Modelling air resistance & drag forces

At *low speeds* e.g. the drag force on a ball bearing falling through a viscous fluid: $F = kv$

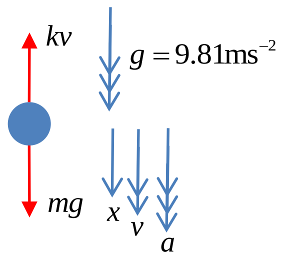
At *subsonic speeds* e.g. the aerodynamic drag on a car, aircraft or parachutist: $F = kv^2$

For 'v^2 drag', the constant k is: $k = \frac{1}{2} c_D \rho A$

- A: cross sectional area of object perpendicular to velocity
- c_D: Drag coefficient
- ρ: fluid density (air is about 1kgm^{-3})

Note **lift** due to a wing (i.e. an *aerofoil*) has a very similar formula $F = \frac{1}{2} c_L \rho A v^2$
In this case we have a *lift coefficient,* which is an empirically determined function of wing shape. Note both drag and lift coefficients are *dimensionless* quantities (i.e. pure numbers).

Low speed drag $F = kv$



$$m\frac{dv}{dt} = mg - kv$$ Newton II

$$\frac{dv}{g - \frac{k}{m}v} = dt$$ Separate the variables

$$\int_u^v \frac{dv}{g - \frac{k}{m}v} = \int_0^t dt$$ Initial velocity u

$$-\frac{m}{k}\int_u^v \frac{-\frac{k}{m}dv}{g - \frac{k}{m}v} = t \qquad \int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$\left[-\frac{m}{k}\ln\left|g - \frac{k}{m}v\right|\right]_u^v = t$$

$$\ln\left|\frac{1 - \frac{k}{mg}v}{1 - \frac{k}{mg}u}\right| = -\frac{kt}{m}$$

$kv \le mg$
$ku \le mg$

Assume drag always less than weight (e.g. a ball dropped into a fluid from rest, rather than propelled into it!

$$1 - \frac{k}{mg}v = \left(1 - \frac{k}{mg}u\right)e^{-\frac{kt}{m}}$$

$$\frac{mg}{k} - v = \left(\frac{mg}{k} - u\right)e^{-\frac{kt}{m}}$$

$$v = \frac{mg}{k} - \left(\frac{mg}{k} - u\right)e^{-\frac{kt}{m}}$$

$$\frac{dx}{dt} = \frac{mg}{k} - \left(\frac{mg}{k} - u\right)e^{-\frac{kt}{m}}$$

$$\int_0^x dx = \int_0^t \left(\frac{mg}{k} - \left(\frac{mg}{k} - u\right)e^{-\frac{kt}{m}}\right)dt$$

$$x = \left[\frac{mg}{k}t + \frac{m}{k}\left(\frac{mg}{k} - u\right)e^{-\frac{kt}{m}}\right]_0^t$$

$$x = \frac{mg}{k}t + \frac{m}{k}\left(\frac{mg}{k} - u\right)e^{-\frac{kt}{m}} - \frac{m}{k}\left(\frac{mg}{k} - u\right)$$

$$x = \frac{mg}{k}t - \frac{m}{k}\left(\frac{mg}{k} - u\right)\left(1 - e^{-\frac{kt}{m}}\right)$$

i.e. asymptotic behaviour is for velocity to tend to towards 'terminal velocity' $v_\infty = \frac{mg}{k}$

Alternative derivation using $a = \frac{dv}{dt} = \frac{dv}{dx} \times \frac{dx}{dt} = \frac{vdv}{dx}$

$$m\frac{vdv}{dx} = mg - kv \Rightarrow \frac{vdv}{g - \frac{k}{m}v} = dx \Rightarrow \int_u^v \frac{vdv}{g - \frac{k}{m}v} = \int_0^t dx$$

$$\frac{v}{g - \frac{k}{m}v} = A + \frac{B}{g - \frac{k}{m}v} = \frac{A\left(g - \frac{k}{m}v\right) + B}{g - \frac{k}{m}v}$$

$$v: \quad 1 = -\frac{k}{m}A \Rightarrow A = -\frac{m}{k}$$

$$v^0: \quad 0 = Ag + B \Rightarrow B = -Ag = \frac{mg}{k}$$

$$\therefore \int_u^v \frac{vdv}{g - \frac{k}{m}v} = \frac{m}{k}\int_u^v \left(-1 + \frac{g}{g - \frac{k}{m}v}\right)dv$$

