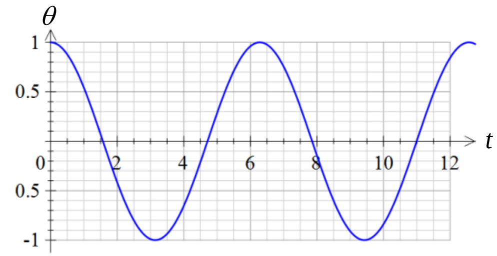
Simple Harmonic Motion (SHM)

The swing of a pendulum, the compression of a spring, the oscillatory movement of charge in a circuit containing a capacitor, inductor and resistor circuit, in fact *all* vibrational phenomena can be described by sinusoidal variation*

$$\theta(t) = \theta_0 \cos\left(\frac{2\pi}{T}t + \phi\right)$$

Maximum amplitude (θ_0), period (T), time (t), Initial phase (ϕ)

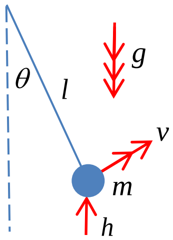


$$\dot{\theta} = \frac{d\theta}{dt} = -\frac{2\pi}{T}\theta_0 \sin\left(\frac{2\pi}{T}t + \phi\right)$$

$$\ddot{\theta} = \frac{d^2\theta}{dt^2} = -\left(\frac{2\pi}{T}\right)^2 \theta_0 \cos\left(\frac{2\pi}{T}t + \phi\right)$$

$$\boxed{\ddot{\theta} = -\left(\frac{2\pi}{T}\right)^2 \theta}$$

If the 'displacement' of a system varies according to an equation $\ddot{\theta} = -\left(\frac{2\pi}{T}\right)^2 \theta$ then we can use this to determine the *period* of oscillations.



Rather than apply variants of *Newton's Second Law* to determine the equation of motion of a system, it is often easier to start with an expression for the **total energy**. If the system does not lose energy to friction, air resistance etc, then the **time derivative of the total energy will be zero**, since the **total energy will be constant**. For systems involving a single type of 'displacement', this mathematical approach will yield the **equation of motion**. For more general systems (e.g. the motion of a *coupled* pendulum), the *Calculus of Variations* and *Lagrangian* (or alternatively *Hamiltonian*) dynamics can be used. The benefit of all of these techniques is they generate the equation of motion of a system from a *scalar*, rather than a *vector* expression. A scalar expression for the total energy of a system is often easier to determine than an expression for the vector sum of forces, torques etc.

Example1: A simple pendulum

$$E = \tfrac{1}{2}mv^2 + mgh$$

$$v = l\dot{\theta}$$

$$h = l - l\cos\theta$$

$$\therefore \boxed{E = \tfrac{1}{2}ml^2\dot{\theta}^2 + mgl(1-\cos\theta)}$$

$$\dot{E} = \tfrac{1}{2}ml^2\left(2\dot{\theta}\ddot{\theta}\right) + mgl\sin\theta\dot{\theta}$$

$$\dot{E} = 0$$

$$\therefore ml^2\dot{\theta}\left(\ddot{\theta} + \frac{g}{l}\sin\theta\right) = 0$$

$$m, \dot{\theta} \neq 0 \;\; \forall\, t$$

$$\therefore \boxed{\ddot{\theta} + \frac{g}{l}\sin\theta = 0}$$

If θ is in radians and $\theta \ll 1$

$$\boxed{\sin\theta \approx \theta}$$

$$\therefore \boxed{\ddot{\theta} \approx -\frac{g}{l}\theta}$$

This is called the "small angle approximation"

$$\ddot{\theta} \approx -\frac{g}{l}\theta \quad \text{Pendulum}$$

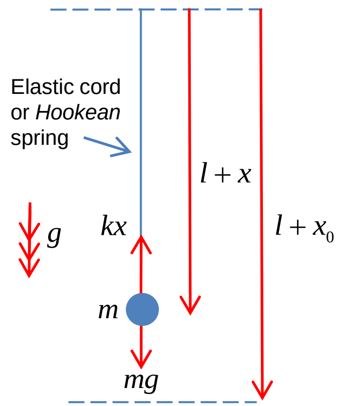
$$\ddot{\theta} = -\left(\frac{2\pi}{T}\right)^2 \theta \quad \text{SHM}$$

