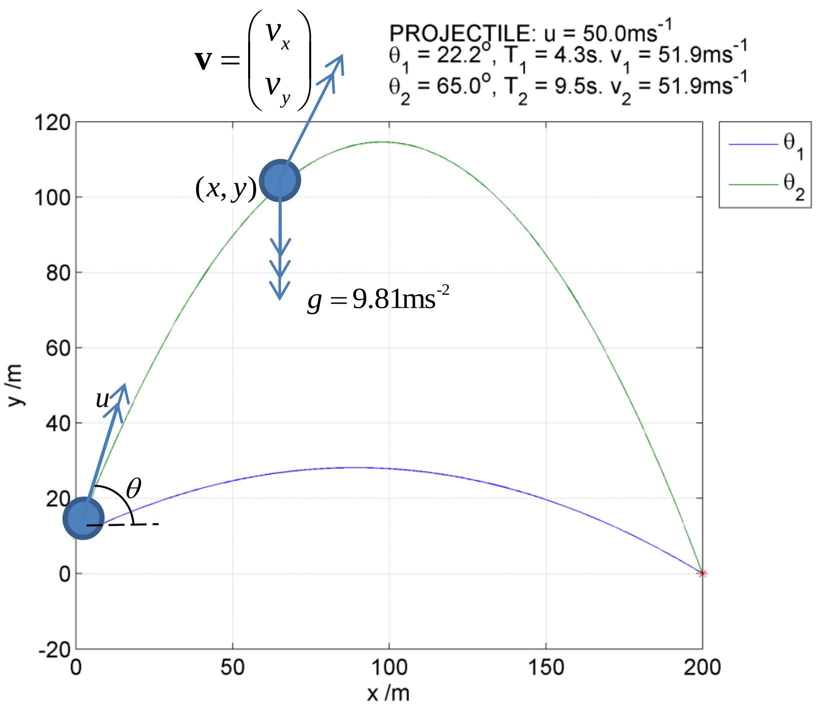
Projectiles are typically modelled as point masses (i.e. 'particles') falling under gravity. In other words, internal motion and rotation is ignored and only the centre of mass of the projectile is considered. *Air resistance is often ignored* to enable analysis to proceed without a computer. Note this assumption may be significantly invalid for many real projectiles! Hence this system reduces to a *two dimensional kinematics problem, where acceleration is constant.*

Let the coordinates of the projectile be (x,y) on a Cartesian grid. Let the initial velocity be u at an elevation of θ and let the projectile be launched from $(0,h)$
Since acceleration is constant:

$$v_x = u\cos\theta$$
$$v_y = u\sin\theta - gt$$
$$v_y^2 = u^2\sin^2\theta - 2g(y-h)$$
$$x = ut\cos\theta$$
$$y = h + ut\sin\theta - \tfrac{1}{2}gt^2$$

Note this means the x direction velocity is *always constant* throughout the motion!



We can therefore combine these equations to find various properties of the projectile's trajectory

$$x = ut\cos\theta$$
$$\therefore t = \frac{x}{u\cos\theta}$$
$$\therefore y = h + x\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)x^2$$

$$\frac{1}{\cos^2\theta} = 1+\tan^2\theta$$

i.e. a projectile trajectory is an *inverted parabola*

If the projectile is required to pass through (or collide with!) a particular coordinate (X,Y), we can solve the quadratic trajectory equation to determine the elevation angle, given speed u is known. This calculation relates to models of all ball sports, gunnery (ballistics) etc.

$$Y = h + X\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)X^2$$
$$a\tan^2\theta + b\tan\theta + c = 0$$

$$a = \frac{g}{2u^2}X^2$$
$$b = -X$$
$$c = Y - h + \frac{gX^2}{2u^2}$$
$$\theta = \tan^{-1}\left(\frac{-b \pm \sqrt{b^2-4ac}}{2a}\right)$$

Note *multiple solutions* are possible, depending on the sign of the *discriminant* $b^2 - 4ac$

Elevation angles which give rise to a *zero discriminant* define the *bounding parabola* for the projectile (see next page).

The *apogee* of the trajectory is when $v_y = 0$

$$v_y = u\sin\theta - gt \quad \therefore \; v_y = 0 \Rightarrow t_a = \frac{u}{g}\sin\theta$$

$$v_y^2 = u^2\sin^2\theta - 2g(y-h) \quad \therefore \; v_y = 0 \Rightarrow y_a = h + \frac{u^2}{2g}\sin^2\theta$$

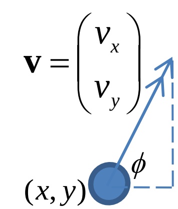
$$x_a = ut_a\cos\theta \quad \therefore \; x_a = \frac{u^2}{g}\sin\theta\cos\theta$$

The speed v of the projectile is:

$$v = \sqrt{v_x^2 + v_y^2}$$
$$v = \sqrt{u^2\cos^2\theta + u^2\sin^2\theta - 2g(y-y_0)}$$
$$v = \sqrt{u^2 - 2g(y-y_0)}$$

Compute angle of velocity using:

$$\phi = \tan^{-1}\left(\frac{v_y}{v_x}\right)$$



Possible values for u and the bounding parabola

$$Y = h + X\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)X^2$$

Trajectory equation

$$a\tan^2\theta + b\tan\theta + c = 0$$

$$a = \frac{g}{2u^2}X^2$$

$$b = -X$$

$$c = Y - h + \frac{gX^2}{2u^2}$$

$$\theta = \tan^{-1}\left(\frac{-b\pm\sqrt{b^2-4ac}}{2a}\right)$$

For real values of θ: $b^2-4ac\ge 0$

Without loss of generality, set a coordinate system such that $h=0$ i.e. vary the target coordinates X,Y instead, by shifting the origin

$$X^2 - 4\left(-\frac{gX^2}{2u^2}\right)\left(-Y-\frac{g}{2u^2}X^2\right)\ge 0$$

$$2u^4X^2 - 2gX^2\left(2Yu^2+gX^2\right)\ge 0$$

$$u^4 - 2Ygu^2 - g^2X^2 \ge 0$$

$$\left(u^2-Yg\right)^2 - Y^2g^2 - g^2X^2 \ge 0$$

$$u^2 \ge Yg + g\sqrt{X^2+Y^2}$$

$$u^2 \le Yg - g\sqrt{X^2+Y^2}$$

← Non physical, since u is real and positive

$$\therefore u \ge \sqrt{g}\sqrt{Y+\sqrt{X^2+Y^2}}$$

The **minimum u parabola** is defined by the trajectory corresponding to the minimum velocity required to generate a projectile trajectory which intersects with (X,Y).

