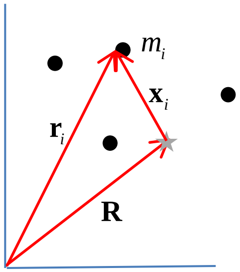
Consider a system of masses with position vectors $\mathbf{r}_i$ and velocities $\dot{\mathbf{r}}_i$



$$M = \sum_i m_i$$

Now write the position vectors and velocities in terms of a vector $\mathbf{R}$ and relative position vector $\mathbf{x}_i$

$$\mathbf{r}_i = \mathbf{R} + \mathbf{x}_i$$

$$\dot{\mathbf{r}}_i = \dot{\mathbf{R}} + \dot{\mathbf{x}}_i$$

The total **angular momentum** in the system is

$$\mathbf{L} = \sum_i m_i \mathbf{r}_i \times \dot{\mathbf{r}}_i$$

$$\mathbf{L} = \sum_i m_i \left(\mathbf{R} + \mathbf{x}_i\right) \times \left(\dot{\mathbf{R}} + \dot{\mathbf{x}}_i\right)$$

$$\mathbf{L} = \sum_i m_i \left(\mathbf{R} \times \dot{\mathbf{R}} + \mathbf{x}_i \times \dot{\mathbf{R}} + \mathbf{R} \times \dot{\mathbf{x}}_i + \mathbf{x}_i \times \dot{\mathbf{x}}_i\right)$$

$$\mathbf{L} = M\mathbf{R} \times \dot{\mathbf{R}} + \sum_i m_i \mathbf{x}_i \times \dot{\mathbf{x}}_i + \sum_i m_i \left(\mathbf{x}_i \times \dot{\mathbf{R}}\right) + \sum_i m_i \mathbf{R} \times \dot{\mathbf{x}}_i$$

Now let us suggest a useful form for $\mathbf{R}$, that is the **mass-weighted average position vector**

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{M} \qquad \therefore \mathbf{R} = \frac{\sum_i m_i \left(\mathbf{R} + \mathbf{x}_i\right)}{M} \Rightarrow \sum_i m_i \mathbf{x}_i = \mathbf{0}$$

For a continuous distribution of mass of density $\rho(\mathbf{r})$

$$\mathbf{R} = \tfrac{1}{M}\int \mathbf{r}\rho(\mathbf{r})dV$$

volume element

Therefore $\sum_i m_i \left(\mathbf{x}_i \times \dot{\mathbf{R}}\right) + \sum_i m_i \mathbf{R} \times \dot{\mathbf{x}}_i$

$$= \left(\sum_i m_i \mathbf{x}_i\right) \times \dot{\mathbf{R}} + \mathbf{R} \times \frac{d}{dt}\left(\sum_i m_i \mathbf{x}_i\right)$$

since it is implied here that the masses *do not* vary with time

$$= \mathbf{0}$$

Therefore the **total angular momentum** is:

$$\mathbf{L} = M\mathbf{R} \times \dot{\mathbf{R}} + \sum_i m_i \mathbf{x}_i \times \dot{\mathbf{x}}_i$$

Sum of angular momenta of masses about $\mathbf{R}$

Angular momentum of a single mass M at position $\mathbf{R}$

Hence:

$$\dot{\mathbf{L}} = M\frac{d}{dt}\left(\mathbf{R} \times \dot{\mathbf{R}}\right) + \sum_i m_i \frac{d}{dt}\left(\mathbf{x}_i \times \dot{\mathbf{x}}_i\right)$$

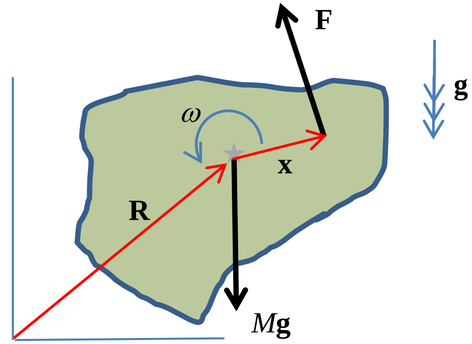
$$\dot{\mathbf{L}} = M\left(\dot{\mathbf{R}} \times \dot{\mathbf{R}} + \mathbf{R} \times \ddot{\mathbf{R}}\right) + \sum_i m_i \left(\dot{\mathbf{x}}_i \times \dot{\mathbf{x}}_i + \mathbf{x}_i \times \ddot{\mathbf{x}}_i\right)$$

$$\dot{\mathbf{L}} = \mathbf{R} \times M\ddot{\mathbf{R}} + \sum_i \mathbf{x}_i \times m_i \ddot{\mathbf{x}}_i$$

i.e. **rate of change of angular momentum equals net *torque*,** since Newton's Second Law equates mass x acceleration with net force.