$$\frac{m}{k}\int_u^v \left(-1 + \frac{g}{g - \frac{k}{m}v}\right)dv = \frac{m}{k}\int_u^v \left(\frac{1}{1 - \frac{k}{mg}v} - 1\right)dv = \frac{m}{k}\int_u^v \left(-\frac{mg}{k}\frac{-\frac{k}{mg}}{1 - \frac{k}{mg}v} - 1\right)dv$$

$$\therefore x = \left[-\frac{m^2 g}{k^2}\ln\left|1 - \frac{k}{mg}v\right| - \frac{mv}{k}\right]_u^v$$

$$x = \frac{m^2 g}{k^2}\ln\left|\frac{1 - \frac{ku}{mg}}{1 - \frac{k}{mg}v}\right| - \frac{m}{k}(v - u)$$

This requires a numeric method to find *v(x)*

For ***Stokes drag***, i.e. the drag force on a small sphere falling in a viscous liquid

$$F = 6\pi\mu r v$$

r is the radius of the sphere and μ is the dynamic viscosity of the fluid (units are kgm^{-1}s^{-1})

Material	μ / kgm^{-1}s^{-1}
Water	8.9×10^{-4}
Blood	3×10^{-3}
Honey	2-10
Ketchup	50-100
Peanut butter	250
Glycerol	1.2
Olive oil	0.08

Material	μ / kgm^{-1}s^{-1}
Castor oil	0.99
Mercury	1.5×10^{-3}
Molten chocolate	10-25
Pitch	2.3×10^{8}
Upper mantle	10^{21}


George Gabriel Stokes (1819-1903)

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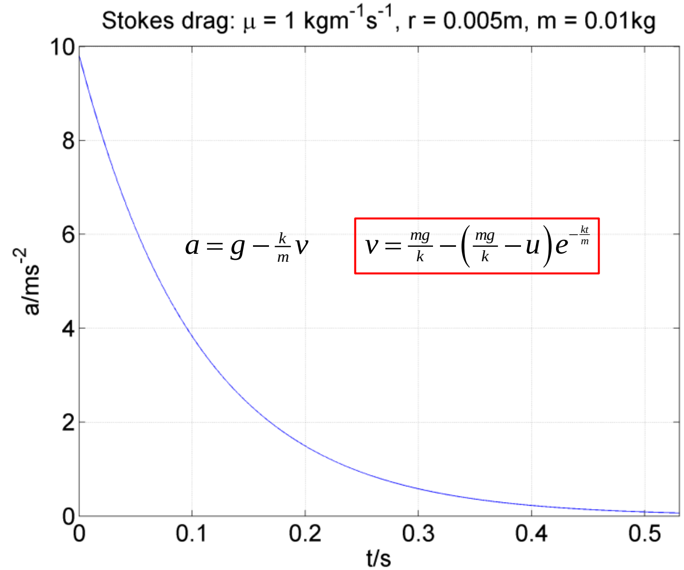

