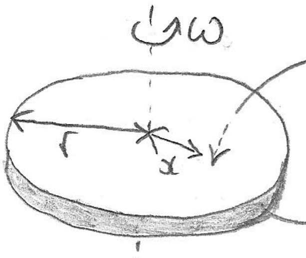
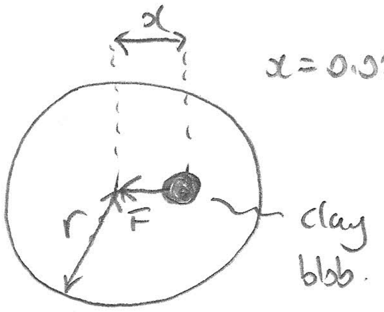
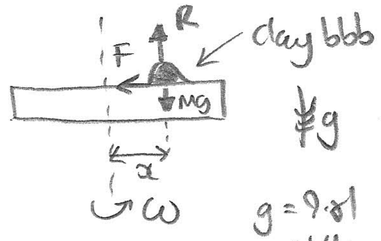
2/







Initial angular momentum of turntable is:

$$\tfrac{1}{2}Mr^2\omega_0 \qquad \left(\omega_0 = \frac{2\pi \times 160}{60}\ \text{rad}\,\text{s}^{-1}\right)$$

Final angular momentum of turntable + clay is:

$$\tfrac{1}{2}Mr^2\omega \ + \ mx^2\omega$$

$\therefore$ By conservation of angular momentum

$$\omega\left(\tfrac{1}{2}Mr^2 + mx^2\right) = \tfrac{1}{2}Mr^2\omega_0$$

So $$\boxed{\omega = \frac{\omega_0}{1 + \dfrac{mx^2}{\frac{1}{2}Mr^2}}} = \boxed{152\ \text{RPM}}$$

Now if clay is on the point of slipping, friction $F = \mu R$
Now $R = mg$ (vertical equilibrium), $\therefore$ $\boxed{F = \mu mg}$
Newton II radially inward, on clay $\Rightarrow$ $\boxed{mx\omega^2 = \mu mg}$

$$\therefore \quad \mu = x\omega^2/g$$

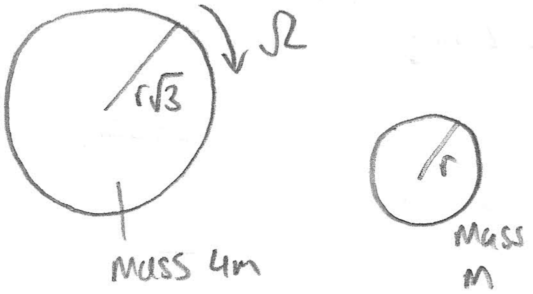
$$\therefore \quad \boxed{\mu = \frac{x}{g}\,\omega_0^2\left(1 + \frac{mx^2}{\frac{1}{2}Mr^2}\right)^{-2}}$$

$$\therefore \quad \mu = \frac{0.05}{9.81}\left(\frac{2\pi \times 160}{60}\right)^2\left(1 + \frac{2.0 \times 5^2}{\frac{1}{2}\,8.0 \times 15^2}\right)^{-2}$$

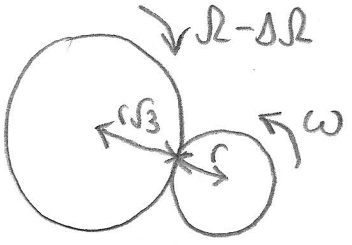
$$= \boxed{1.28}$$

[i.e. would slide on a plane at 52.1°]

3/



BEFORE



AFTER

original rotational KE: $E_0 = \frac{1}{2}\left(\frac{1}{2}4m(r\sqrt{3})^2\right)\Omega^2$

$$\boxed{E_0 = 3mr^2\Omega^2}$$

$\therefore$ $\boxed{\Delta E = 3\varepsilon m r^2 \Omega^2}$ ie energy lost

Final rotational KE: $E_1 = \frac{1}{2}\left(\frac{1}{2}4m(r\sqrt{3})^2\right)(\Omega-\Delta\Omega)^2$

$$+ \frac{1}{2}\left(\frac{1}{2}mr^2\right)\omega^2$$

$$\boxed{E_1 = 3mr^2(\Omega-\Delta\Omega)^2 + \frac{1}{4}mr^2\omega^2}$$

Now $E_1 = E_0 - \Delta E$

$\therefore$ $3mr^2(\Omega-\Delta\Omega)^2 + \frac{1}{4}mr^2\omega^2 = 3mr^2\Omega^2 - 3\varepsilon mr^2\Omega^2$

$$(\Omega-\Delta\Omega)^2 + \frac{1}{12}\omega^2 = \Omega^2 - \varepsilon\Omega^2 = \Omega^2(1-\varepsilon)$$