Hence the period of (small) pendulum oscillations is:

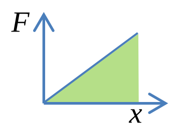
$$\left(\frac{2\pi}{T}\right)^2 = \frac{g}{l}$$

$$\therefore \boxed{T = 2\pi\sqrt{\frac{l}{g}}}$$

Example2: Mass on a Hookean spring or elastic cord



If a spring or elastic cord obeys Hooke's Law, then the restoring force experienced is in direct proportion to the amount it is stretched beyond its natural length



$$\boxed{F = kx = \lambda\frac{x}{l}}$$

k is the spring constant
λ is the elastic modulus

$$\boxed{\lambda = kl}$$

When a spring is stretched, the work done to achieve this is

$$W = \int_0^x F\,dx = \int_0^x kx\,dx = \tfrac{1}{2}kx^2$$

i.e. the area of the green triangle!

The potential energy in a stretched spring is therefore

$$\boxed{\tfrac{1}{2}kx^2}$$

Robert Hooke
1635-1703
Born in Freshwater
Isle of Wight

If the spring hangs in equilibrium $mg = kx_0 \Rightarrow x_0 = \frac{mg}{k}$

The equilibrium displacement from the hanging point of the mass is therefore $l + \frac{mg}{k}$

The total energy of the mass-spring system above is:

$$E = \tfrac{1}{2}m\dot{x}^2 + mg\left(l + \frac{mg}{k} - l - x\right) + \tfrac{1}{2}kx^2 = \tfrac{1}{2}m\dot{x}^2 + mg\left(\frac{mg}{k} - x\right) + \tfrac{1}{2}kx^2$$

*In general a *superposition* of sine waves of different amplitude and frequency

Example2: Mass on a Hookean spring or elastic cord continued ...

$$E = \tfrac{1}{2}m\dot{x}^2 + mg\left(\frac{mg}{k} - x\right) + \tfrac{1}{2}kx^2$$

$$\dot{E} = m\dot{x}\ddot{x} - mg\dot{x} + kx\dot{x}$$

$$\dot{E} = m\dot{x}\left(\ddot{x} - g + \frac{kx}{m}\right)$$

Assume system is lossless

$$\therefore \dot{E} = 0$$

$$\Rightarrow \ddot{x} - g + \frac{kx}{m} = 0$$

Define a new displacement, from the equilibrium position

$$x = z + \frac{mg}{k}$$

$$\ddot{x} = \ddot{z}$$

$$\ddot{x} - g + \frac{kx}{m} = 0$$

$$\Rightarrow \ddot{z} - g + \frac{k}{m}\left(z + \frac{mg}{k}\right) = 0$$

$$\therefore \boxed{\ddot{z} = -\frac{k}{m}z}$$

$$\ddot{\theta} = -\left(\frac{2\pi}{T}\right)^2 \theta \quad \text{SHM}$$

Hence spring oscillations will have period

$$\left(\frac{2\pi}{T}\right)^2 = \frac{k}{m}$$

$$\therefore \boxed{T = 2\pi\sqrt{\frac{m}{k}}}$$

Note in this case, Newton's Second Law can also be used to derive the equation of motion in a fairly straightforward fashion. In this case it is the most efficient method!