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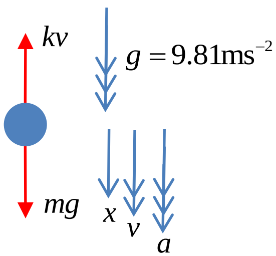
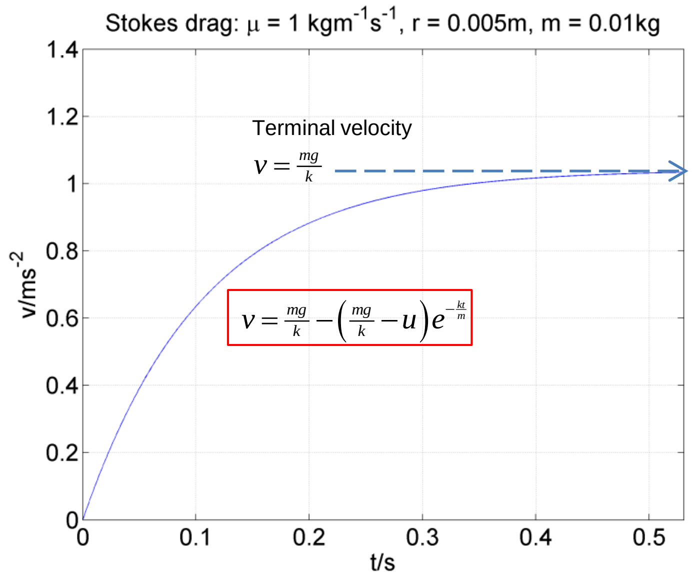
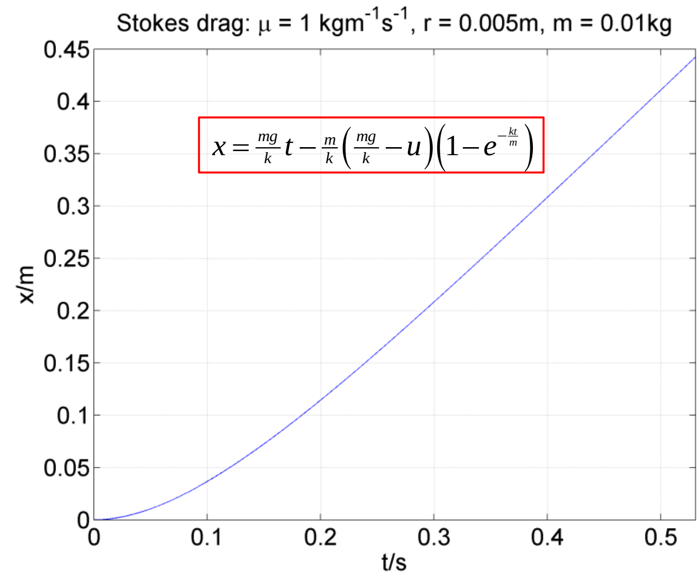
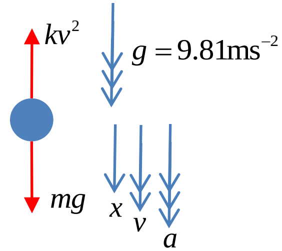






Subsonic drag $F = kv^2$



$k = \frac{1}{2}c_D \rho A$ ← cross sectional area of object perpendicular to velocity

Drag coefficient (c_D)

fluid density (air is about 1kgm-3) (ρ)

Newton II

$$m\frac{dv}{dt} = mg - kv^2$$

$$\int_u^v \frac{dv}{g - \frac{k}{m}v^2} = \int_0^t dt$$

$$\int_u^v \frac{dv}{1 - \frac{k}{mg}v^2} = gt$$

$$z^2 = \frac{k}{mg}v^2 \Rightarrow z = v\sqrt{\frac{k}{mg}}$$

$$dv = \sqrt{\frac{mg}{k}}dz$$

$$\int_{u\sqrt{\frac{k}{mg}}}^{v\sqrt{\frac{k}{mg}}} \sqrt{\frac{mg}{k}}\frac{dz}{1 - z^2} = gt$$

$$\left[\sqrt{\frac{mg}{k}}\tanh^{-1} z\right]_{u\sqrt{\frac{k}{mg}}}^{v\sqrt{\frac{k}{mg}}} = gt$$

$$\tanh^{-1} v\sqrt{\frac{k}{mg}} = \sqrt{\frac{k}{mg}}gt + \tanh^{-1}\left(u\sqrt{\frac{k}{mg}}\right)$$

$$\boxed{v = \sqrt{\frac{mg}{k}}\tanh\left(\sqrt{\frac{k}{mg}}gt + \tanh^{-1}\left(u\sqrt{\frac{k}{mg}}\right)\right)}$$

$$x = \int_0^t v\,dt = \sqrt{\frac{mg}{k}}\int_0^t \tanh\left(\sqrt{\frac{k}{mg}}gt + \tanh^{-1}\left(u\sqrt{\frac{k}{mg}}\right)\right)$$

$$x = \frac{1}{g}\sqrt{\frac{mg}{k}}\sqrt{\frac{mg}{k}}\left[\ln\left(\cosh\left(\sqrt{\frac{k}{mg}}gt + \tanh^{-1}\left(u\sqrt{\frac{k}{mg}}\right)\right)\right)\right]_0^t$$

$$\boxed{x = \frac{m}{k}\ln\left(\frac{\cosh\left(\sqrt{\frac{k}{mg}}gt + \tanh^{-1}\left(u\sqrt{\frac{k}{mg}}\right)\right)}{\cosh\left(\tanh^{-1}\left(u\sqrt{\frac{k}{mg}}\right)\right)}\right)}$$