Now if contact between cylinders is slip free

$r\sqrt{3}(\Omega-\Delta\Omega) = r\omega$ (ie match speeds at cylinder edge)

$\therefore$ $\boxed{\Omega - \Delta\Omega = \frac{\omega}{\sqrt{3}}}$

So $\frac{\omega^2}{3} + \frac{1}{12}\omega^2 = \Omega^2(1-\varepsilon)$

$$\omega^2\left(\frac{4+1}{12}\right) = \Omega^2(1-\varepsilon)$$

$$\omega^2 = \Omega^2(1-\varepsilon)\times 12/5$$

$\therefore$ $\boxed{\omega = \sqrt{\frac{12}{5}}\,\Omega(1-\varepsilon)^{\frac{1}{2}}}$

Now $\Omega - \Delta\Omega = \frac{\omega}{\sqrt{3}} \Rightarrow \Delta\Omega = \Omega - \frac{\omega}{\sqrt{3}}$

$$\therefore \Delta\Omega = \Omega - \sqrt{\tfrac{12}{3\times5}}\,\Omega(1-\varepsilon)^{\frac{1}{2}}$$

$$\boxed{\Delta\Omega = \Omega\left(1 - \sqrt{\tfrac{4}{5}}\,\Omega(1-\varepsilon)^{\frac{1}{2}}\right)}$$

(3)

let ΔL be angular impulse provided to system:

$$\tfrac{1}{2}4m(r\sqrt{3})^2\Omega + \Delta L = \tfrac{1}{2}4m(r\sqrt{3})^2(\Omega - \Delta\Omega) - \tfrac{1}{2}mr^2\omega$$

$$\therefore\quad 6mr^2\Omega + \Delta L = 6mr^2(\Omega - \Delta\Omega) - \tfrac{1}{2}mr^2\omega$$

$$\therefore\quad 6mr^2\Omega + \Delta L = 6mr^2\frac{\omega}{\sqrt{3}} - \tfrac{1}{2}mr^2\omega$$

$$\therefore\quad \Delta L = \left(\frac{6mr^2}{\sqrt{3}} - \tfrac{1}{2}mr^2\right)\sqrt{\tfrac{12}{5}}\,\Omega(1-\varepsilon)^{\frac{1}{2}} - 6mr^2\Omega$$

$$\therefore\quad \Delta L = mr^2\Omega\left(\left(\frac{6}{\sqrt{3}} - \frac{1}{2}\right)(1-\varepsilon)^{\frac{1}{2}}\sqrt{\tfrac{12}{5}} - 6\right)$$

$$\Delta L = 6mr^2\Omega\left(\left(\frac{1}{\sqrt{3}} - \frac{1}{12}\right)(1-\varepsilon)^{\frac{1}{2}}\sqrt{\tfrac{12}{5}} - 1\right)$$

$$\frac{1}{\sqrt{3}} - \frac{1}{12} = \frac{12-\sqrt{3}}{12\sqrt{3}}$$

$$\left(\frac{1}{\sqrt{3}} - \frac{1}{12}\right)\sqrt{\tfrac{12}{5}} = \frac{12-\sqrt{3}}{\sqrt{12\times3\times5}} = \frac{12-\sqrt{3}}{\sqrt{3^2\times2^2\times5}} = \boxed{\frac{12-\sqrt{3}}{6\sqrt{5}}}$$

$$\therefore\quad \Delta L = 6mr^2\Omega\left(\underbrace{\frac{12-\sqrt{3}}{6\sqrt{5}}(1-\varepsilon)^{\frac{1}{2}} - 1}_{\text{clearly -ve since } 0<\varepsilon<1}\right)$$

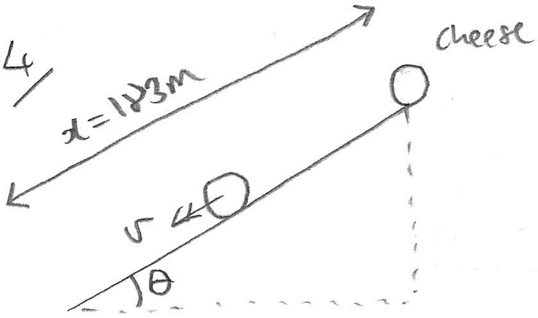
$$\Rightarrow \boxed{\Delta L = mr^2\Omega\left(6 - \frac{12-\sqrt{3}}{\sqrt{5}}(1-\varepsilon)^{\frac{1}{2}}\right)}$$

can $\Delta L = 0$? $\Rightarrow (1-\varepsilon)^{\frac{1}{2}} = \frac{6\sqrt{5}}{12-\sqrt{3}} \Rightarrow \varepsilon = 1 - \frac{36\times5}{(12-\sqrt{3})^2}$

