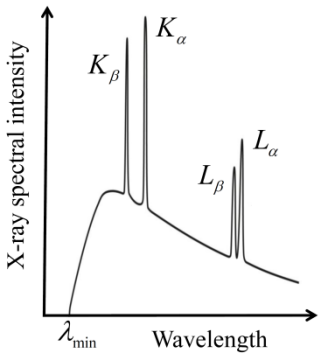
Medical imaging is to create meaningful visual representations of the interior structures of a living organism, humans in particular. It is hugely beneficial for this process to *not* require the body to be physically opened, as this can risk infection and will cause significant pain. Medical practitioners have a wide range of imaging tools at their disposal, and this problem sheet will explore several in common usage. Whereas lower energy forms of X-rays and gamma rays are typically used for *diagnostic purposes*, higher energy forms can be used *therapeutically*, e.g. for the destruction of tumors.



Wilhelm Röntgen (1845-1923) discovered **X-rays**, a much higher frequency form of electromagnetic radiation than visible and ultra-violet light. "Soft" X-rays have a vacuum wavelength of between 120pm to 10nm, whereas "Hard" X-rays, which are more penetrating, have wavelengths between about 10pm to 120pm. Mostly Hard X-rays are used for medical imaging.

X-rays can be produced by accelerating electrons via a strong electric field in a near-vacuum, and smashing the high velocity electrons into a metal target such as tungsten. A medical X-ray imaging system may require typical voltages of between 60kV to 120kV.

The *spectrum* of X-rays produced by a metal varies with accelerating potential. There are two key features: The first is a *continuous* spectrum due to *Bremsstrahlung*. X-rays are radiated from the *decelerated* electrons as they interact with charged particles, i.e. the electrons and protons in the metal target. The second are the K and L lines which correspond to *discrete electron energy levels* in the metal. Although the distinction is over simplistic, the 'braking radiation' is mostly a *classical* effect, whereas the K and L lines are *quantum mechanical* in nature. The K and L lines are *always at the same wavelength*, whereas changing the accelerating potential will shift and stretch the *Bremsstrahlung* component of the spectrum. Note the spectrum must have a *minimum wavelength cutoff.* This is when an X-ray corresponds to 100% of the accelerated electron kinetic energy. i.e. $eV = hc/\lambda_{\min} \Rightarrow \lambda_{\min} = \dfrac{1,240}{V/\text{kV}} \times 10^{-12}\,\text{m}$. Planck's constant $h = 6.626 \times 10^{-34}\,\text{Js}$; speed of light in a vacuum $c = 2.998 \times 10^{8}\,\text{ms}^{-1}$; charge on the electron $e = -1.602 \times 10^{-19}\,\text{C}$. Note 1pm = $10^{-12}\,\text{m}$.

X-rays are only partially *attenuated* by bone, and much less so by muscle and lower density organic tissues such as fat and muscle. X-rays can therefore be passed through a human body and darken a photographic plate (or in a modern setup, result in a digital encoding of X-ray intensity). Bone structures will appear brighter due to the higher attenuation. A *contrast medium* such as *iodine* or a *barium meal* can be ingested by a patient to widen the range of attenuation, and hence increase the clarity of X-ray imaging of soft tissues such as heart blood vessels or intestinal structures.

X-rays performed at multiple orientations can be combined in a **Computer Assisted Tomography (CAT) scan** to generate in software the equivalent of a human being placed in a meat slicer and the slices photographed. Combination of the slices can result in a three-dimensional representation of the internal structures of a living organism. A hazard of a CAT (or simply 'CT') scan is a *high dose* of X-rays, so it must be used sparingly.

Attenuation of X-rays by human tissues will have a $1/x^2$ reduction of intensity I with range x from a source, and *absorption* which can be modeled via an *exponential* variation: $I = I_0 e^{-\mu x}$. The attenuation coefficient is μ. The Half Value Thickness (HVT) x_{HVT} is when $I = \frac{1}{2} I_0$ i.e. $\dfrac{\ln 2}{\mu} = x_{\text{HVT}}$.

Gamma rays, which have even higher photon energies (and smaller wavelengths) than X-rays can be used to image internal organs. These are produced from a *radioactive tracer* (such as Technetium-99) that is injected into the patient. As the tracer moves with the blood-flow in the body, it will indicate its presence via the production of gamma rays which can be detected via a *gamma camera*, consisting of a *scintillator* (which converts gamma rays to visible light) and *photomultiplier tube* array detector (which converts visible light photons to a cascade of electrons, which in turn allows very low light levels to be detected in terms of voltage measurements).

