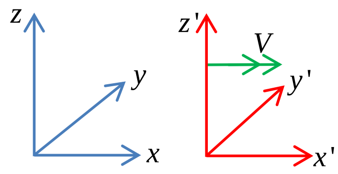
Special Relativity is a theory of dynamics proposed by **Albert Einstein** in 1905. The key mathematical element is the use of the ***Lorentz Transform*.** This extends the equations of *Galilean Relativity*, which relate the Cartesian *x,y,z* coordinates of an object to coordinates of the same object as viewed in a frame of reference moving at velocity *V* in the positive *x* direction relative to the *x,y,z* system. Let *S* denote the *x,y,z* coordinate system and *S'* denote the *x',y',z'* coordinates of the moving frame.

The Lorentz transform incorporates the strange (but seeming true!) fact that the speed of light is the **same** for both S and S' frames. In other words, if a torch is shone from frame *S*, the speed of the light observed by *S'* would be the same speed as in *S*, and *not* the speed of light minus *V*.

The consequence of this effect is *profound*. It results in *length contraction*, *time dilation* and *time synchronisation* changes between the *S* and *S'* frames.

Galilean relativity

$$\boxed{\begin{aligned} x &= x' + Vt \\ x' &= x - Vt \\ y &= y' \\ z &= z' \\ t &= t' \end{aligned}}$$



Galilean relativity appears to work just fine in normal scenarios on Earth, i.e. when $V \ll c$ where the speed of light $c = 2.998 \times 10^8 \text{ ms}^{-1}$
The effects of Special relativity are *only significant* when *V* is close to *c*.

Consider the following candidates for the Lorentz transform of the spatial coordinates between the *S* and *S'* frames:

$$\boxed{\begin{aligned} x &= \gamma(x' + Vt') \\ x' &= \gamma(x - Vt) \\ y &= y' \\ z &= z' \end{aligned}}$$

γ is a function of *V*. In order to be consistent with Galilean relativity, it must be *unity* when $V \ll c$

Hence:

$$x = \gamma(x' + Vt') \qquad x' = \gamma(x - Vt)$$

$$\frac{x}{\gamma} = x' + Vt' \qquad \frac{x'}{\gamma} = x - Vt$$

$$t' = \frac{x}{\gamma V} - \frac{x'}{V} \qquad t = \frac{x}{V} - \frac{x'}{\gamma V}$$

$$t' = \frac{x}{\gamma V} - \frac{\gamma(x - Vt)}{V} \qquad t = \frac{\gamma(x' + Vt')}{V} - \frac{x'}{\gamma V}$$

$$\therefore \boxed{t' = \gamma\left(t - \frac{x}{V}\left(1 - \frac{1}{\gamma^2}\right)\right)} \qquad \therefore \boxed{t = \gamma\left(t' + \frac{x'}{V}\left(1 - \frac{1}{\gamma^2}\right)\right)}$$

Now consider a *spherical light pulse* emitted when *x' = x*. Since it radiates out at speed *c* in ***both*** *S* and *S'* from their (respective) origins, we can compare the radii *r,r'* of the pulse as observed from *S* and *S'*

$$r'^2 = c^2 t'^2 = x'^2 + y'^2 + z'^2$$
$$r^2 = c^2 t^2 = x^2 + y^2 + z^2$$

Since $y = y', \; z = z'$ this means $c^2 t'^2 - x'^2 = c^2 t^2 - x^2$

Now when $x' = 0, \quad x = Vt$

Hence $c^2 t'^2 = c^2 t^2 - V^2 t^2$

$$\Rightarrow t' = t\sqrt{1 - \frac{V^2}{c^2}}$$

Now using $t = \gamma\left(t' + \frac{x'}{V}\left(1 - \frac{1}{\gamma^2}\right)\right)$ when $x' = 0 \;\; \Rightarrow t = \gamma t\sqrt{1 - \frac{V^2}{c^2}}$