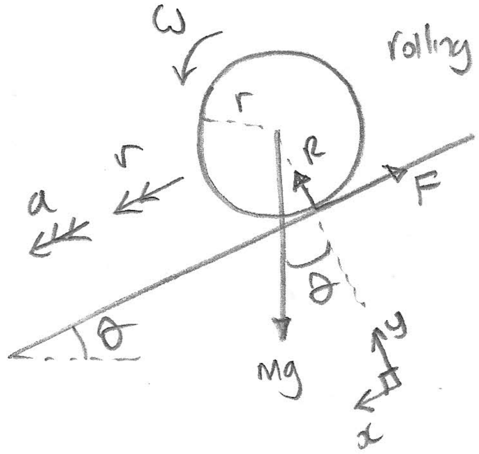
$\Rightarrow \varepsilon = -0.707$. which is unphysical.

$\varepsilon = 1$: $\boxed{\Delta L = 6mr^2\Omega}$ $\quad\varepsilon = 0$: $\boxed{\Delta L = mr^2\Omega\left(6 - \frac{12-\sqrt{3}}{\sqrt{5}}\right)}$ $\approx 1.41 \times mr^2\Omega$.

4/



$g = 9.81$ N/kg



rolling cheese of mass M

Moment of inertia I

Coopers Hill has a gradient of 1 in 2

$\therefore \tan\theta = \frac{1}{2}$

$\Rightarrow \boxed{\theta = 26.6^\circ}$

friction force

Newton II: $/\!/x: \quad Ma = Mg\sin\theta - F$ (1)

$/\!/y: \quad 0 = R - Mg\cos\theta$ (2)

Using "torque = moment of inertia $\times\ \dot\omega$"

$$Fr = I\dot\omega$$

Now if no slip with hill

$v = r\omega \quad \therefore \quad a = r\dot\omega$

$$\therefore \quad Fr = \frac{Ia}{r}$$

$$\therefore \quad F = Ia/r^2 \quad \Rightarrow \text{in (1)}: \quad Ma = Mg\sin\theta - Ia/r^2$$

$$\Rightarrow \quad a\left(M + I/r^2\right) = Mg\sin\theta$$

$$\therefore \quad \boxed{a = \frac{g\sin\theta}{1 + I/Mr^2}}$$

Since constant acceleration motion, (starting from rest), after distance x, $v^2 = 2ax$

$$\therefore \quad \boxed{v = \left(\frac{2gx\sin\theta}{1 + I/Mr^2}\right)^{\frac{1}{2}}}$$

Now if no slip: $F \leq \mu R$

From (2): $R = Mg\cos\theta$, and since $F = \frac{Ia}{r^2}$

$\therefore \frac{I}{r^2} \frac{g\sin\theta}{1 + I/Mr^2} \leq \mu Mg\cos\theta$

$$\therefore \boxed{\mu \geq \frac{I/Mr^2 \tan\theta}{1 + I/Mr^2}}$$

[Better: $\boxed{\mu \geq \frac{\tan\theta}{Mr^2/I + 1}}$]

So using $r = 0.18\,m$, $M = 4.0\,kg$, $\theta = \tan^{-1}\frac{1}{2}$, $x = 183\,m$

if cheese was cylindrical: $I = \frac{1}{2}Mr^2$

so $I/Mr^2 = \boxed{\frac{1}{2}}$ if cheese was spherical $I = \frac{2}{5}Mr^2$

$\therefore I/Mr^2 = \boxed{\frac{2}{5}}$

Note for a spherical cheese r would be different if M the same, but ratio I/Mr^2 would still be $2/5$.

$\boxed{\text{Cylindrical cheese:}}$ $v = \left(\frac{2 \times 9.81 \times 183 \times \sin(26.6°)}{1 + \frac{1}{2}}\right)^{\frac{1}{2}}$

$= \boxed{32.7\,m/s} \approx 118\,km/h$ (!)