Note asymptotic behaviour is for velocity to tend to towards ‘terminal velocity’ $\boxed{v_\infty = \sqrt{\frac{mg}{k}}}$

$$\int \frac{a}{1 - a^2x^2} = \tanh^{-1} ax + c$$

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Alternative derivation using $\boxed{a = \frac{dv}{dt} = \frac{dv}{dx}\times\frac{dx}{dt} = \frac{vdv}{dx}}$

$$m\frac{vdv}{dx} = mg - kv^2 \qquad \text{Newton II}$$

$$\int_u^v \frac{vdv}{g - \frac{k}{m}v^2} = \int_0^x dx \qquad \int \frac{f'(x)}{f(x)}dx = \ln|f(x)| + c$$

$$-\frac{m}{2k}\int_u^v \frac{\left(-\frac{2k}{m}\right)vdv}{g - \frac{k}{m}v^2} = \int_0^x dx$$

$$\left[-\frac{m}{2k}\ln\left|g - \frac{k}{m}v^2\right|\right]_u^v = x$$

$$\ln\left|\frac{g - \frac{k}{m}v^2}{g - \frac{k}{m}u^2}\right| = -\frac{2kx}{m}$$

$$\ln\left|\frac{1 - \frac{k}{mg}v^2}{1 - \frac{k}{mg}u^2}\right| = -\frac{2kx}{m}$$

$$\left|\frac{1 - \frac{k}{mg}v^2}{1 - \frac{k}{mg}u^2}\right| = e^{-\frac{2kx}{m}}$$

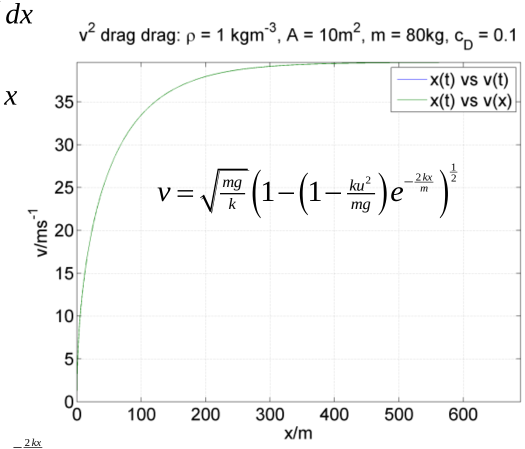
$$1 - \frac{k}{mg}v^2 = \left(1 - \frac{k}{mg}u^2\right)e^{-\frac{2kx}{m}}$$

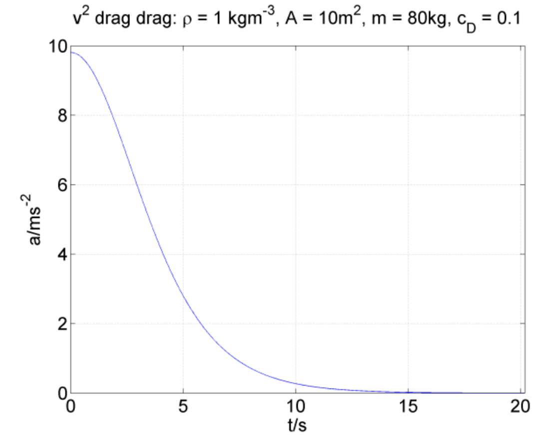
$$\boxed{v = \sqrt{\frac{mg}{k}}\left(1 - \left(1 - \frac{ku^2}{mg}\right)e^{-\frac{2kx}{m}}\right)^{\frac{1}{2}}}$$

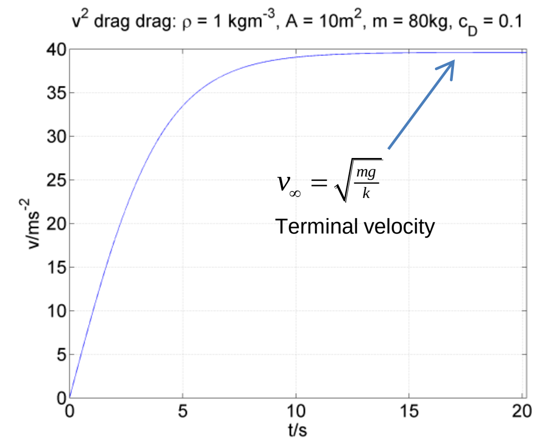
Assume drag always less than weight i.e.