$$\therefore \boxed{\gamma = \left(1 - \frac{V^2}{c^2}\right)^{-\frac{1}{2}}}$$

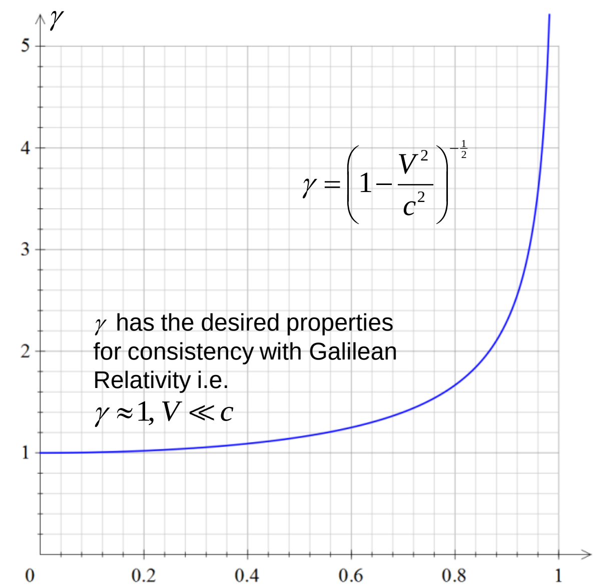
$$\therefore \; 1 - \frac{1}{\gamma^2} = 1 - 1 - \frac{V^2}{c^2} = \frac{V^2}{c^2}$$

$$\therefore \frac{1}{V}\left(1 - \frac{1}{\gamma^2}\right) = \frac{V}{c^2}$$

The **Lorentz Transform** is now revealed!

$$\boxed{\begin{aligned} x &= \gamma(x' + Vt') & x' &= \gamma(x - Vt) \\ y &= y' & y &= y' \\ z &= z' & z &= z' \\ t &= \gamma\left(t' + \frac{Vx'}{c^2}\right) & t' &= \gamma\left(t - \frac{Vx}{c^2}\right) \end{aligned}}$$

So ***lengths contract*** and ***time dilates and shifts*** when *V* becomes close to *c*



The **Lorentz Transform** can be applied to relate **other dynamical parameters** between the S and S' frames

Velocity

$$v_x = \frac{dx}{dt} = \frac{\gamma\left(dx'+Vdt'\right)}{\gamma\left(dt'+\frac{V}{c^2}dx'\right)}$$

$$v_x = \frac{\frac{dx'}{dt'}+V}{1+\frac{V}{c^2}\frac{dx'}{dt'}}$$

$$\boxed{v_x = \frac{v_x'+V}{1+\frac{v_x'V}{c^2}}}$$

$$v_y = \frac{dy}{dt} = \frac{dy'}{\gamma\left(dt'+\frac{V}{c^2}dx'\right)}$$

$$v_y = \frac{\frac{dy'}{dt'}}{\gamma\left(1+\frac{V}{c^2}\frac{dx'}{dt'}\right)}$$

$$\boxed{v_y = \frac{v_y'}{\gamma\left(1+\frac{v_x'V}{c^2}\right)}}$$

$$v_z = \frac{dz}{dt} = \frac{dz'}{\gamma\left(dt'+\frac{V}{c^2}dx'\right)}$$

$$v_z = \frac{\frac{dz'}{dt'}}{\gamma\left(1+\frac{V}{c^2}\frac{dx'}{dt'}\right)}$$

$$\boxed{v_z = \frac{v_z'}{\gamma\left(1+\frac{v_z'V}{c^2}\right)}}$$

and by an equivalent argument

$$\boxed{\begin{aligned} v_x' &= \frac{v_x - V}{1-\frac{v_xV}{c^2}} \\ v_y' &= \frac{v_y}{\gamma\left(1-\frac{v_xV}{c^2}\right)} \\ v_z' &= \frac{v_z}{\gamma\left(1-\frac{v_xV}{c^2}\right)} \end{aligned}}$$

If the velocity of a *photon* of light is defined in plane polars

$$\boxed{\begin{aligned} v_x &= c\cos\theta \\ v_x' &= c\cos\theta' \\ \cos\theta &= \frac{\cos\theta'+\frac{V}{c}}{1+\frac{V}{c}\cos\theta'} \\ \cos\theta' &= \frac{\cos\theta-\frac{V}{c}}{1-\frac{V}{c}\cos\theta} \end{aligned}}$$

This is called 'relativistic aberration'

Doppler shift

Consider a receding wave source of frequency f' in the S' frame. It crosses the x axis of the S frame at angle θ. and speed u. The velocity of waves emitted is w, in S.