We can therefore decompose a system of masses **into bulk motion of the centre of mass** plus **internal motion (e.g. rotation etc) about the centre of mass**

torque i.e. the 'moment' of a force $\boldsymbol{\tau} = \mathbf{r} \times \mathbf{f}$



We can analyse the dynamics of a irregular, *but rigid, mass* in terms of:

(i) **Acceleration** of the centre of mass

$$M\ddot{\mathbf{R}} = M\mathbf{g} + \mathbf{F}$$

(ii) **Rotation** about the centre of mass

Note forces passing through the centre of mass *only* effect the motion of the mass centre. They *cannot* cause any rotation of the rigid body.

$$\mathbf{x} \times \mathbf{F} = \frac{d\mathbf{L}}{dt} = \mathbf{I}\dot{\boldsymbol{\omega}} = \mathbf{I}\frac{d\boldsymbol{\omega}}{dt}$$

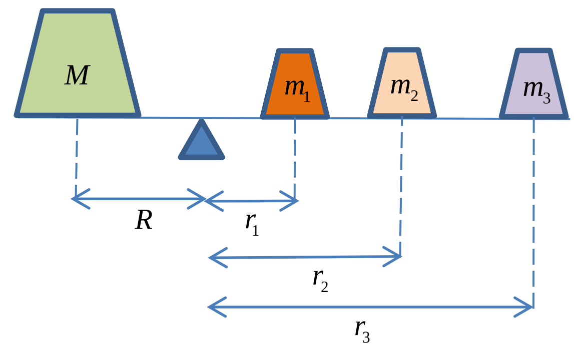
Torque about centre of mass

Inertia tensor x rate of change of angular velocity

In this case, the angular momentum $\mathbf{L}$ is relative to the centre of mass being stationary.

$$\mathbf{I}\frac{d\boldsymbol{\omega}}{dt} = \sum_i \mathbf{x}_i \times m_i \ddot{\mathbf{x}}_i$$

A simple 1D example of centre of mass



In order for the above system to be in equilibrium, the total moments about the pivot must be zero.

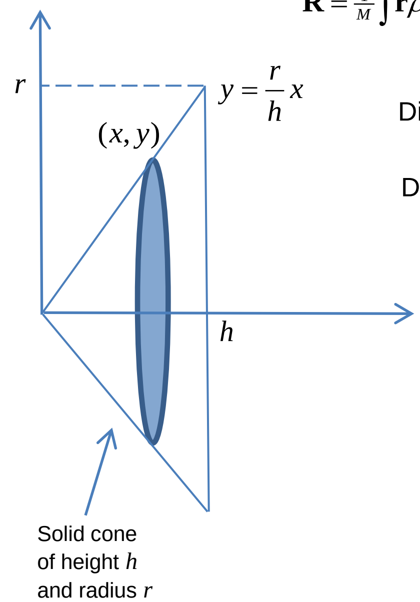
$$MR = m_1 r_1 + m_2 r_2 + m_3 r_3$$

$$\therefore R = \frac{\sum_i m_i r_i}{M}$$

If M is the sum of all the masses, then we can think of the three-mass system *being equivalent* to a single mass M a distance R from the fulcrum.

Solid	Centre of mass	From
Solid hemisphere, radius *r*	$d = \frac{3r}{8}$	sphere centre
Hemispherical shell, radius *r*	$d = \frac{r}{2}$	sphere centre
Sector of disk, radius *r*, angle 2θ	$d = \frac{2}{3} r \frac{\sin\theta}{\theta}$	disk centre
Arc of circle, radius *r*, angle 2θ	$d = r\frac{\sin\theta}{\theta}$	circle centre
Arbitrary triangular lamina, height *h* (perpendicular distance between base and apex)	$d = \frac{h}{3}$	perpendicular from base
Solid cone or pyramid, height *h*	$d = \frac{h}{4}$	perpendicular from base
Solid spherical cap, height *h*, sphere radius *r*	$d = \frac{3}{4}\frac{(2r-h)^2}{3r-h}$	sphere centre
Spherical cap shell, height *h*, sphere radius *r*	$d = r - \frac{h}{2}$	sphere centre
Semi-elliptical lamina, height *h*	$d = \frac{4h}{3\pi}$	from base