Positron Emmision Tomography (PET) scanners work in a similar fashion to a gamma camera in the sense of detection of gamma rays that are produced inside the body via a radioactive tracer. In a PET scan a *positron-emitter* is ingested. A popular form is *flurodeoxyglucose* (FDG), which is a glucose molecule with on oxygen atom replaced by Fluorine-18. This is a positron emitter with a half life of about 110minutes. ${}^{18}_{9}\text{F} \rightarrow {}^{18}_{8}\text{O} + {}^{0}_{+1}e^{+} + \nu_e + \gamma$

The positrons will travel about a mm before they *annihilate* with a electron, ${}^{0}_{+1}e^{+} + {}^{0}_{-1}e^{-} \rightarrow 2\gamma$, producing a pair of 0.51MeV gamma rays. These can be detected by a ring of gamma detectors, and hence used to triangulate the source of the annihilation. A computer image of the 3D location of gamma sources can be used to map the movement of the FDG tracer in the body, and hence the internal structure of the body itself. PET scanners are very expensive, not least because the FDG must be prepared in a laboratory with a particle accelerator in close proximity to the scanner.

Magnetic Resonance Imaging (MRI) is an amazing, but expensive and complex technique that can create a detailed 3D representation of the human body in a similar fashion to a CAT scanner. However instead of using X-rays, which can be harmful in large doses, a MRI scanner detects the *magnetic field* produced by protons that are caused to *precess* about a direction set by a very strong (1 to 2 Tesla) magnetic field. The precession frequency (called the *Larmor frequency*) is given by $f = \frac{1}{2\pi}\gamma B$ where B is the magnetic field strength and γ is the *gyromagnetic ratio*, which for protons is $\gamma = 42.57\text{MHzT}^{-1}$. A measurable magnetic field is produced by causing protons to *resonate* by providing a driving radio-frequency pulse at the Larmor frequency. MRI scanners are very expensive as *superconducting coils* are typically used to generate the high magnetic fields. To achieve this, very low temperatures (about 4 Kelvin) are needed which means liquid helium, which is very expensive to manufacture and store. The use of a very strong magnetic field in a MRI scanner is additionally a major hazard if a patient contains any implant that is made of magnetizable metal such as earrings, fracture pins, artificial joint, pacemaker etc.

Ultrasound, pulses of high frequency acoustic waves with frequencies between 1 to 15 MHz (note human range of hearing is 20Hz to 20kHz) can be used for imaging the interior of a human body. Ultrasound is typically much cheaper than X-rays, PET scans, CAT scans and MRI scans, and is particularly useful for imaging delicate structures such as fetuses or sexual organs that could suffer cell damage if exposed to X-rays or injected with a radioactive tracer.

An *Amplitude (A) scan* works by transmitted a ultrasonic pulse through a body and, between pulses, listening for reflections. This is effectively *sonar*, and can be used to make accurate internal range measurements such as the width of an eyeball from lens to optic nerve. Ultrasound devices use a *transducer* head based upon a *piezoelectric material*. An electrical pulse will cause the material to undergo strain, which will produce the ultrasound. In receive mode, an incoming ultrasonic pulse will cause the material to strain, which will in turn induce a voltage which can be measured. Quartz is one of the original piezoelectric materials, but these days lead zirconate titanate (a ceramic) or polyvinylidene fluoride (a polymer plastic) are used.

Computerized combinations of A-scans at different angles can be used to generate 2D, and also 3D images. These are called *Brightness (B) scans.* Most fetal scans of pregnant women are B-scans.

Ultrasound works on the basis of a balance between transmitted and reflected waves at the boundary of body structures that have an *acoustic impedance* Z contrast. $Z = \rho c$ where ρ is density and c is the speed of acoustic waves. At a boundary, the power of reflected waves is: $I_r = I_0 \times \left(\frac{Z_1 - Z_2}{Z_1 + Z_2} \right)^2$ where I_0 is the intensity (power per unit area) of ultrasound incident to the boundary. Note the power of a wave of amplitude A and frequency f is $I \propto ZA^2 f^2$. The intensity of transmitted ultrasound is $I_t = I_0 - I_r$. Note this means a **wave amplitude reflection coefficient** of $R = \frac{Z_1 - Z_2}{Z_1 + Z_2}$. If $Z_2 \gg Z_1$ then $R \approx -1$ which implies *inversion* of a wave incident to a low to high impedance boundary. This explains why coupling gel is used during ultrasound scanning. The air-skin boundary has such a large impedance contrast, that most of the ultrasound would be otherwise reflected. Note *ultrasound frequencies are conserved across an impedance boundary*, so wavelengths must vary in the same ratio to speeds. $c = f\lambda \quad \therefore c_1/\lambda_1 = c_2/\lambda_2$.