$$u^2 = g\left(Y+\sqrt{X^2+Y^2}\right)$$

$$y = x\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)x^2$$

Trajectory equation for minimum u parabola

$$a\tan^2\theta + b\tan\theta + c = 0$$

$$a = \frac{g}{2u^2}X^2,\quad b = -X,\quad c = Y + \frac{g}{2u^2}X^2$$

$$b^2 - 4ac = 0$$

$$\therefore \theta = \tan^{-1}\left(\frac{-b}{2a}\right)$$

$$\theta = \tan^{-1}\left(\frac{X}{\frac{g}{u^2}X^2}\right)$$

$$\theta = \tan^{-1}\left(\frac{u^2}{gX}\right)$$

$$\theta = \tan^{-1}\left(\frac{Y+\sqrt{X^2+Y^2}}{X}\right)$$

minimum u parabola elevation angle

$$\therefore \tan\theta = \frac{Y+\sqrt{X^2+Y^2}}{X}$$

$$y = x\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)x^2$$

$$y = x\left(\frac{Y+\sqrt{X^2+Y^2}}{X}\right) - \frac{g}{2g\left(Y+\sqrt{X^2+Y^2}\right)}\left(1+\frac{\left(Y+\sqrt{X^2+Y^2}\right)^2}{X^2}\right)x^2$$

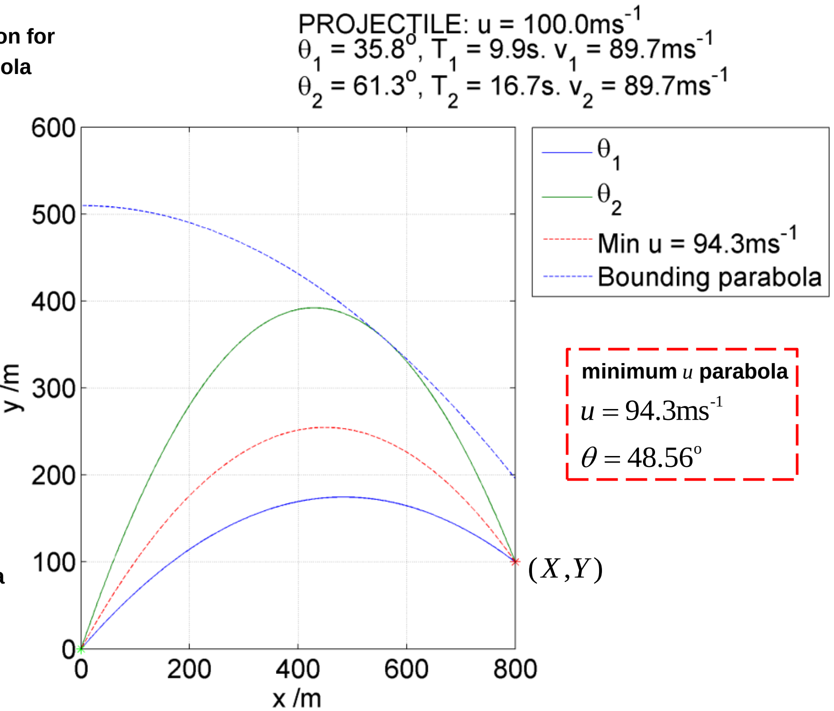
$$y = x\left(\frac{Y+\sqrt{X^2+Y^2}}{X}\right) - \frac{g}{2g\left(Y+\sqrt{X^2+Y^2}\right)}\left(\frac{X^2+Y^2+2Y\sqrt{X^2+Y^2}+X^2+Y^2}{X^2}\right)x^2$$

$$y = x\left(\frac{Y+\sqrt{X^2+Y^2}}{X}\right) - \frac{1}{Y+\sqrt{X^2+Y^2}}\left(\frac{X^2+Y^2+Y\sqrt{X^2+Y^2}}{X^2}\right)x^2$$

$$y = x\left(\frac{Y+\sqrt{X^2+Y^2}}{X}\right) - \frac{\sqrt{X^2+Y^2}}{Y+\sqrt{X^2+Y^2}}\left(\frac{\sqrt{X^2+Y^2}+Y}{X^2}\right)x^2$$

$$y = x\left(\frac{Y+\sqrt{X^2+Y^2}}{X}\right) - \frac{\sqrt{X^2+Y^2}}{X^2}x^2$$

minimum u parabola. Only *one* value of θ is possible, since the trajectory equation discriminant is zero.



The **bounding parabola** is slightly different – this bounds the possible set of trajectories *given* a value of u

$$y = x\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)x^2$$

$$2u^2y = 2u^2x\tan\theta - gx^2 - gx^2\tan^2\theta$$

$$gx^2\tan^2\theta - 2u^2x\tan\theta + 2u^2y + gx^2 = 0$$

For positive discriminant

$$4u^4x^2 - 4gx^2\left(2u^2y+gx^2\right)\ge 0$$

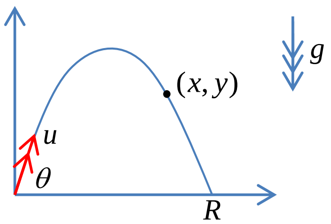
$$\frac{u^4}{g} \ge 2u^2y + gx^2$$

$$y \le \frac{u^2}{2g} - \frac{g}{2u^2}x^2$$

Bounding parabola

$$y = \frac{u^2}{2g} - \frac{g}{2u^2}x^2$$

The maximum range problem



Given a fixed projectile launch speed what angle maximises range?