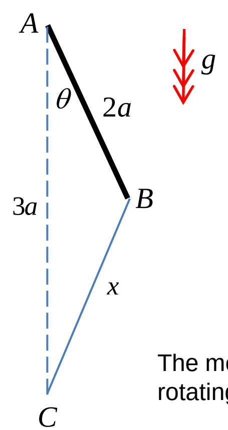
$$m\ddot{x} = mg - kx$$

$$x = z + \frac{mg}{k}$$ z is the displacement from equilibrium

$$\therefore m\ddot{z} = mg - k\left(z + \frac{mg}{k}\right)$$

$$\boxed{\ddot{z} = -\frac{k}{m}z}$$

Example3: Hinged rod and light elastic cord*



Consider a rigid rod of mass m and length $2a$ freely hinged at A.

A light elastic cord is also attached to A and threaded through the rod. It is attached to a fixed point at C directly below A. The elastic cord has natural length $2a$ and elastic modulus $\lambda = 5mg$

The moment of inertia of the rod rotating about one end is

$$I = \int_0^{2a}\left(\frac{m}{2a}dx \times x^2\right)$$

$$I = \frac{m}{2a}\int_0^{2a} x^2 dx = \frac{m}{2a}\tfrac{1}{3}(2a)^3 = \tfrac{4}{3}ma^2$$

Hence total energy is:

$$E = \tfrac{1}{2}\left(\tfrac{4}{3}ma^2\right)\dot{\theta}^2 + \tfrac{1}{2}(5mg)\frac{x^2}{2a} + mg\left(a - a\cos\theta\right)$$

Kinetic Elastic Gravitational

Cosine rule:

$$x^2 = (3a)^2 + (2a)^2 - 2(3a)(2a)\cos\theta$$

$$x^2 = (13 - 12\cos\theta)a^2$$

$$E = \tfrac{2}{3}ma^2\dot{\theta}^2 + \tfrac{5}{4}mga\left(13 - 12\cos\theta\right) + mg\left(a - a\cos\theta\right)$$

$$E = \tfrac{2}{3}ma^2\dot{\theta}^2 + mga\left(\tfrac{5}{4}\times 13 + 1 - \left(\tfrac{5}{4}\times 12 + 1\right)\cos\theta\right)$$

$$\boxed{E = \tfrac{2}{3}ma^2\dot{\theta}^2 + mga\left(17\tfrac{1}{4} - 16\cos\theta\right)}$$

$$\dot{E} = \tfrac{4}{3}ma^2\dot{\theta}\ddot{\theta} + 16mga\sin\theta\dot{\theta}$$

$$\boxed{\dot{E} = \tfrac{4}{3}ma^2\dot{\theta}\left(\ddot{\theta} + \frac{12g}{a}\sin\theta\right)}$$

$$\dot{E} = 0$$

$$\therefore \ddot{\theta} = -\frac{12g}{a}\sin\theta$$

$$\theta \ll 1 \therefore \sin\theta \approx \theta$$

$$\boxed{\ddot{\theta} \approx -\frac{12g}{a}\theta}$$

SHM $\ddot{\theta} = -\left(\frac{2\pi}{T}\right)^2 \theta$

$$\therefore \left(\frac{2\pi}{T}\right)^2 = \frac{12g}{a}$$

$$\therefore \boxed{T = 2\pi\sqrt{\frac{a}{12g}}}$$

Generalized SHM via a potential $V(x)$

$m\ddot{x} = -\dfrac{dV}{dx} = -V'$ Force is proportional to gradient of some potential V

$x = x_0 + \delta, \; \therefore \ddot{x} = \ddot{\delta}$ Small perturbation about equilibrium

$V'(x) = V'(x_0) + V''(x_0)\delta + \tfrac{1}{2}V'''(x_0)\delta^2 + ...$ Taylor expand

$$V'(x_0) = 0 \quad \therefore V'(x) \approx V''(x_0)\delta$$

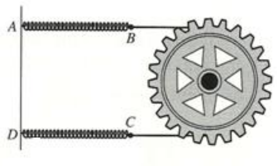
$$\therefore m\ddot{\delta} \approx -V''(x_0)\delta$$

$\ddot{\delta} = -\omega^2\delta$ Compare to SHM

$$\therefore \boxed{\omega \approx \sqrt{\frac{V''(x_0)}{m}}}$$

*Adapted from OCR Mechanics 4 textbook *Small Oscillations & Stability*

Example 4: Light springs and a sprocket wheel



Sprocket radius r

Consider a frictionless sprocket wheel of moment of inertia I connected via light inextensible chains to identical light springs, each of elastic modulus λ and natural length l.