$\mu \geq \frac{1/2}{2+1} \Rightarrow \boxed{\mu \geq \frac{1}{6}}$ (i.e. $\mu \geq 0.17$)

[Recall $\tan\theta = \frac{1}{2}$]

spherical cheese:

$$v = \left(\frac{2 \times 9.81 \times 183 \times \sin(26.6^\circ)}{1 + \frac{2}{5}} \right)^{\frac{1}{2}}$$

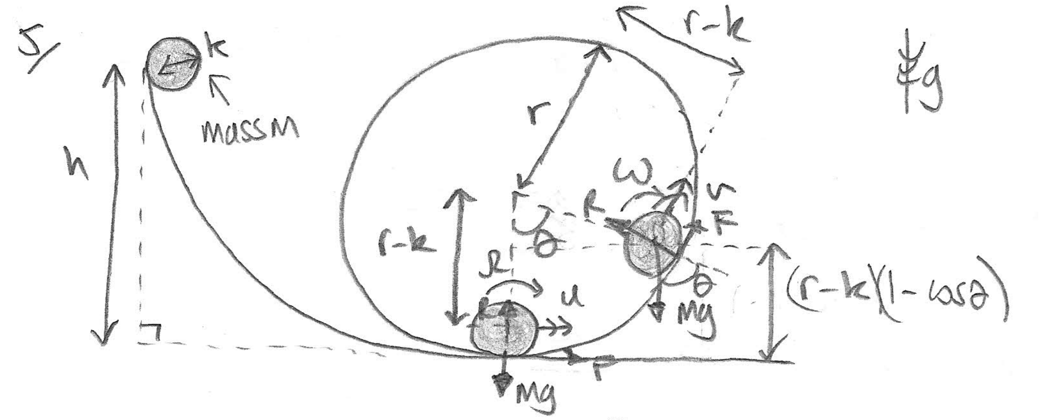
$$v = 33.9 \text{ m/s} \approx \boxed{122 \text{ km/h}}$$

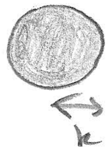
$$\mu \geq \frac{\frac{1}{2}}{\frac{5}{2} + 1}$$

$$\mu \geq \frac{1}{5 + 2}$$

$$\boxed{\mu \geq \frac{1}{7}} \quad \text{ie} \quad \boxed{\mu \geq 0.14}$$

5/





Sphere of mass M radius k

Moment of inertia

$$\boxed{I = \frac{2}{5} m k^2}$$

Conservation of energy:

$$\boxed{\begin{aligned} mgh &= \tfrac{1}{2} m u^2 + \tfrac{1}{2} I \Omega^2 \\ mgh &= \tfrac{1}{2} m v^2 + \tfrac{1}{2} I \omega^2 + mg(r-k)(1-\cos\theta) \end{aligned}}$$

← Bottom of loop the loop

Newton II radially inward: $\boxed{\dfrac{mv^2}{r-k} = R - mg\cos\theta}$

Rolling condition (no slip): $\boxed{v = k\omega}$

Also $F \le \mu R$

$\omega = \frac{v}{k}$

so from energy equation: $mgh = \frac{1}{2} m v^2 + \frac{1}{2} I \frac{v^2}{k^2} + mg(r-k)(1-\cos\theta)$

$$\Rightarrow \quad v^2 \left(\tfrac{1}{2} I/k^2 + \tfrac{1}{2} m \right) = mgh - mg(r-k)(1-\cos\theta)$$

$$\Rightarrow \quad \boxed{v = \left(\frac{2mg\left(h - (r-k)(1-\cos\theta)\right)}{I/k^2 + m} \right)^{\frac{1}{2}}}$$

Now $\boxed{R = \dfrac{mv^2}{r-k} + mg\cos\theta.}$ To maintain contact with the loop the loop track, $R \ge 0$

$$\therefore \quad \frac{m \times 2\cancel{mg}\left(h - (r-k)(1-\cos\theta)\right)}{\left(I/k^2 + m\right)(r-k)} + \cancel{mg}\cos\theta \ge 0$$

$$\frac{h - (r-k)(1-\cos\theta)}{\left(\frac{I}{2mk^2} + \frac{1}{2}\right)(r-k)} + \frac{\left(\frac{I}{2mk^2} + \frac{1}{2}\right)(r-k)\cos\theta}{\left(\frac{I}{2mk^2} + \frac{1}{2}\right)(r-k)} \ge 0$$

$$\Rightarrow \quad h - (r-k)(1-\cos\theta) + \left(\frac{I}{2mk^2} + \frac{1}{2}\right)(r-k)\cos\theta \geq 0$$