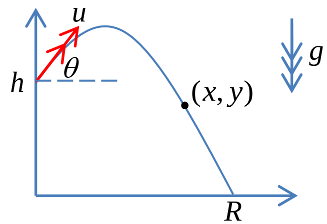
$$x = ut\cos\theta$$
$$y = ut\sin\theta - \tfrac{1}{2}gt^2$$

$$x = R, y = 0$$
$$\therefore 0 = t\left(u\sin\theta - \tfrac{1}{2}gt\right)$$
$$t > 0 \Rightarrow u\sin\theta - \tfrac{1}{2}gt = 0$$
$$\therefore t = \frac{2u\sin\theta}{g}$$
$$R = ut\cos\theta$$
$$\therefore R = \frac{2u^2}{g}\sin\theta\cos\theta$$
$$\boxed{R = \frac{u^2}{g}\sin 2\theta}$$

Hence maximum range is:

$$\boxed{R_{\max} = \frac{u^2}{g}, \quad \theta = 45^{\circ}}$$

Let us now extend the problem to a starting height which is *not* at ground level.



$$x = ut\cos\theta$$
$$y = ut\sin\theta - \tfrac{1}{2}gt^2 + h$$

$$x = R, y = 0$$
$$\therefore 0 = ut\sin\theta - \tfrac{1}{2}gt^2 + h$$
$$t^2 - \frac{2ut}{g}\sin\theta - \frac{2h}{g} = 0$$
$$\left(t - \frac{u\sin\theta}{g}\right)^2 - \frac{u^2\sin^2\theta}{g^2} - \frac{2gh}{g^2} = 0$$
$$t = \frac{u\sin\theta}{g} + \frac{u}{g}\sqrt{\sin^2\theta + \frac{2gh}{u^2}}$$

positive root since $t > 0$

$$R = ut\cos\theta$$
$$\boxed{R = \frac{u^2}{g}\left(\sin\theta\cos\theta + \cos\theta\sqrt{\sin^2\theta + \frac{2gh}{u^2}}\right)}$$

To maximize R we need to find θ such that

$$\frac{d}{d\theta}\left(\frac{Rg}{u^2}\right) = 0$$

For brevity define $\boxed{\alpha = \frac{2gh}{u^2}}$

Note: $\alpha = \frac{2gh}{u^2} = \frac{mgh}{\tfrac{1}{2}mu^2} = \frac{\text{GPE}}{\text{KE}}$

$$\frac{d}{d\theta}\left(\sin\theta\cos\theta + \cos\theta\sqrt{\sin^2\theta + \alpha}\right) = 0$$
$$\sin\theta(-\sin\theta) + \cos\theta(\cos\theta) + \frac{\tfrac{1}{2}\cos\theta}{\sqrt{\sin^2\theta + \alpha}}(2\sin\theta\cos\theta) - \sin\theta\sqrt{\sin^2\theta + \alpha} = 0$$
$$-\sin^2\theta + \cos^2\theta = \sin\theta\sqrt{\sin^2\theta + \alpha} - \frac{\sin\theta\cos^2\theta}{\sqrt{\sin^2\theta + \alpha}}$$

$\cos^2\theta = 1 - \sin^2\theta$

$$\sqrt{\sin^2\theta + \alpha}\left(1 - 2\sin^2\theta\right) = \sin\theta\left(\sin^2\theta + \alpha\right) - \sin\theta\left(1 - \sin^2\theta\right)$$
$$\sqrt{\sin^2\theta + \alpha} = \frac{2\sin^3\theta + (\alpha - 1)\sin\theta}{1 - 2\sin^2\theta}$$
$$\sin^2\theta + \alpha = \frac{4\sin^6\theta + 4(\alpha - 1)\sin^4\theta + (\alpha - 1)^2\sin^2\theta}{1 - 4\sin^2\theta + 4\sin^4\theta}$$
$$\left(1 - 4\sin^2\theta + 4\sin^4\theta\right)\left(\sin^2\theta + \alpha\right) = 4\sin^6\theta + 4(\alpha - 1)\sin^4\theta + (\alpha - 1)^2\sin^2\theta$$
$$\sin^2\theta + \alpha - 4\sin^4\theta - 4\alpha\sin^2\theta + 4\sin^6\theta + 4\alpha\sin^4\theta =$$
$$4\sin^6\theta + 4\alpha\sin^4\theta - 4\sin^4\theta + (\alpha - 1)^2\sin^2\theta$$

$$\alpha + (1 - 4\alpha)\sin^2\theta = (\alpha - 1)^2\sin^2\theta$$
$$\alpha = \left(\alpha^2 - 2\alpha + 1 - 1 + 4\alpha\right)\sin^2\theta$$
$$\alpha = \left(\alpha^2 + 2\alpha\right)\sin^2\theta$$
$$\frac{1}{\alpha + 2} = \sin^2\theta$$
$$\boxed{\sin\theta = \frac{1}{\sqrt{2 + \alpha}}}$$
$$\cos\theta = \sqrt{1 - \tfrac{1}{2+\alpha}}$$
$$\boxed{\cos\theta = \sqrt{\frac{1 + \alpha}{2 + \alpha}}}$$

The range maximizing angle is therefore:

$$\boxed{\theta = \sin^{-1}\left(\frac{1}{\sqrt{2 + \alpha}}\right)}$$

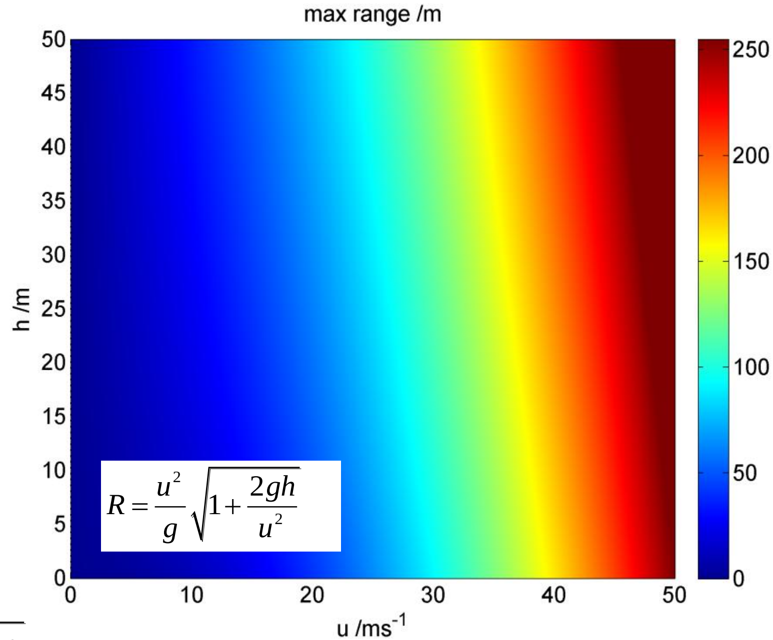
$$\frac{Rg}{u^2} = \sin\theta\cos\theta + \cos\theta\sqrt{\sin^2\theta + \alpha}$$
$$\frac{Rg}{u^2} = \frac{1}{\sqrt{2 + \alpha}}\sqrt{\frac{1 + \alpha}{2 + \alpha}} + \sqrt{\frac{1 + \alpha}{2 + \alpha}}\sqrt{\frac{1}{2 + \alpha} + \alpha}$$
$$\frac{Rg}{u^2} = \frac{\sqrt{1 + \alpha}}{2 + \alpha} + \sqrt{\frac{1 + \alpha}{2 + \alpha}}\sqrt{\frac{1 + 2\alpha + \alpha^2}{2 + \alpha}}$$
$$\frac{Rg}{u^2} = \frac{\sqrt{1 + \alpha}}{2 + \alpha} + \frac{\sqrt{1 + \alpha}}{2 + \alpha}\sqrt{(1 + \alpha)^2}$$
$$\frac{Rg}{u^2} = \frac{\sqrt{1 + \alpha}}{2 + \alpha}(1 + 1 + \alpha) = \frac{\sqrt{1 + \alpha}}{2 + \alpha}(2 + \alpha)$$
$$\frac{Rg}{u^2} = \sqrt{1 + \alpha}$$

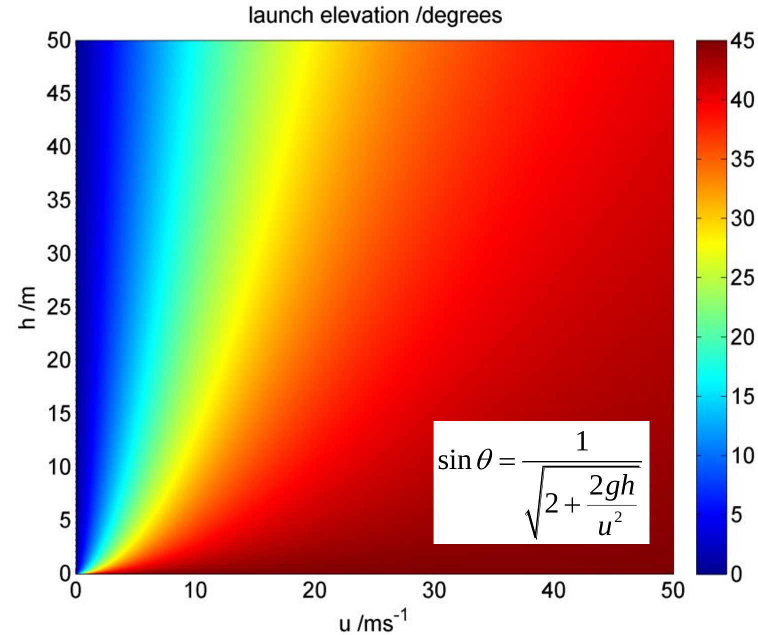
$$\boxed{R = \frac{u^2}{g}\sqrt{1 + \frac{2gh}{u^2}}, \quad \sin\theta = \frac{1}{\sqrt{2 + \frac{2gh}{u^2}}}}$$

Given the maximum range problem involves *two parameters*, to visualize possible solutions we need to plot a *surface* graph.

In the example plots, *colour* is used to indicate the height of the surface.

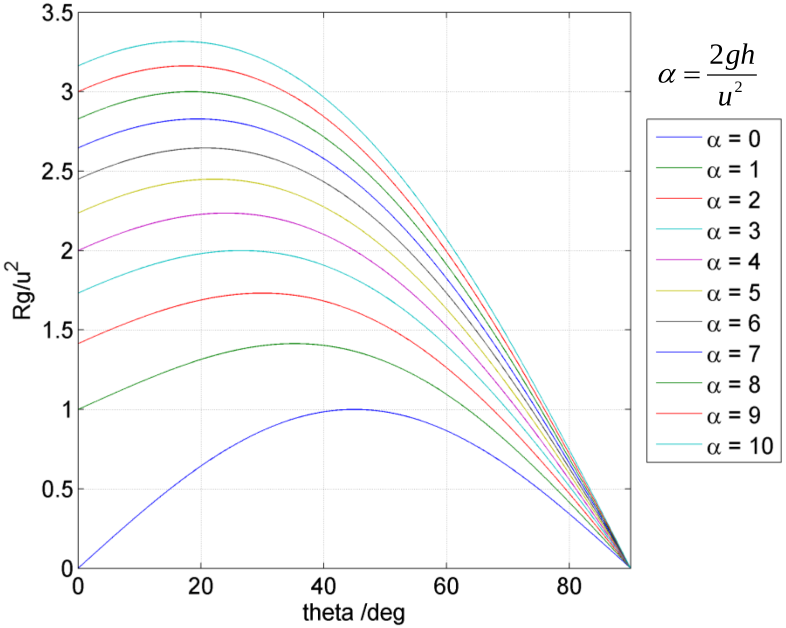
In all examples $g = 9.81\text{ms}^{-2}$





$$\frac{Rg}{u^2} = \sin\theta\cos\theta + \cos\theta\sqrt{\sin^2\theta + \alpha}$$