In equilibrium $AB = BC = d + l$

Let sprocket be rotated clockwise such that AB is stretched by a further distance a from equilibrium. $a < d$

The system is released, and the displacement of B right of the equilibrium is x.

Total energy of the system is:

$$E = \tfrac{1}{2}I\left(\frac{\dot{x}}{r}\right)^2 + \tfrac{1}{2}\frac{\lambda}{l}(d+x)^2 + \tfrac{1}{2}\frac{\lambda}{l}(d-x)^2$$

Note angular speed ω of the sprocket is related to the movement of point B via $\dot{x} = r\omega$

$$E = \frac{I}{2r^2}\dot{x}^2 + \tfrac{1}{2}\frac{\lambda}{l}\left(d^2 + x^2 + 2dx + d^2 - 2dx + x^2\right)$$

$$E = \frac{I}{2r^2}\dot{x}^2 + \frac{\lambda}{l}\left(d^2 + x^2\right)$$

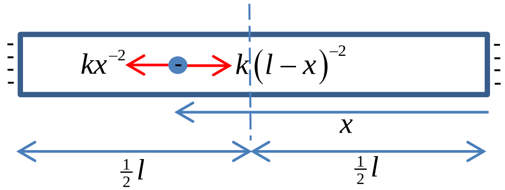
$$\dot{E} = \frac{I}{r^2}\dot{x}\ddot{x} + \frac{2\lambda x\dot{x}}{l}$$

$$\dot{E} = 0 \Rightarrow \frac{I\dot{x}}{r}\left(\ddot{x} + \frac{2\lambda r^2 x}{lI}\right) = 0$$

$$\ddot{x} = -\frac{2\lambda r^2}{lI}x$$

SHM, period $T = 2\pi\sqrt{\dfrac{lI}{2\lambda r^2}}$

Example 5: charge in a tube



A charge is placed in a tube of length l.
It is repelled with an inverse square law from identical immobile charges placed at the ends of the tube.

If the change has mass m, by Newton II:

$$m\ddot{x} = kx^{-2} - k(l-x)^{-2}$$

Define a small perturbation z from equilibrium at $x = 0.5l$

$$x = \tfrac{1}{2}l + z$$

$$m\ddot{z} = k\left(\tfrac{1}{2}l + z\right)^{-2} - k(l - \tfrac{1}{2}l - z)^{-2}$$

$$m\ddot{z} = k\left(\tfrac{1}{2}l + z\right)^{-2} - k(\tfrac{1}{2}l - z)^{-2}$$

$$m\ddot{z} = k\left(\tfrac{1}{2}l\right)^{-2}\left(1 + \frac{2z}{l}\right)^{-2} - k\left(\tfrac{1}{2}l\right)^{-2}\left(1 - \frac{2z}{l}\right)^{-2}$$

$$z \ll l$$

$$\therefore \left(1 + \frac{2z}{l}\right)^{-2} \approx 1 - \frac{4z}{l}$$

$$\therefore \left(1 - \frac{2z}{l}\right)^{-2} \approx 1 + \frac{4z}{l}$$

Generalized Binomial expansion. Ignore higher order terms.

