approx -\left(\frac{g}{R}\left(\sin\theta_0 + \frac{g}{R\omega^2}\delta\right) - \omega^2\sin\theta_0\frac{g}{R\omega^2} - \omega^2\delta\left(1 - 2\sin^2\theta_0\right)\right)$$

$$\therefore \ddot{\delta} \approx -\left(\frac{g^2\delta}{R^2\omega^2} - \omega^2\delta + \omega^2\delta\sin^2\theta_0 + \omega^2\delta\sin^2\theta_0\right)$$

$$\therefore \ddot{\delta} \approx -\left(\frac{g^2\delta}{R^2\omega^2} - \omega^2\delta + \omega^2\delta - \omega^2\delta\cos^2\theta_0 + \omega^2\delta\sin^2\theta_0\right)$$

$$\therefore \ddot{\delta} \approx -\left(\frac{g^2\delta}{R^2\omega^2} - \omega^2\delta\frac{g^2}{R^2\omega^4} + \omega^2\delta\sin^2\theta_0\right)$$

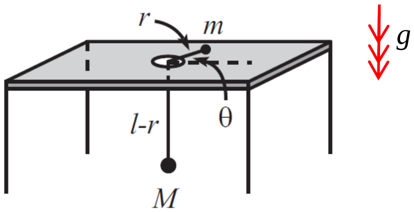
$$\therefore \ddot{\delta} \approx -\omega^2\delta\sin^2\theta_0 = -\omega^2\left(1 - \frac{g^2}{R^2\omega^4}\right)\delta = -\left(\omega^2 - \frac{g^2}{R^2\omega^2}\right)\delta$$

which is SHM of frequency:

$$f = \tfrac{1}{2\pi}\sqrt{\omega^2 - \frac{g^2}{R^2\omega^2}} \qquad \leftarrow \qquad \delta(t) = \delta_0\cos\left(2\pi ft - \phi\right)$$

* Morin, *Classical Mechanics* problem 6.11

Example #8: Small oscillations of whirly bungs on a table*



Mass m can slide without friction on a horizontal table. It is connected via a light an inextensible string of length l to mass M through a hole in the table. It is assumed that M only moves vertically.

The kinetic energy of the system is:

$$T=\tfrac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)+\tfrac{1}{2}M\dot{r}^2$$

Note since the string is inextensible, the downward velocity of M is $-\dot{r}$

Taking zero GPE to be the level of the table

$$V=-Mg(l-r)$$

Hence the Lagrangian for the system is:

$$L=T-V$$

$$L=\tfrac{1}{2}m\left(\dot{r}^2+r^2\dot{\theta}^2\right)+\tfrac{1}{2}M\dot{r}^2+Mgl-Mgr$$

$$L=\tfrac{1}{2}\left(m+M\right)\dot{r}^2+\tfrac{1}{2}mr^2\dot{\theta}^2+Mgl-Mgr$$

Applying the Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}}\right)=\frac{\partial L}{\partial\theta}$$

$$\therefore\frac{d}{dt}\left(mr^2\dot{\theta}\right)=0$$

$$\therefore mr^2\dot{\theta}=J$$

i.e. conservation of **angular momentum** J

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{r}}\right)=\frac{\partial L}{\partial r}$$

$$\tfrac{1}{2}\left(m+M\right)\dot{r}^2+\tfrac{1}{2}mr^2\dot{\theta}^2+Mgl-Mgr$$

$$\therefore\frac{d}{dt}\left(\left(m+M\right)\dot{r}\right)=mr\dot{\theta}^2-Mg$$

$$\therefore\left(m+M\right)\ddot{r}=mr\dot{\theta}^2-Mg$$

$$\dot{\theta}=\frac{J}{mr^2}$$

$$\therefore\left(m+M\right)\ddot{r}=mr\frac{J^2}{m^2r^4}-Mg$$

$$\therefore\left(m+M\right)\ddot{r}=\frac{J^2}{mr^3}-Mg$$

Equilibrium when:

$$\frac{J^2}{mr^3}-Mg=0$$

$$\therefore r_0=\left(\frac{J^2}{mMg}\right)^{\frac{1}{3}}$$

i.e. circular motion of mass m

Consider small perturbation in r about this value:

$$r_0^3=\frac{J^2}{mMg}$$

$$r=r_0+\delta,\quad\delta\ll 1$$

$$\left(m+M\right)\ddot{r}=\frac{J^2}{mr^3}-Mg$$

$$\therefore\left(m+M\right)\ddot{\delta}=\frac{J^2}{mr_0^3\left(1+\frac{\delta}{r_0}\right)^3}-Mg$$

$$\therefore\left(m+M\right)\ddot{\delta}\approx\frac{J^2}{m}\frac{mMg}{J^2}\left(1-\tfrac{3\delta}{r_0}\right)-Mg$$

$$\therefore\ddot{\delta}\approx\frac{Mg}{m+M}\left(1-\tfrac{3\delta}{r_0}\right)-\frac{Mg}{m+M}$$

$$\therefore\ddot{\delta}\approx-\frac{3M}{m+M}\frac{g}{r_0}\delta$$

which is SHM of frequency:

$$f=\tfrac{1}{2\pi}\sqrt{\frac{3M}{m+M}}\sqrt{\frac{g}{r_0}}$$

$$r(t)=r_0+\delta_0\cos\left(2\pi ft-\phi\right)$$

* Morin, *Classical Mechanics* pp 240-241