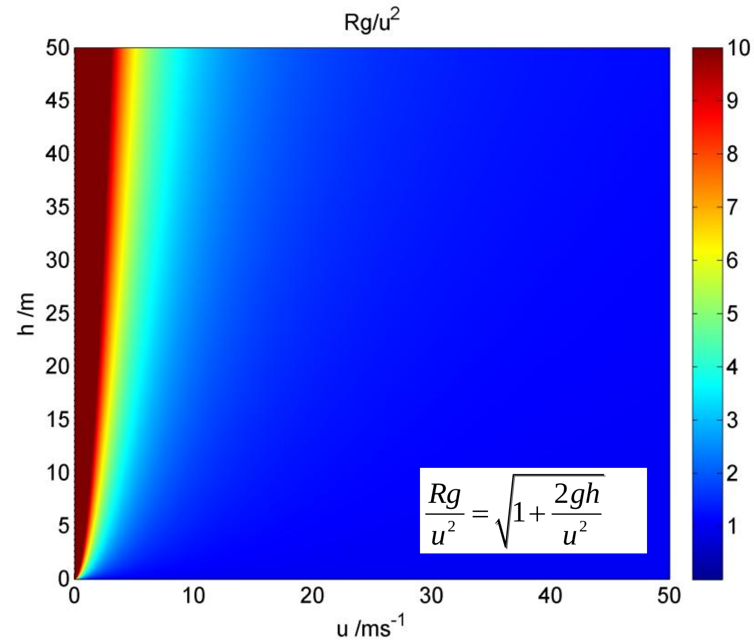
$$\alpha = \frac{2gh}{u^2}$$



This graph demonstrates that range has a maximum value as the launch elevation is varied.

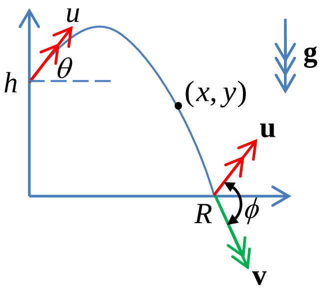
The angle which results in the maximum range is given by

$$\sin\theta = \frac{1}{\sqrt{2 + \alpha}}$$



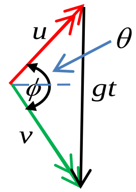
An elegant solution to the maximum range problem

There is an alternative, more *geometric*, method that arrives at the solution to the maximum range problem without so much trigonometric horror!



The velocity at maximum range R is given by the vector equation:

$$\boxed{\mathbf{v} = \mathbf{u} + \mathbf{g}t}$$



The area A of the vector triangle can be computed in *two* different ways:

$$A = \tfrac{1}{2}uv\sin\phi$$

$$A = \tfrac{1}{2}gt \times u\cos\theta$$

$$\therefore uv\sin\phi = gut\cos\theta$$

Since the projectile moves at constant speed horizontally: $\boxed{R = ut\cos\theta}$

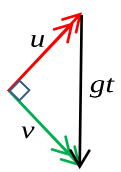
By conservation of energy: $mgh + \tfrac{1}{2}mu^2 = \tfrac{1}{2}mv^2 \quad \therefore \boxed{v = \sqrt{2gh + u^2}}$

Hence: $uv\sin\phi = gut\cos\theta \;\Rightarrow \dfrac{u}{g}\sin\phi\sqrt{2gh + u^2} = R$

$$\therefore R = \frac{u^2}{g}\sqrt{1 + \frac{2gh}{u^2}}\sin\phi$$

The largest R possible corresponds to $\sin\phi = 1 \Rightarrow \boxed{\phi = 90^\circ}$

$$\boxed{R = \frac{u^2}{g}\sqrt{1 + \frac{2gh}{u^2}}}$$



At maximum range the velocity triangle is *right angled*, so using Pythagoras' theorem we can calculate the time of flight corresponding to the maximum range

$$g^2t^2 = u^2 + v^2 \quad \therefore g^2t^2 = u^2 + 2gh + u^2$$

$$\therefore \boxed{t = \frac{u}{g}\sqrt{2 + \frac{2gh}{u^2}}}$$

We can use this result, combined with the expression for R, to find the required elevation angle to result in maximum range.

$$R = ut\cos\theta$$

$$\therefore \frac{u^2}{g}\sqrt{1 + \frac{2gh}{u^2}} = u\frac{u}{g}\sqrt{2 + \frac{2gh}{u^2}}\cos\theta$$

$$\therefore \cos\theta = \frac{\sqrt{1 + \frac{2gh}{u^2}}}{\sqrt{2 + \frac{2gh}{u^2}}}$$

$$\sin^2\theta = 1 - \cos^2\theta$$

$$\therefore \sin^2\theta = 1 - \frac{1 + \frac{2gh}{u^2}}{2 + \frac{2gh}{u^2}}$$

$$\therefore \sin^2\theta = \frac{2 + \frac{2gh}{u^2} - 1 - \frac{2gh}{u^2}}{2 + \frac{2gh}{u^2}}$$

$$\therefore \sin^2\theta = \frac{1}{2 + \frac{2gh}{u^2}}$$

$$\boxed{\therefore \theta = \sin^{-1}\left(\frac{1}{2 + 2gh/u^2}\right)}$$

Example 1: A projectile is fired from a canon.
The elevation is 45^o and the muzzle velocity is 100ms^{-1} .
Assuming the projectile is fired on level ground from (0,0), calculate
(i) the apogee coordinates; (ii) the ground range; (iii) time of flight.
Assume air resistance is ignorable and set $g = 9.81\text{ms}^{-2}$