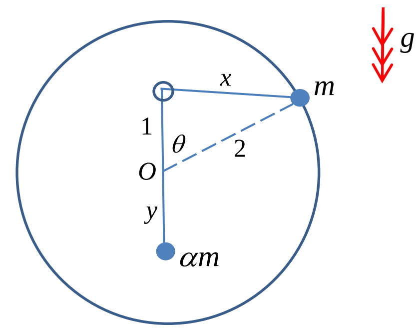
$$m\ddot{z} \approx k\left(\tfrac{1}{2}l\right)^{-2}\left(1 - \frac{4z}{l}\right) - k\left(\tfrac{1}{2}l\right)^{-2}\left(1 + \frac{4z}{l}\right)$$

$$m\ddot{z} \approx k\frac{4}{l^2}\left(-\frac{8z}{l}\right) \Rightarrow \ddot{z} \approx -\frac{32k}{ml^3}z$$

SHM, with period $T \approx 2\pi\sqrt{\dfrac{ml^3}{32k}}$

Example 6: Bead on a frictionless circular wire

Stability test



A bead of mass m is threaded on a frictionless circular wire mounted vertically. It is connected by a light inextensible string to a mass αm via a frictionless ring.

The other mass hangs vertically. The ring is one unit above the circle centre and the circle has radius 2 units. The string connecting the two masses has length 4 units.

Consider gravitational potential energies relative to the circle centre.
Total energy of the system is therefore

$$E = \tfrac{1}{2}m\left(2\dot{\theta}\right)^2 + mg(2\cos\theta) + \tfrac{1}{2}\alpha m\left(2\dot{\theta}\right)^2 - \alpha mgy$$

Note vertically hanging mass must be moving with the same velocity as the bead, since the string which connects them is inextensible

$$y + 1 + x = 4$$
$$y = 3 - x$$
$$x^2 = 1^2 + 2^2 - 2(1)(2)\cos\theta$$
$$x^2 = 5 - 4\cos\theta$$

$$E = 2m(1+\alpha)\dot{\theta}^2 + 2mg\cos\theta - \alpha mg\left(3 - \sqrt{5 - 4\cos\theta}\right)$$

Is this system stable? Is there an equilibrium point? If there is will small perturbations result in SHM?

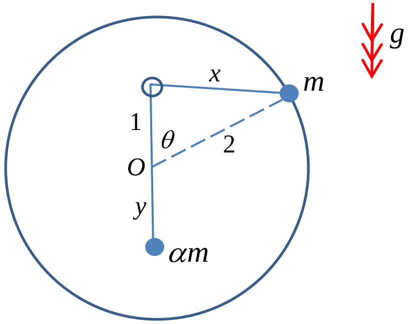
We can write the total energy (assumed constant) as

$$E = T + V$$

$$T = 2m(1+\alpha)\dot{\theta}^2 \quad \text{Kinetic energy}$$

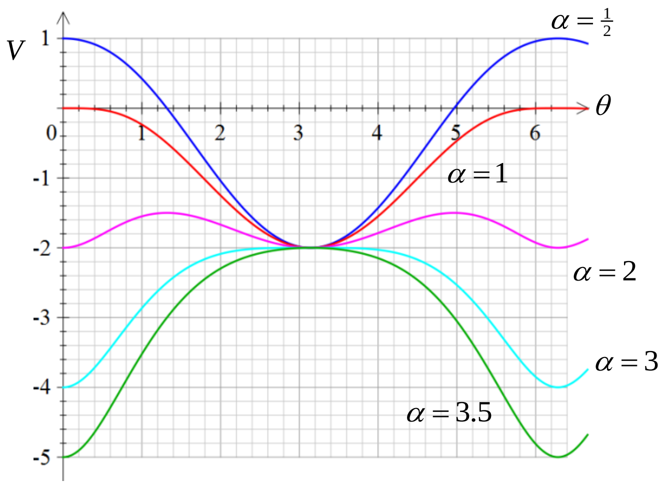
$$V = 2mg\cos\theta - \alpha mg\left(3 - \sqrt{5 - 4\cos\theta}\right) \quad \text{Potential energy}$$

See next page ....