Example: Finding the centre of mass of a solid cone



$$\mathbf{R} = \tfrac{1}{M}\int \mathbf{r}\rho(\mathbf{r})dV$$

By symmetry, centre of mass must be along the *x* axis

Disc shape volume element $dV = \pi y^2 dx$

Density $\rho = \frac{M}{\frac{1}{3}\pi r^2 h}$

Another common notation for centre of mass i.e. an 'average' *x* position

$$\bar{x} = \tfrac{1}{M}\int_{x=0}^{r} x\rho dV$$

$$\bar{x} = \tfrac{1}{M}\frac{M}{\frac{1}{3}\pi r^2 h}\int_0^h x\pi\left(\frac{r}{h}x\right)^2 dx$$

$$\bar{x} = \frac{3}{h^3}\int_0^h x^3 dx$$

$$\bar{x} = \frac{3}{h^3}\left[\tfrac{1}{4}x^4\right]_0^h$$

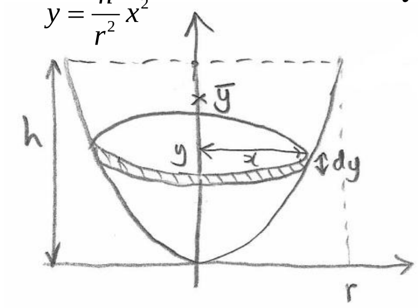
$$\bar{x} = \tfrac{3}{4}h$$

$$\therefore d = h - \bar{x} = \boxed{\tfrac{1}{4}h}$$

Require centre of mass distance from base, rather than apex of cone

Example: Finding the centre of mass of a solid paraboloid

$$y = \frac{h}{r^2}x^2$$



$$\mathbf{R} = \tfrac{1}{M}\int \mathbf{r}\rho(\mathbf{r})dV$$

By symmetry, centre of mass must be along the *x* axis

Disc shape volume element $dV = \pi x^2 dy$

Volume of paraboloid is

$$V = \pi\int_{y=0}^{h} \frac{r^2 y}{h} dy$$

$$V = \frac{\pi r^2}{h}\left[\tfrac{1}{2}y^2\right]_0^h$$

$$\boxed{V = \tfrac{1}{2}\pi r^2 h}$$

Density: $\boxed{\rho = \frac{M}{V} = \frac{2M}{\pi r^2 h}}$

Therefore centre of mass (from apex, or 'nose' of paraboloid) is:

$$d = \tfrac{1}{M}\int_{y=0}^{h} y\rho dV$$

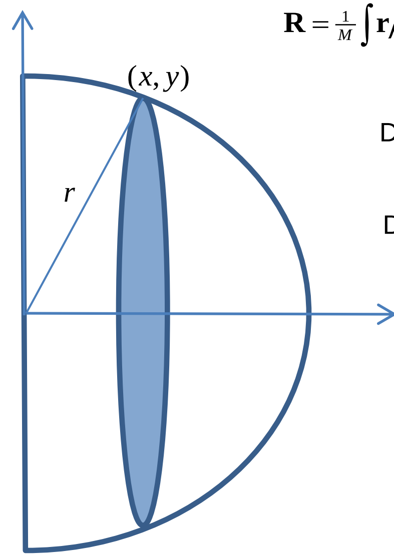
$$d = \tfrac{1}{M}\frac{2M}{\pi r^2 h}\pi\int_0^h y\frac{r^2 y}{h}dy$$