The relevant equations are:

$$y_a = \frac{u^2 \sin^2\theta}{2g}$$

$$x_a = \frac{u^2 \sin\theta\cos\theta}{g}$$

i.e. $x = x_{\max},\ y = 0$

$$0 = x_{\max}\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)x_{\max}^2$$

$$\therefore x_{\max} = \frac{2u^2\tan\theta}{g(1+\tan^2\theta)} = \frac{2u^2\sin\theta\cos\theta}{g} = \boxed{\frac{u^2\sin 2\theta}{g}}$$

$$t = \frac{x_{\max}}{u\cos\theta}$$

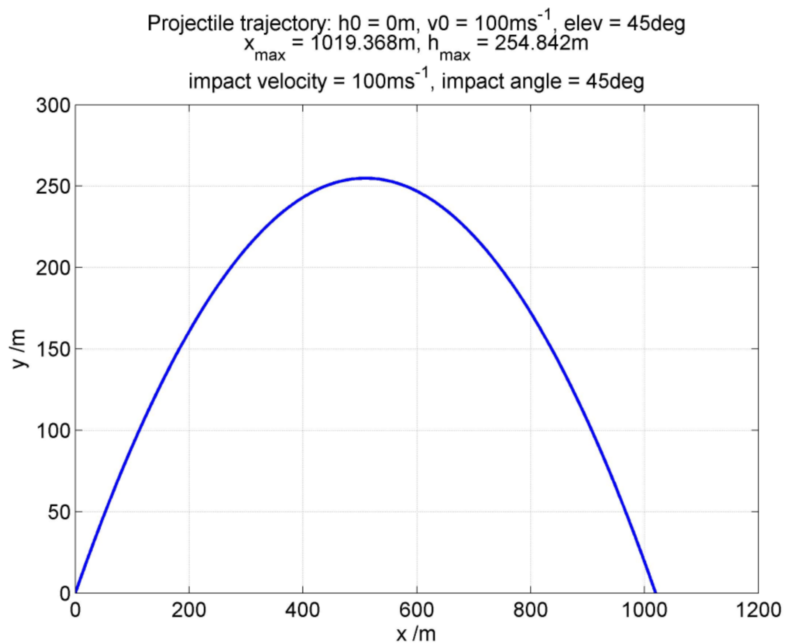
Hence:

$$y_a = \frac{100^2\times\frac{1}{2}}{2\times 9.81} = \boxed{254.8\text{m}}$$

$$x_a = \frac{100^2\times\frac{1}{\sqrt{2}}\times\frac{1}{\sqrt{2}}}{9.81} = \boxed{509.7\text{m}}$$

$$x_{\max} = \frac{100^2}{9.81} = \boxed{1019.4\text{m}}$$

$$t = \frac{x_{\max}}{100\times\frac{1}{\sqrt{2}}} = \boxed{14.4\text{s}}$$



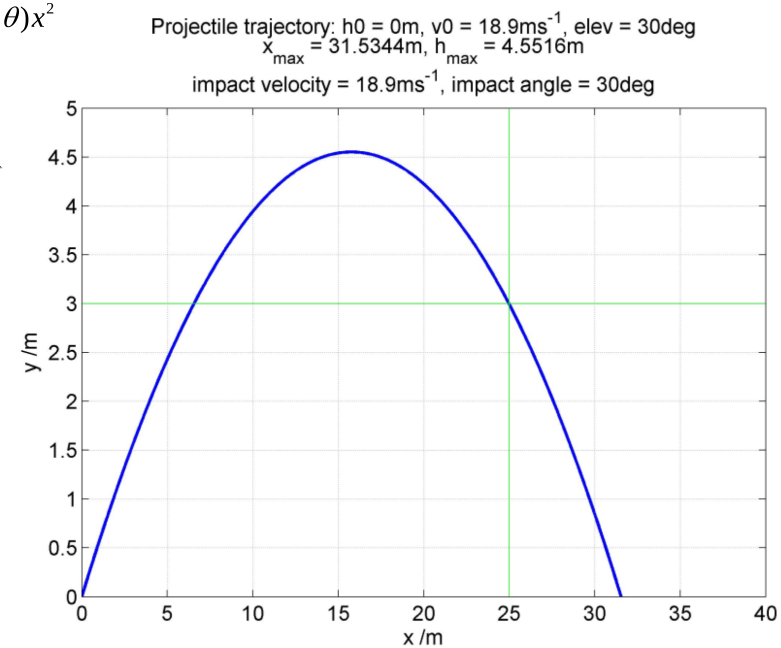
Example 2: In the 2003 Rugby Union World Cup final, Jonny Wilkinson scored the winning drop goal for England. If he struck the ball from ground level at an angle of thirty degrees, what velocity must he have kicked the ball, given he cleared the goals (height 3m) from a range of 25m? Ignore air resistance and set $g = 9.81\text{ms}^{-2}$

$$y = x\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)x^2$$

$$u = \sqrt{\frac{g(1+\tan^2\theta)x^2}{2\left(x\tan\theta - y\right)}}$$

$$\therefore u > \sqrt{\frac{9.81\times(1+\frac{1}{3})\times 25^2}{2\times\left(\frac{25}{\sqrt{3}}-3\right)}}$$

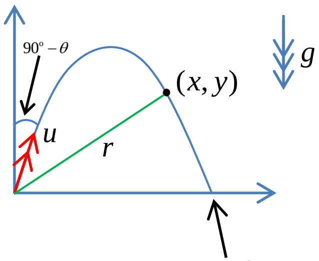
$$\therefore u > \boxed{18.9\text{ms}^{-1}}$$



Projectile range

The distance r of a particle undergoing projectile motion from (0,0) is given by:

$$r^2 = x^2 + y^2$$
$$y = ut\sin\theta - \tfrac{1}{2}gt^2$$
$$x = ut\cos\theta$$



$$R = \frac{u^2}{g}\sin 2\theta$$

Hence:

$$r^2 = u^2t^2\cos^2\theta + \left(ut\sin\theta - \tfrac{1}{2}gt^2\right)^2$$
$$r^2 = u^2t^2\cos^2\theta + u^2t^2\sin^2\theta - gt^2ut\sin\theta + \tfrac{1}{4}g^2t^4$$
$$r^2 = u^2t^2\left(\cos^2\theta + \sin^2\theta\right) - gt^3u\sin\theta + \tfrac{1}{4}g^2t^4$$
$$r^2 = u^2t^2 - gt^3u\sin\theta + \tfrac{1}{4}g^2t^4$$

$$\therefore r = \sqrt{u^2t^2 - gt^3u\sin\theta + \tfrac{1}{4}g^2t^4}$$

Is it possible to have a maximum or minimum in a graph of r vs t (and hence, since they are proportional) x? Ignore 'obvious' minimum when $t = 0$.