Stability and the potential energy function V

$$E = T + V$$
$$T = 2m(1+\alpha)\dot{\theta}^2$$
$$V = 2mg\cos\theta - \alpha mg\left(3 - \sqrt{5-4\cos\theta}\right)$$



The graph of V vs θ exhibits *general features.*

If there is a **local minima**, **equilibrium is *stable* at the minima. Small perturbations about this minima will result in SHM.**

If there is a **local maxima**, this means an **unstable equilibrium.** A small perturbation from this point will result in divergence towards any nearby stable minima (if there are any).

$$\frac{dV}{d\theta} = -2mg\sin\theta + \tfrac{1}{2}\alpha mg\left(5-4\cos\theta\right)^{-\frac{1}{2}}\left(4\sin\theta\right)$$

$$\frac{dV}{d\theta} = 2mg\sin\theta\left(\frac{\alpha}{\sqrt{5-4\cos\theta}} - 1\right)$$

$$\frac{dV}{d\theta} = 0 \Rightarrow \sin\theta = 0 \Rightarrow \theta = 0, \pi$$

$$\frac{dV}{d\theta} = 0 \Rightarrow \frac{\alpha}{\sqrt{5-4\cos\theta}} - 1 = 0$$

$$\therefore \alpha^2 = 5 - 4\cos\theta$$

$$\therefore \cos\theta = \frac{5-\alpha^2}{4} \qquad \theta = \cos^{-1}\left(\frac{5-\alpha^2}{4}\right)$$

$$\cos\theta \geq -1$$
$$\therefore \frac{5-\alpha^2}{4} \geq -1$$
$$5-\alpha^2 \geq -4$$
$$9 \geq \alpha^2 \Rightarrow \alpha \leq 3$$

$$\cos\theta \leq 1$$
$$\therefore \frac{5-\alpha^2}{4} \leq 1$$
$$5-\alpha^2 \leq 4$$
$$1 \leq \alpha^2 \Rightarrow \alpha \geq 1$$

$$\frac{d^2V}{d\theta^2} > 0$$ at the stationary point of V means a local minima.

$$\frac{d^2V}{d\theta^2} = 2mg\cos\theta\left(\frac{\alpha}{\sqrt{5-4\cos\theta}} - 1\right) + \ldots$$

$$\ldots + 2mg\sin\theta\left(-\tfrac{1}{2}\frac{4\alpha\sin\theta}{\left(5-4\cos\theta\right)^{\frac{3}{2}}}\right)$$

Consider situation $1 < \alpha < 3,\ \theta \ll 1$

$$\sin\theta \approx \theta,\ \cos\theta \approx 1$$

$$\frac{dV}{d\theta} \approx 2mg\theta(\alpha - 1)$$

$$\dot{E} = 4m(1+\alpha)\dot{\theta}\ddot{\theta} + \frac{dV}{d\theta}\dot{\theta}$$

$$\dot{E} \approx 4m(1+\alpha)\dot{\theta}\left(\ddot{\theta} + \frac{2mg\theta(\alpha-1)}{4m(1+\alpha)}\right)$$

$$\dot{E} = 0 \Rightarrow \ddot{\theta} = -\frac{g(\alpha-1)}{2(1+\alpha)}\theta$$ SHM

$$T = 2\pi\sqrt{\frac{2(\alpha+1)}{g(\alpha-1)}}$$

	Stable equilibrium	Unstable equilibrium
$\alpha < 1$	$\theta = \pi$	$\theta = 0$
$1 < \alpha < 3$	$\theta = 0$ $\theta = \pi$	$\theta = \cos^{-1}\left(\frac{5-\alpha^2}{4}\right)$ $\quad \theta = 2\pi - \cos^{-1}\left(\frac{5-\alpha^2}{4}\right)$
$\alpha > 3$	$\theta = 0$	$\theta = \pi$