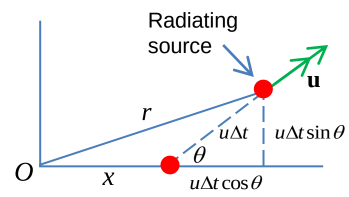
The period T of waves received by an observer (in the x direction) at the origin O of the S frame is:

$$T = \Delta t + \frac{r-x}{w}$$

time between wave crests at *source*

wave speed

extra distance travelled by source between wave crests



From geometry:

$$r = \sqrt{\left(x+u\Delta t\cos\theta\right)^2 + u^2\Delta t^2\sin^2\theta}$$

$$r = \sqrt{x^2 + u^2\Delta t^2\cos^2\theta + 2ux\Delta t\cos\theta + u^2\Delta t^2\sin^2\theta}$$

$$r = \sqrt{x^2 + u^2\Delta t^2 + 2ux\Delta t\cos\theta}$$

$$r = x\sqrt{1+2\cos\theta\frac{u\Delta t}{x}+\left(\frac{u\Delta t}{x}\right)^2}$$

If $u\Delta t \ll x$ $\quad r \approx x\sqrt{1+2\cos\theta\frac{u\Delta t}{x}} \approx x\left(1+\cos\theta\frac{u\Delta t}{x}\right) = x+u\Delta t\cos\theta$

$$\therefore \boxed{r-x \approx u\Delta t\cos\theta}$$

Hence frequency of radiation received at O is $f = 1/T$ where:

$$\frac{1}{f} = \Delta t + \frac{u\Delta t\cos\theta}{w} = \Delta t\left(1+\frac{u\cos\theta}{w}\right)$$

$$\gamma = \left(1-\frac{u^2}{c^2}\right)^{-\frac{1}{2}}$$

Using the **generalized Lorentz Transform** $\quad \Delta t = \gamma\left(\Delta t'+\frac{\mathbf{u}\cdot\Delta\mathbf{r}'}{c^2}\right)$

Now since the source is stationary in the S' frame $\quad \Delta\mathbf{r}' = 0$

Therefore $\Delta t = \gamma\Delta t' = \frac{\gamma}{f'}$

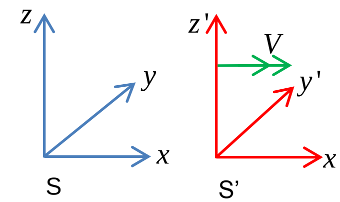
$$\frac{1}{f} = \frac{\gamma}{f'}\left(1+\frac{u\cos\theta}{w}\right) \Rightarrow$$

Hence:

$$\boxed{f = \frac{f'}{\gamma\left(1+\frac{u\cos\theta}{w}\right)}}$$

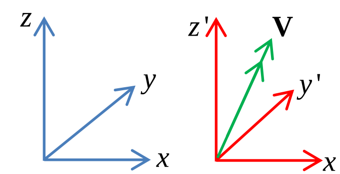
The Lorentz Transform

$$\boxed{\begin{aligned} x &= \gamma\left(x'+Vt'\right) & x' &= \gamma\left(x-Vt\right) \\ y &= y' & y' &= y \\ z &= z' & z' &= z \\ t &= \gamma\left(t'+\frac{Vx'}{c^2}\right) & t' &= \gamma\left(t-\frac{Vx}{c^2}\right) \end{aligned}}$$



We can generalize to an S' velocity which is not parallel to the x axis of the S frame

$$\boxed{\begin{aligned} &\mathbf{r} = \left(x,y,z\right), \quad \mathbf{r}' = \left(x',y',z'\right) \\ &\mathbf{r} = \mathbf{r}' + \left(\frac{\gamma-1}{V^2}\left(\mathbf{V}\cdot\mathbf{r}'\right)+\gamma t'\right)\mathbf{V} \\ &t = \gamma\left(t'+\frac{\mathbf{V}\cdot\mathbf{r}'}{c^2}\right) \\ &\mathbf{r}' = \mathbf{r} + \left(\frac{\gamma-1}{V^2}\left(\mathbf{V}\cdot\mathbf{r}\right)-\gamma t'\right)\mathbf{V} \\ &t' = \gamma\left(t-\frac{\mathbf{V}\cdot\mathbf{r}}{c^2}\right) \\ &V = \left|\mathbf{V}\right| \\ &\gamma = \left(1-\frac{V^2}{c^2}\right)^{-\frac{1}{2}} \end{aligned}}$$