Doppler shifts of ultrasonic pulses from moving fluids such as blood can be used to measure flow rates internal to the body. The Doppler frequency shift (note a *two-way process* since the moving fluid reflects the acoustic waves) is given by:

$\Delta f = \frac{2fv\cos\theta}{c}$ where θ is the angle of the incident beam from the blood flow velocity vector, and v is the blood flow speed. c is the ultrasound speed and f is the ultrasound frequency.

Question 1

(i) An X-ray source used by a dentist accelerates electrons into a metal target, inside a vacuum tube. The voltage between the cathode and anode of the tube is 70.0kV.
(a) Calculate the smallest in-vacuum wavelength of the X-rays.
(b) The current in the X-ray tube is 8.0mA, and 0.8% of the kinetic energy of electrons is, on average, converted to X-rays. Calculate the total energy (in J) of the X-rays, if the exposure time is 0.320s.

(ii) Calculate the photon energies (in keV) of the K lines in tungsten, if their respective (vacuum) wavelengths are 0.0184nm and 0.0209nm. A silver (Ag) anode has K line photon energies of 22.2keV and 24.9keV respectively. By sketching spectra, one underneath the other with the same wavelength scale, compare a 120kV tungsten anode tube, with a 70kV silver anode tube. (*Hint*: first make a table of wavelengths $\lambda_\alpha, \lambda_\beta, \lambda_{\min}$, in pm, for each metal).

(iii) In a PET scanner, a positron annihilates with an electron to produce gamma radiation. Explain why *two* gamma rays moving in *opposite* directions is the outcome. Use the rest mass equation $E = mc^2$ to calculate the minimum energy of each gamma ray in MeV. The mass of an electron (or a positron) is $m_e = 9.109 \times 10^{-31}\text{kg}$.

(iv) In a PET scan, a patient is injected with is *flurodeoxyglucose* (FDG), which contains a radioactive Fluorine-18 atom, which undergoes *positron decay* with a half life of about 110 minutes. Assume it takes 30minutes to prepare an FDG tracer injection from the manufacture of FDG (in a particle accelerator from Oxygen-18), and the PET scan itself takes 30minutes. Calculate the fraction of the initial activity of 'fresh' FTG before and after the scan.

(v) A piezoelectric ultrasonic sensor delivers a 5.00MHz acoustic wave for a duration of $20.0\mu\text{s}$, when driven by an electrical pulse. This is the *transmit* mode. The transceiver then works in reverse, i.e. produces a voltage as it receives reflections. 4000 pulses are produced per second.
(a) If the speed of sound in fat is 1450m/s, and the speed of sound in bone is 4,000m/s, calculate the maximum thickness of each (pure) substance such that reflections will be received before another pulse is transmitted.
(b) If soft tissue has an average sound speed of 1540m/s, what is the minimum depth of a body feature with a significant impedance contrast that can be detected? Assume you need *to receive a entire reflected pulse to declare a detection.* How many oscillation periods comprise a single $20.0\mu\text{s}$ pulse?

(vi) Use the table below to answer the following questions. (a) Why is a *coupling gel* needed between an ultrasonic transceiver and soft tissue rather than air to soft tissue? (b) Calculate the fraction of incident ultrasound *intensity* at a muscle to bone boundary that is reflected and transmitted, assuming normal incidence. (c) Calculate the *critical angle* of ultrasound to the normal of the muscle-bone interface. What happens when the ultrasound exceeds this?

Substance	Density /kgm^{-3}	Speed of sound /ms^{-1}
Air	1.3	340
Soft tissue (average)	1,060	1,540
Muscle	1,070	1,580
Bone	1,900	4,000

(vii) *Haemodynamics* is the study of blood flow. Blood in the *pulmonary artery* (which carries deoxygenated blood from the right ventricle of the heart to the lungs) can travel at speed up to 57cm/s. Calculate the Doppler shift of a 12MHz ultrasonic pulse that is transmitted at 50^o to normal incidence to the arterial blood flow direction. Take the speed of sound to be 1,540m/s. What fraction of the ultrasound frequency is this?