$$d = \frac{2}{h^2}\int_0^h y^2 dy$$

$$d = \frac{2}{h^2}\left[\tfrac{1}{3}y^3\right]_0^h$$

$$\boxed{d = \tfrac{2}{3}h}$$

Example: Finding the centre of mass of a solid hemisphere



$$\mathbf{R} = \tfrac{1}{M}\int \mathbf{r}\rho(\mathbf{r})dV$$

By symmetry, centre of mass must be along the *x* axis

Disc shape volume element $dV = \pi y^2 dx$

Density: $\rho = \frac{M}{\tfrac{2}{3}\pi r^3}$

$$d = \tfrac{1}{M}\int_{x=0}^{r} x\rho dV$$

$$d = \tfrac{1}{M}\frac{M\pi}{\tfrac{2}{3}\pi r^3}\int_0^r xy^2 dx$$

$$d = \frac{1}{\tfrac{2}{3}r^3}\int_0^r x\left(r^2 - x^2\right)dx$$

$$d = \frac{3}{2r^3}\left[\tfrac{1}{2}x^2r^2 - \tfrac{1}{4}x^4\right]_0^r$$

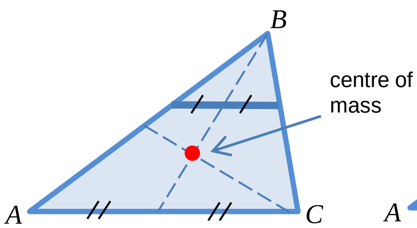
$$d = \frac{3r^4}{2r^3}\left(\tfrac{1}{2} - \tfrac{1}{4}\right)$$

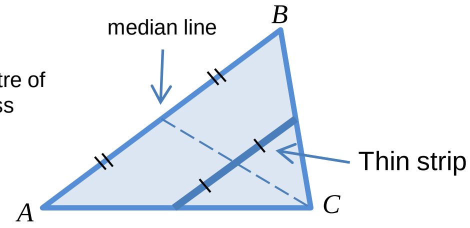
$$\boxed{d = \frac{3r}{8}}$$

Centre of mass of a triangular lamina

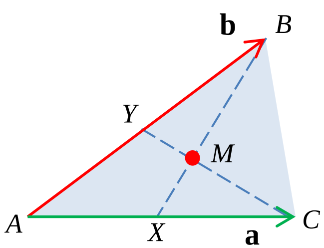
$\bullet\ \overrightarrow{AM} = \tfrac{1}{3}(\mathbf{a}+\mathbf{b})$

The centre of mass of a uniform triangular lamina must be on the intersection of *median lines.* (i.e. *angle bisectors* of a given vertex).





This is because we could construct the triangle from thin strips, parallel to one of the sides. The centre of mass of each strip must line on the median line, so therefore the overall centre of mass must also be on this line



$$\mathbf{a} = \overrightarrow{AC} \qquad \mathbf{b} = \overrightarrow{AB}$$

$$\tfrac{1}{2}\mathbf{a} = \overrightarrow{AX} \qquad \tfrac{1}{2}\mathbf{b} = \overrightarrow{AY}$$

$$\bullet\ \overrightarrow{AM} = \mathbf{b} + \lambda\overrightarrow{BX}$$

$$\overrightarrow{BX} = \tfrac{1}{2}\mathbf{a} - \mathbf{b}$$

$$\therefore \overrightarrow{AM} = \mathbf{b} + \lambda\left(\tfrac{1}{2}\mathbf{a} - \mathbf{b}\right)$$

$$\bullet\ \overrightarrow{AM} = \mathbf{a} + \mu\overrightarrow{CY}$$

$$\overrightarrow{CY} = -\mathbf{a} + \tfrac{1}{2}\mathbf{b}$$

$$\therefore \overrightarrow{AM} = \mathbf{a} + \mu\left(-\mathbf{a} + \tfrac{1}{2}\mathbf{b}\right)$$

Hence:

$$\mathbf{a} + \mu\left(-\mathbf{a} + \tfrac{1}{2}\mathbf{b}\right) = \mathbf{b} + \lambda\left(\tfrac{1}{2}\mathbf{a} - \mathbf{b}\right)$$

$$\mathbf{a}\left(1 - \mu - \tfrac{1}{2}\lambda\right) + \mathbf{b}\left(\tfrac{1}{2}\mu - 1 + \lambda\right) = \mathbf{0}$$

This must be true for *all* possible **a, b**. Therefore:

$$1 - \mu - \tfrac{1}{2}\lambda = 0 \quad \therefore \mu = 1 - \tfrac{1}{2}\lambda$$