Relativistic Doppler shift cont ....

$$f=\frac{f'}{\gamma\left(1+\frac{u\cos\theta}{w}\right)}$$

Define **Doppler frequency shift**

$$\boxed{\Delta f=f-f'}$$

$$\Delta f=\frac{f'}{\gamma\left(1+\frac{u\cos\theta}{w}\right)}-f'$$

$$\boxed{\frac{\Delta f}{f'}=\frac{1}{\gamma\left(1+\frac{u\cos\theta}{w}\right)}-1}$$

Note if waves are *electromagnetic* $w=c$

The *classical* formula can easily be recovered by setting $\gamma = 1$

$$\frac{\Delta f}{f'}\approx\frac{1}{1+\frac{u\cos\theta}{w}}-1=\frac{1-1-\frac{u\cos\theta}{w}}{1+\frac{u\cos\theta}{w}}$$

$$\frac{\Delta f}{f'}\approx-\frac{\frac{u\cos\theta}{w}}{1+\frac{u\cos\theta}{w}}$$

If $u\cos\theta\ll w$

$$\boxed{\frac{\Delta f}{f'}\approx-\frac{u\cos\theta}{w}}$$

Unlike the classical formula, we get a *transverse Doppler effect* when $\theta = 90^\circ$ in the relativistic version

$$\frac{\Delta f}{f'}=\frac{1}{\gamma}-1$$

The Doppler shift is also related to the '***redshift***' z of a moving, radiating source

$$z=\frac{f'-f}{f}=\frac{f'}{f}-1$$

$$\boxed{z=\gamma\left(1+\frac{u\cos\theta}{w}\right)-1}$$

Momentum

We might expect 'force = rate of change of momentum' to be true in a relativistic sense as well as in the classical. However, the speed limit of c would imply an *upper limit on the amount of momentum a given mass could* have, if we use the classical momentum formula

$$\mathbf{p}=m\mathbf{u}$$

This would be *counter to reality* – we could easily devise a theoretical system which does a finite amount of work, indefinitely, upon fixed mass system. e.g. a ball rolling down a infinitely long slope!
To get around this problem, let us *redefine* momentum such that it *can* become infinite as velocity tends towards c. i.e. multiply by γ....

$$\boxed{\begin{array}{l}\mathbf{p}=\gamma m\mathbf{u}\\ \gamma=\left(1-\frac{\mathbf{u}\cdot\mathbf{u}}{c^2}\right)^{-\frac{1}{2}}=\left(1-\frac{u^2}{c^2}\right)^{-\frac{1}{2}}\end{array}}$$

Some useful derivatives involving γ

$$\frac{d\gamma}{dt}=-\frac{1}{2}\left(1-\frac{\mathbf{u}\cdot\mathbf{u}}{c^2}\right)^{-\frac{3}{2}}\left(-2\frac{\mathbf{u}}{c^2}\cdot\frac{d\mathbf{u}}{dt}\right)$$

$$\boxed{\frac{d\gamma}{dt}=\gamma^3\frac{\mathbf{a}\cdot\mathbf{u}}{c^2}} \qquad \mathbf{a}=\frac{d\mathbf{u}}{dt}$$

acceleration

$$\frac{d\gamma}{du}=-\frac{1}{2}\left(1-\frac{u^2}{c^2}\right)^{-\frac{3}{2}}\left(-\frac{2u}{c^2}\right)$$

$$\boxed{\frac{d\gamma}{du}=\gamma^3\frac{u}{c^2}}$$

Force, work & energy

$$\boxed{1+\frac{\gamma^2u^2}{c^2}=\gamma^2}$$

$$\Rightarrow\gamma^2\left(1-\frac{u^2}{c^2}\right)=1\Rightarrow\gamma=\left(1-\frac{u^2}{c^2}\right)^{-\frac{1}{2}}$$

$$\boxed{\mathbf{f}=\frac{d}{dt}(\gamma m\mathbf{u})}$$

$$\mathbf{f}=m\gamma\frac{d\mathbf{u}}{dt}+m\mathbf{u}\frac{d\gamma}{dt}$$

$$\boxed{\mathbf{f}=m\gamma\mathbf{a}+m\gamma^3\left(\frac{\mathbf{a}\cdot\mathbf{u}}{c^2}\right)\mathbf{u}}$$