$$\frac{dr^2}{dt} = 2r\frac{dr}{dt} \quad \therefore \text{ if } r > 0 \text{ then } \frac{dr}{dt} = 0 \text{ if } \frac{dr^2}{dt} = 0$$
$$r^2 = u^2t^2 - gt^3u\sin\theta + \tfrac{1}{4}g^2t^4$$
$$\therefore \frac{dr^2}{dt} = 2u^2t - 3gt^2u\sin\theta + g^2t^3$$
$$\therefore \frac{dr^2}{dt} = 0 \Rightarrow 2u^2t - 3gt^2u\sin\theta + g^2t^3 = 0$$
$$\therefore t\left(2u^2 - 3gtu\sin\theta + g^2t^2\right) = 0$$

Since $t > 0$: $\quad 2u^2 - 3gtu\sin\theta + g^2t^2 = 0$

$$\therefore t^2 - \frac{3u}{g}\sin\theta t + \frac{2u^2}{g^2} = 0$$
$$\therefore \left(t - \frac{3u}{2g}\sin\theta\right)^2 - \frac{9u^2}{4g^2}\sin^2\theta + \frac{2u^2}{g^2} = 0$$
$$\therefore t_\pm = \frac{3u}{2g}\sin\theta \pm \sqrt{\frac{9u^2}{4g^2}\sin^2\theta - \frac{2u^2}{g^2}}$$

$$\therefore t_\pm = \frac{3u}{2g}\left(\sin\theta \pm \sqrt{\sin^2\theta - \tfrac{8}{9}}\right)$$

Real roots (i.e. there are times when the graph of r vs t is indeed at a maxima or minima) occur when:

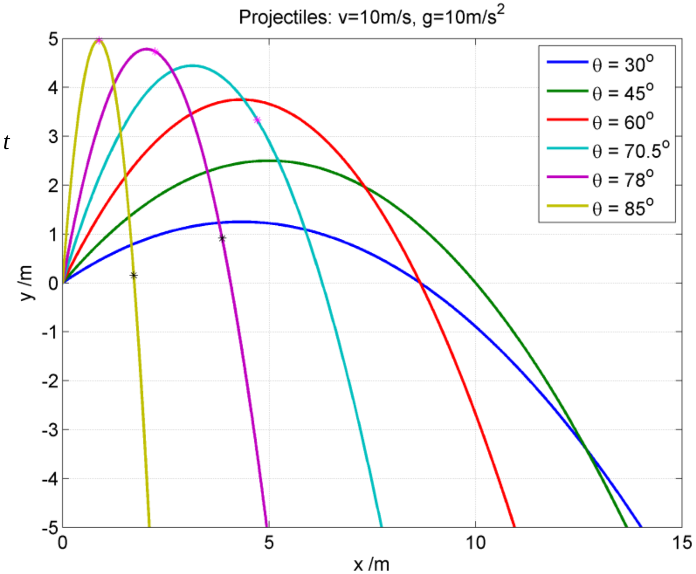
$$\sin^2\theta > \tfrac{8}{9} \quad \therefore \sin\theta > \tfrac{2\sqrt{2}}{3} \approx 70.5^\circ \quad \text{since } 0 \le \theta \le 90^\circ$$

The critical angle for stationary points of r vs t is when the above equality holds.

$$\sin\theta = \tfrac{2\sqrt{2}}{3} \Rightarrow \theta \approx 70.5^\circ$$
$$\therefore t_\pm = \frac{3u}{2g}\sin\theta = \frac{3u}{2g}\tfrac{2\sqrt{2}}{3}$$
$$\therefore t_\pm = \frac{u}{g}\sqrt{2}$$

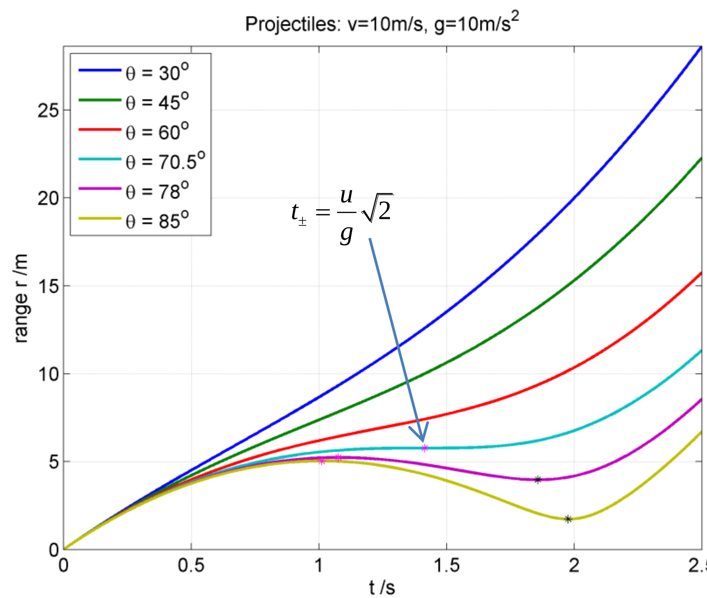
which is a nice result, since the maximum horizontal range when $\theta = 45^\circ$ is:

$$R_{\max} = \frac{u}{g}$$



* a maxima in r vs t * a minima in r vs t

$$t_\pm = \frac{3u}{2g}\left(\sin\theta \pm \sqrt{\sin^2\theta - \tfrac{8}{9}}\right)$$
$$\theta \ge \sin^{-1}\left(\tfrac{2\sqrt{2}}{3}\right)$$



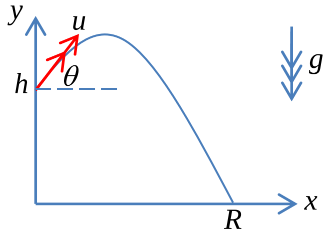
You can clearly see a maximum and minimum in a graph of r vs t for elevation angles greater than 70.5º.