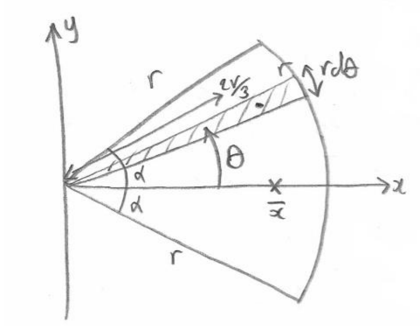
$$\tfrac{1}{2}\mu - 1 + \lambda = 0 \quad \therefore \mu = 2 - 2\lambda$$

$$\therefore 1 - \tfrac{1}{2}\lambda = 2 - 2\lambda \quad \therefore \tfrac{3}{2}\lambda = 1$$

$$\therefore \lambda = \tfrac{2}{3} \quad \therefore \overrightarrow{AM} = \mathbf{b} + \tfrac{2}{3}\left(\tfrac{1}{2}\mathbf{a} - \mathbf{b}\right) = \boxed{\tfrac{1}{3}(\mathbf{a}+\mathbf{b})}$$

$$\boxed{\bullet\ \overrightarrow{AM} = \tfrac{1}{3}(\mathbf{a}+\mathbf{b})}$$

Centre of mass of a sector of a circular lamina



By symmetry, the centre of mass must lie on the x axis

If we treat the 'sectorettes' as triangles, the x coordinate of their centre of mass will be at

$$x = \tfrac{2}{3} r\cos\theta$$

Note both sides of x axis

Mass of 'sectorette'

$$\bar{x} = \frac{2\int_0^\alpha \tfrac{2}{3} r\cos\theta \times \rho\tfrac{1}{2}r^2 d\theta}{\tfrac{1}{2}r^2(2\alpha)\rho}$$

Total mass of sector (uniform density ρ)

Angles in **radians**

$$\bar{x} = \frac{2r}{3\alpha}\int_0^\alpha \cos\theta d\theta$$

$$\bar{x} = \frac{2r}{3\alpha}\left[\sin\theta\right]_0^\alpha$$

$$\boxed{\bar{x} = \frac{2r\sin\alpha}{3\alpha}}$$

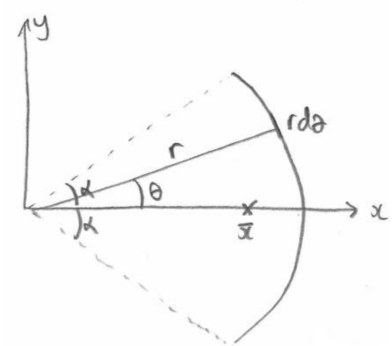
The same idea can be used to find the centre of mass of a **wire arc** of mass per unit length ρ

$$\bar{x} = \frac{2\int_0^\alpha r\cos\theta \times \rho r d\theta}{2r\alpha\rho}$$

$$\bar{x} = \frac{r}{\alpha}\int_0^\alpha \cos\theta d\theta$$

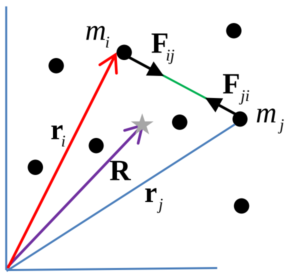
$$\bar{x} = \frac{r}{\alpha}\left[\sin\theta\right]_0^\alpha$$

$$\boxed{\bar{x} = \frac{r\sin\alpha}{\alpha}}$$



General properties of many-body systems with 'reciprocal' internal forces.

Consider a system of masses with position vectors $\mathbf{r}_i$ relative to an arbitrary fixed coordinate system origin.
Assume they interact with a force which acts along the radial separation between the masses. e.g. like gravitational and electric forces (but *not* magnetism).
Assume there are no external forces i.e. the forces acting on one mass are sourced from the other masses.