'Relativistic Newton's Second Law'

$$\boxed{W=\int\mathbf{f}\cdot d\mathbf{r}=\int\mathbf{f}\cdot\mathbf{u}dt}$$

Work done

$$W=m\int\left(\gamma\mathbf{a}\cdot\mathbf{u}+\gamma^3\left(\frac{\mathbf{a}\cdot\mathbf{u}}{c^2}\right)u^2\right)dt$$

$$W=m\int\gamma(\mathbf{a}\cdot\mathbf{u})\left(1+\frac{\gamma^2u^2}{c^2}\right)dt$$

$$W=m\int\gamma^3(\mathbf{a}\cdot\mathbf{u})dt$$

$$1+\frac{\gamma^2u^2}{c^2}=\gamma^2$$

$$W=mc^2\int\gamma^3\frac{(\mathbf{a}\cdot\mathbf{u})}{c^2}dt$$

$$\frac{d\gamma}{dt}=\gamma^3\frac{\mathbf{a}\cdot\mathbf{u}}{c^2}$$

$$W=mc^2\int\frac{d\gamma}{dt}dt$$

$$W=mc^2\int_{\gamma_0}^{\gamma_1}d\gamma$$

$$\boxed{W=(\gamma_1-\gamma_0)mc^2}$$

So the **total energy** of a mass m is

$$\boxed{E=\gamma mc^2}$$

and when the *mass is at rest*

$$\boxed{\begin{array}{l}\gamma=1\\ E_0=mc^2\end{array}}$$

Hence **kinetic energy** is

$$\boxed{E_k=(\gamma-1)mc^2}$$

Now in **classical limit**

$u\ll u$

$$\gamma\approx1+\tfrac{1}{2}\frac{u^2}{c^2}$$

$$\therefore(\gamma-1)mc^2=\boxed{\tfrac{1}{2}mu^2}$$

Energy, momentum invariant

Consider the following quantity:

$$k = E^2 - |\mathbf{p}|^2 c^2$$
$$k = \left(\gamma mc^2\right)^2 - \left(\gamma m\mathbf{u}\right)\cdot\left(\gamma m\mathbf{u}\right)c^2$$
$$k = \gamma^2 m^2 c^4 - \gamma^2 m^2 u^2 c^2$$
$$k = m^2 c^4 \gamma^2 \left(1 - \frac{u^2}{c^2}\right)$$
$$k = m^2 c^4 \left(1 - \frac{u^2}{c^2}\right)^{-1} \left(1 - \frac{u^2}{c^2}\right)$$
$$k = m^2 c^4$$

This is clearly an invariant, *regardless* of the frame of reference.

$$\boxed{E^2 - |\mathbf{p}|^2 c^2 = m^2 c^4}$$

Application: "A particle with rest mass 2*m* strikes a stationary particle with rest mass 3*m*. The 2*m* particle had kinetic energy mc^2, and the result was an inelastic collision. Find the rest mass *M* of the resulting particle in terms of *m*"

$$(\gamma - 1)mc^2 = mc^2$$
$$\therefore \gamma = 2$$
$$\therefore 2 = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}}$$
$$\Rightarrow \tfrac{1}{4} = 1 - \frac{u^2}{c^2}$$
$$\Rightarrow \frac{u^2}{c^2} = \tfrac{3}{4}$$
$$\Rightarrow u = \tfrac{\sqrt{3}}{2}c \approx 0.866c$$
$$\therefore E = 2 \times 2mc^2 + 3mc^2 = 7mc^2$$
$$|\mathbf{p}|^2 = \gamma^2 (2m)^2 u^2 = 16m^2 \times \tfrac{3}{4}c^2 = 12m^2c^2$$
$$E^2 - |\mathbf{p}|^2 c^2 = M^2 c^4$$
$$\therefore 49m^2c^4 - 12m^2c^4 = M^2c^4$$
$$M^2 = 37m^2$$
$$\boxed{M = m\sqrt{37}}$$

Using **conservation of energy,** we can also find the *velocity* of the resulting particle

$$\left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} mc^2 \sqrt{37} = 7mc^2$$
$$1 - \frac{v^2}{c^2} = \frac{37}{49}$$
$$\therefore \frac{v^2}{c^2} = \frac{12}{49}$$
$$\boxed{\begin{aligned} v &= \frac{\sqrt{12}}{7}c \\ v &\approx 0.495c \end{aligned}}$$