Projectile distance travelled

The distance travelled by a particle undergoing projectile motion from $(0,h)$ is given by:

$$s=\int_0^X \sqrt{(dx)^2+(dy)^2}$$

$$s=\int_0^X \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$$



$$R=\frac{u^2}{g}\left(\sin\theta\cos\theta+\cos\theta\sqrt{\sin^2\theta+\frac{2gh}{u^2}}\right)$$

Now trajectory equation is:

$$y=h+x\tan\theta-\frac{g}{2u^2}(1+\tan^2\theta)x^2$$

$$\therefore \frac{dy}{dx}=\tan\theta-\frac{gx}{u^2}(1+\tan^2\theta)$$

$$\therefore s=\int_0^X \sqrt{1+\left(\tan\theta-\frac{gx}{u^2}(1+\tan^2\theta)\right)^2}\,dx$$

Consider a substitution:

$$z=\tan\theta-\frac{gx}{u^2}(1+\tan^2\theta) \quad \therefore dz=-\frac{g}{u^2}(1+\tan^2\theta)dx$$

$$\therefore s=-\frac{u^2}{g(1+\tan^2\theta)}\int_{\tan\theta}^{\tan\theta-\frac{gX}{u^2}(1+\tan^2\theta)}\sqrt{1+z^2}\,dz$$

Note standard integral:

$$\int\sqrt{1+z^2}\,dz=\tfrac{1}{2}\ln\left|\sqrt{1+z^2}+x\right|+\tfrac{1}{2}z\sqrt{1+z^2}+c$$

$$\therefore s=\frac{u^2}{g(1+\tan^2\theta)}\left[\tfrac{1}{2}\ln\left|\sqrt{1+z^2}+z\right|+\tfrac{1}{2}z\sqrt{1+z^2}\right]_{\tan\theta-\frac{gX}{u^2}(1+\tan^2\theta)}^{\tan\theta}$$

Which can be calculated easily using MATLAB/Python/Excel etc, and checked with a numeric approximate calculation using a small discrete value of Δx.

Consider a special case when projectile is launched from the origin (i.e. $h=0$), and $X=R=\frac{2u^2}{g}\sin\theta\cos\theta$ i.e. when the inverted parabolic trajectory crosses the horizontal axis after launch.

$$\therefore \tan\theta-\frac{gX}{u^2}(1+\tan^2\theta)=\tan\theta-2\sin\theta\cos\theta(1+\tan^2\theta)$$

$$=\tan\theta-\frac{2\sin\theta\cos\theta}{\cos^2\theta}=-\tan\theta$$

$$\therefore s=\frac{u^2}{g(1+\tan^2\theta)}\left[\tfrac{1}{2}\ln\left|\sqrt{1+z^2}+z\right|+\tfrac{1}{2}z\sqrt{1+z^2}\right]_{-\tan\theta}^{\tan\theta}$$

$$=\tfrac{1}{2}\frac{u^2}{g(1+\tan^2\theta)}\left(\ln\left|\sqrt{1+\tan^2\theta}+\tan\theta\right|+\tan\theta\sqrt{1+\tan^2\theta}-\ln\left|\sqrt{1+\tan^2\theta}-\tan\theta\right|+\tan\theta\sqrt{1+\tan^2\theta}\right)$$

$$=\frac{u^2}{g(1+\tan^2\theta)}\left(\tfrac{1}{2}\ln\left|\frac{\sqrt{1+\tan^2\theta}+\tan\theta}{\sqrt{1+\tan^2\theta}-\tan\theta}\right|+\tan\theta\sqrt{1+\tan^2\theta}\right)$$

$$=\frac{u^2\cos^2\theta}{g}\left(\tfrac{1}{2}\ln\left|\frac{\frac{1}{\cos\theta}+\frac{\sin\theta}{\cos\theta}}{\frac{1}{\cos\theta}-\frac{\sin\theta}{\cos\theta}}\right|+\frac{\sin\theta}{\cos\theta}\frac{1}{\cos\theta}\right)=\frac{u^2\cos^2\theta}{g}\left(\tfrac{1}{2}\ln\left|\frac{1+\sin\theta}{1-\sin\theta}\right|+\frac{\sin\theta}{\cos^2\theta}\right)$$

$$\frac{1+\sin\theta}{1-\sin\theta}=\frac{(1+\sin\theta)^2}{1-\sin^2\theta}=\frac{(1+\sin\theta)^2}{\cos^2\theta}$$

$$\therefore s=\frac{u^2}{g}\left(\ln\left(\frac{1+\sin\theta}{\cos\theta}\right)\cos^2\theta+\sin\theta\right)$$

When R is maximized: $\theta=\frac{\pi}{4},\ \sin\theta=\cos\theta=\frac{1}{\sqrt{2}}=\frac{\sqrt{2}}{2},\ R=\frac{2u^2}{g},$

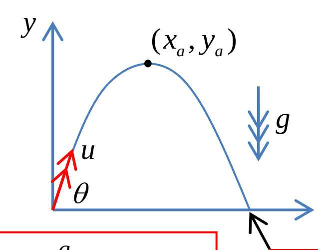
$$\therefore s=\frac{u^2}{g}\left(\ln\left|\frac{1+\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}}\right|\times\tfrac{1}{2}+\tfrac{\sqrt{2}}{2}\right)$$

$$\therefore s=\tfrac{1}{2}\frac{u^2}{g}\left(\ln\left(1+\sqrt{2}\right)+\sqrt{2}\right)$$

$$\therefore s\approx 1.15\frac{u^2}{g}$$

$$\ln\left(1+\sqrt{2}\right)+\sqrt{2}\approx 2.296$$

Universal parabola constant



$$x_a=\tfrac{1}{2}R,\quad y_a=\frac{u^2}{2g}\sin^2\theta$$

$$s=\frac{u^2}{g}\left(\ln\left(\frac{1+\sin\theta}{\cos\theta}\right)\cos^2\theta+\sin\theta\right)$$

$$t=\frac{x}{u\cos\theta},\quad T=\frac{R}{u\cos\theta}$$

$$y=x\tan\theta-\frac{g}{2u^2}(1+\tan^2\theta)x^2$$

$$R=\frac{u^2}{g}\sin 2\theta$$