Force on mass i due to mass j is therefore:

$$\mathbf{F}_{ij} = \begin{cases} F_{ij}\dfrac{\mathbf{r}_j - \mathbf{r}_i}{|\mathbf{r}_j - \mathbf{r}_i|} & i \neq j \\ \mathbf{0} & i = j \end{cases}$$

Assume the forces are reciprocal or "equal and opposite", as for contact forces described by *Newton's Third Law*

$$\mathbf{F}_{ij} = -\mathbf{F}_{ji}$$

The total force on mass i is: $\mathbf{F}_i = \sum_{j \neq i} \mathbf{F}_{ij}$ and therefore the total force on the system is: $\mathbf{F}_{tot} = \sum_{i \neq j} \mathbf{F}_i = \sum_{i \neq j}\sum_{j \neq i} \mathbf{F}_{ij}$

Now i and j are simply labels so we can also write: $\mathbf{F}_{tot} = \sum_{j \neq i}\sum_{i \neq j} \mathbf{F}_{ji}$

$$M = \sum_i m_i$$

$$\mathbf{R} = \frac{\sum_i m_i \mathbf{r}_i}{M}$$

Centre of mass

$$\therefore 2\mathbf{F}_{tot} = \sum_{i \neq j}\sum_{j \neq i} \mathbf{F}_{ij} + \sum_{j \neq i}\sum_{i \neq j} \mathbf{F}_{ji}$$

$$\therefore 2\mathbf{F}_{tot} = \sum_{i \neq j}\sum_{j \neq i} \mathbf{F}_{ij} - \sum_{j \neq i}\sum_{i \neq j} \mathbf{F}_{ij} \quad \leftarrow \quad \mathbf{F}_{ij} = -\mathbf{F}_{ji}$$

$$\therefore 2\mathbf{F}_{tot} = \mathbf{0}$$

So there is **no net force** on a system with reciprocal internal forces like this.

Let us apply Newton's second law to this system:

$$m_i \ddot{\mathbf{r}}_i = \sum_{j \neq i} \mathbf{F}_{ij}$$

$$\therefore \sum_i m_i \ddot{\mathbf{r}}_i = \mathbf{F}_{tot} = \mathbf{0}$$

$$M\ddot{\mathbf{R}} = \sum_i m_i \ddot{\mathbf{r}}_i$$

$$\therefore \ddot{\mathbf{R}} = \mathbf{0}$$

$$\therefore \dot{\mathbf{R}} = \text{constant}$$

So the velocity of the centre of mass of the system **must move at constant velocity** i.e. *zero acceleration*.

If an external force acts on the system we can model the system as a *particle* of mass M and apply Newton II to the centre of mass to determine overall *translational* motion.

See page 1 →

What about rotation and angular momentum?

If all internal forces are of the form $\mathbf{F}_{ij} = \begin{cases} F_{ij}\dfrac{\mathbf{r}_j - \mathbf{r}_i}{|\mathbf{r}_j - \mathbf{r}_i|} & i \neq j \\ \mathbf{0} & i = j \end{cases}$ then there is *no net torque* $\boldsymbol{\tau}$.

Torque is the rate of change of **angular momentum**

$$\boldsymbol{\tau}_i = \frac{d\mathbf{L}_i}{dt} = \frac{d}{dt}\left(\mathbf{r}_i \times m_i \dot{\mathbf{r}}_i\right)$$

Proof is obvious really, since 'radial' forces have no turning moment. Mathematically, this fact is revealed via the cross product of a vector with itself being zero.

$$\boldsymbol{\tau}_i = \sum_{j \neq i}\left(\mathbf{r}_j - \mathbf{r}_i\right) \times \mathbf{F}_{ij} = \sum_{j \neq i}\left(\mathbf{r}_j - \mathbf{r}_i\right) \times F_{ij}\frac{\mathbf{r}_j - \mathbf{r}_i}{|\mathbf{r}_j - \mathbf{r}_i|}$$

$$\boldsymbol{\tau}_i = \sum_{j \neq i}\left(\mathbf{r}_j - \mathbf{r}_i\right) \times \left(\mathbf{r}_j - \mathbf{r}_i\right)\frac{F_{ij}}{|\mathbf{r}_j - \mathbf{r}_i|} = \sum_{j \neq i} \mathbf{0} \times \frac{F_{ij}}{|\mathbf{r}_j - \mathbf{r}_i|}$$

$$\boldsymbol{\tau}_i = \mathbf{0}$$

Hence the **total angular momentum (and indeed the angular momentum of each mass) must be a constant.**

To change the angular momentum of this system there must be an *external torque* applied to the system, or a *non radial form force law.*

This explains why orbiting planets interacting gravitationally (i.e. a 'Kepler problem') have angular momentum as a *constant* of the motion.

Note $\mathbf{L}$ and $\mathbf{J}$ are both typical symbols for angular momentum. $\mathbf{J}$ is usually the total angular momentum.