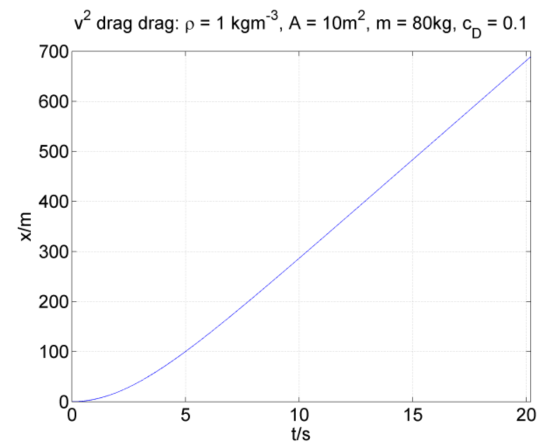
$$kv^2 \le mg$$

$$ku^2 \le mg$$



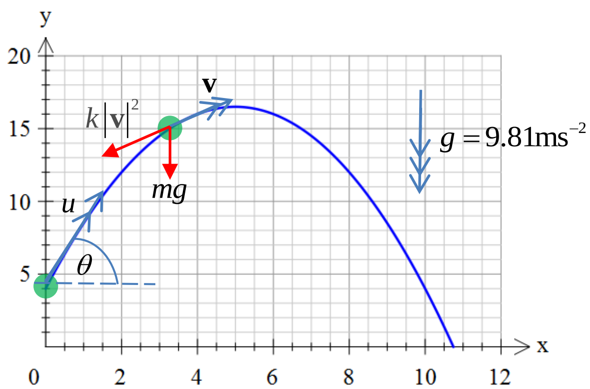






$$\int \tanh(ax + b)dx = \frac{1}{a}\ln(\cosh(ax + b)) + c$$

Projectile motion incorporating drag



Newton II

$$m\frac{d\mathbf{v}}{dt} = -mg\hat{\mathbf{y}} - k|\mathbf{v}|^2\,\hat{\mathbf{v}}$$

The fact that the drag force always opposes the velocity vector makes this equation difficult to integrate *analytically*. i.e. you cannot separate x and y components of velocity

Initial conditions

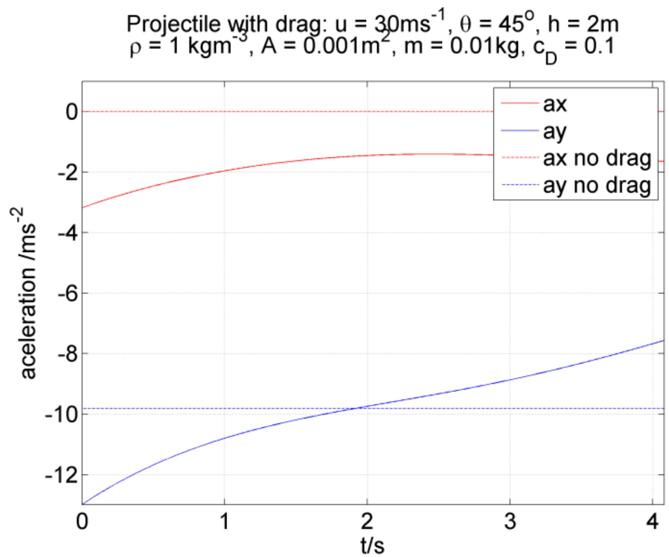
$$t = 0$$
$$x = 0$$
$$y = h$$
$$\mathbf{v}_0 = u\cos\theta\hat{\mathbf{x}} + u\sin\theta\hat{\mathbf{y}}$$

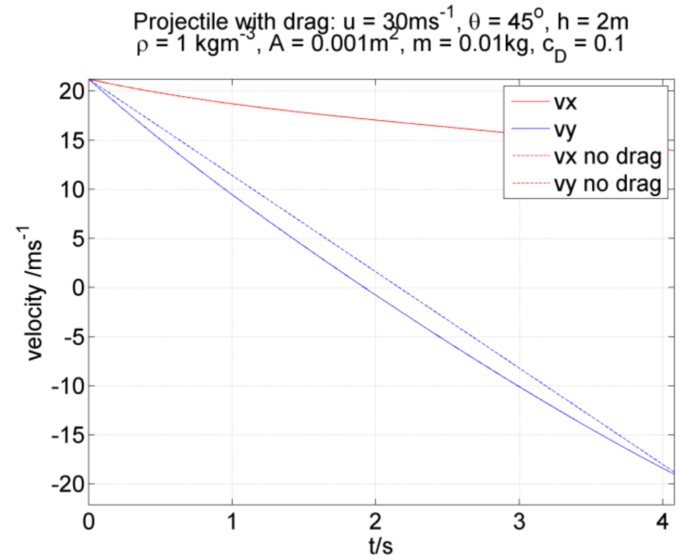
Simple numerical method for finding velocity and x,y coordinates