(viii) Titanium is a metal which is often used in hip replacements. Ultrasonic waves travelling from muscle to titanium results in 78% of incident power being reflected. *Note reflections are inverted*. If the density of titanium is 4500kgm^{-3}, calculate the speed of sound in titanium. Hence calculate the fraction of incident ultrasound power *transmitted* across a titanium to bone boundary. In this case, will reflected waves be inverted, or not?

(ix) Draw an annotated diagram to explain the operation of a photomultiplier tube used to detect gamma rays in a PET scanner or gamma camera.

Question 2

Draw up a table to compare and contrast various medical imaging technologies. Use the column and row headings below:

Medical imaging technology	**Wave source**	**What can be imaged, and how?**	**Disadvantages / hazards**
X-ray (standard photography)			
X-ray CAT scanner			
Gamma camera			
PET scanner			
Ultrasound A-scan			
Ultrasound B-scan			
MRI			

Question 3

A collimated beam of hard X-rays is used to destroy a cancerous growth, or *malignant tumour*. The tumour, with cross section 2.2mm × 2.2mm is at a depth of 15.2mm in soft tissue, which has a Half Value Thickness (HVT) of 54.3mm for these X-rays. Assume 234J of absorbed X-rays will destroy the tumour, and the tumour absorbs 11% of the X-ray energy. If the X-ray exposure lasts 42 minutes, calculate the required intensity (in W/m^2) of the X-ray beam incident to the soft tissue.

A simple rule for therapeutic use of X-rays is that the maximum dose is delivered *at a distance into the body (in cm) roughly equal to ¼ of the beam energy (in MeV).* Using $E = hc/\lambda$, calculate the minimum wavelength of X-rays than can be used.

Question 4

The attenuation constant of X-rays produced by a 100kV source is $\mu_m = 0.21\text{cm}^{-1}$ for muscle and $\mu_b = 0.60\text{cm}^{-1}$ for bone. Beam intensity I W/m^2 is modeled to reduce *exponentially* with tissue thickness x via a relation $I(x) = I_0 e^{-\mu x}$, and also as an *inverse-square law with distance from source*. Dr French fractures his left radial forearm bone and requires an X-ray. If his arm is 5.00cm thick, consisting of 2.00cm of muscle, 1.00cm of bone and then 2.00cm of muscle, calculate the X-ray intensity expected to emerge from the arm. 20Wm^{-2} is received with the source and detector 5.0cm apart, but no arm in between. Assume no reflections of X-rays at each muscle, bone boundary.

Repeat the calculation, but with a leg with thickness 14.0cm with the middle 2.34cm being the femur bone. Assume the rest is muscle. Assume the X-ray source delivers the same radiative power, and in both cases the X-ray source is placed *right next to the limb*. Assume in both cases the limb is in contact with the detector. In the leg case, calculate the detected X-ray intensity relative to the without-arm detection in (a).

Question 5

Use following tissue depth x (cm) and X-ray intensity I (W/m^2) measurements to plot (a) Intensity vs tissue depth and (b) an *appropriate straight-line graph* that enables you to calculate the attenuation constant μ (in cm^{-1}) and Half Value Thickness (HVT) (in cm). You may wish to use a spreadsheet to answer this question.

x = [0.00, 0.89, 1.78, 2.67, 3.56, 4.44, 5.33, 6.22, 7.11, 8.00]

I = [20, 14.7, 10.8, 7.94, 5.83, 4.29, 3.15, 2.31, 1.70, 1.25]

Question 6

Ultrasonic waves travel through muscle tissue and reflect off a muscle-bone boundary. The transmitted fraction of the incident waves continue through 1.23cm of bone and then reflect off the bone-muscle boundary, travel back through the bone, and are eventually transmitted through the bone-muscle boundary. If an ultrasonic transducer is connected to an oscilloscope such that both reflections are clearly visible, calculate the *voltages* and *time delay* between the pair of reflections, and hence carefully sketch the anticipated A-scan trace. (Note: plot voltages, not intensities!)

Assume the voltage across the piezoelectric sensor associated with ultrasound *incident* on the muscle-bone boundary is 0.70V. The oscilloscope time base is 1.0 μs , the pulse duration is 1.2 μs , the voltage divisions are 0.1V and the oscilloscope display is a 10 by 10 grid, from 0 to 10 μs , and from -0.5V to 0.5V. Assume the muscle-bone reflection is at 2.0 μs . If *higher frequency ultrasound is used, suggest qualitatively how the A-scan might change.*

Substance	Density /kgm^{-3}	Speed of sound /ms^{-1}
Muscle	1,070	1,580
Bone	1,900	4,000