$$t_{n+1} = t_n + \Delta t$$
$$\mathbf{v}_{n+1} = \mathbf{v}_n + \Delta\mathbf{v}_n$$
$$\Delta\mathbf{v}_n = \left(-g\hat{\mathbf{y}} - \frac{k}{m}|\mathbf{v}_n|\mathbf{v}_n\right)\Delta t$$
$$\hat{\mathbf{v}}_n = \frac{\mathbf{v}_n}{|\mathbf{v}_n|} \quad \therefore |\mathbf{v}_n|^2\,\hat{\mathbf{v}}_n = |\mathbf{v}_n|\mathbf{v}_n$$
$$x_{n+1} = x_n + (\mathbf{v}_n \cdot \hat{\mathbf{x}})\Delta t$$
$$y_{n+1} = y_n + (\mathbf{v}_n \cdot \hat{\mathbf{y}})\Delta t$$

$$\frac{d\mathbf{v}}{dt} = -g\hat{\mathbf{y}} - \frac{k}{m}|\mathbf{v}|^2\,\hat{\mathbf{v}}$$

The idea is we fix a small, constant time step Δt, and consider constant acceleration and velocity between time steps. We can refine our technique by using a more accurate fit. e.g. constant acceleration motion between steps (**Verlet**) or a 'fourth-order method' (i.e. errors in Δt^4) such as **Runge-Kutta**.



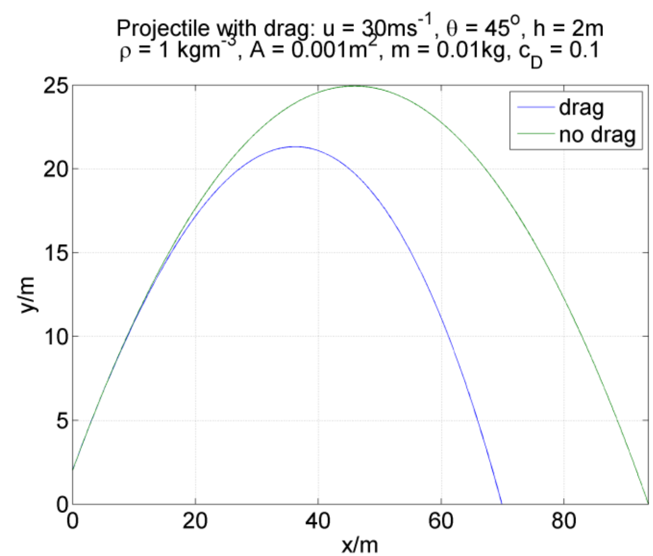


$$k = \tfrac{1}{2}c_D\rho A$$

cross sectional area of object perpendicular to velocity

Drag coefficient

fluid density (air is about 1kgm^{-3})



Verlet method

$$t_{n+1} = t_n + \Delta t$$
$$\mathbf{v}_{n+1} = \mathbf{v}_n + \Delta\mathbf{v}_n$$
$$\Delta\mathbf{v}_n = \left(-g\hat{\mathbf{y}} - \frac{k}{m}|\mathbf{v}_n|\mathbf{v}_n\right)\Delta t$$
$$\hat{\mathbf{v}}_n = \frac{\mathbf{v}_n}{|\mathbf{v}_n|} \quad \therefore |\mathbf{v}_n|^2\,\hat{\mathbf{v}}_n = |\mathbf{v}_n|\mathbf{v}_n$$
$$x_{n+1} = x_n + (\mathbf{v}_n \cdot \hat{\mathbf{x}})\Delta t + \tfrac{1}{2}\left(-\frac{k}{m}|\mathbf{v}_n|\mathbf{v}_n \cdot \hat{\mathbf{x}}\right)\Delta t^2$$
$$y_{n+1} = y_n + (\mathbf{v}_n \cdot \hat{\mathbf{y}})\Delta t + \tfrac{1}{2}\left(-g - \frac{k}{m}|\mathbf{v}_n|\mathbf{v}_n \cdot \hat{\mathbf{y}}\right)\Delta t^2$$

The Verlet method uses extra terms for the x,y computation to take into account the approximation of constant acceleration between time steps.