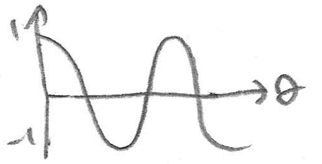
$$(r-k)\cos\theta\left(1 + \frac{1}{2} + \frac{I}{2mk^2}\right) \geq r-k-h$$

$$\Rightarrow \cos\theta \geq \frac{r-k-h}{(r-k)\left(1+\frac{1}{2}+\frac{I}{2mk^2}\right)}$$

$$\Rightarrow \cos\theta \geq \frac{1-\frac{h}{r-k}}{\frac{3}{2}+\frac{I}{2mk^2}}$$

$$\Rightarrow \cos\theta \geq \frac{2}{3}\left(\frac{1-\frac{h}{r-k}}{1+\frac{I}{3mk^2}}\right) \qquad (*)$$

Now



$\cos\theta \geq -1$

So $(*)$ is always true if $\frac{2}{3}\left(\frac{1-\frac{h}{r-k}}{1+\frac{I}{3mk^2}}\right) < -1$

$$\Rightarrow \quad 1 - \frac{h}{r-k} < -\frac{3}{2}\left(1 + \frac{I}{3mk^2}\right)$$

$$1 + \frac{3}{2} + \frac{I}{2mk^2} < \frac{h}{r-k}$$

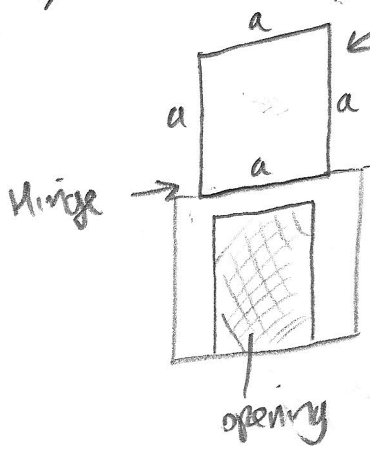
$$\Rightarrow \quad \boxed{h > (r-k)\left(\frac{5}{2} + \frac{I}{2mk^2}\right)}$$

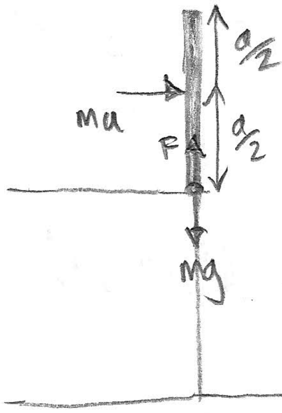
Now $I = \frac{2}{5}mk^2$ so $\frac{I}{2mk^2} = \frac{1}{5}$. $\frac{5}{2} + \frac{1}{5} = \frac{27}{10}$

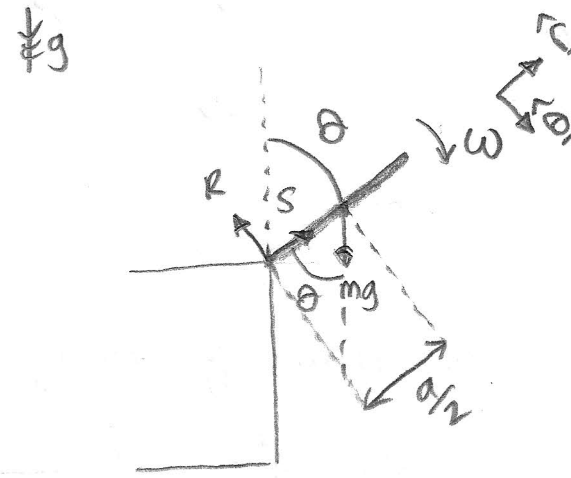
$$\therefore \quad \boxed{h > \frac{27}{10}(r-k)}$$

6/

Metal door of mass M

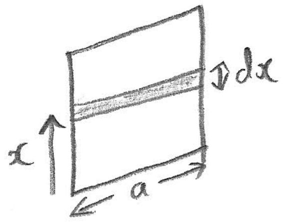






Magnitude of hinge force $\boxed{F = \sqrt{R^2+S^2}}$

Moment of inertia of door about axis along bottom edge

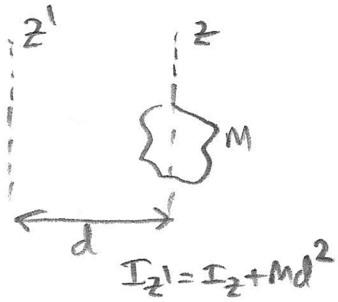


$$I = \int \delta m\, x^2 = \int_0^a \frac{M}{a^2}\, a\, dx\, x^2$$

$$= \frac{M}{a}\left[\frac{1}{3}x^3\right]_0^a$$

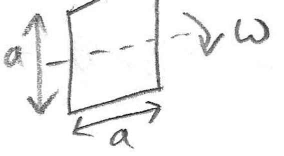
$$= \boxed{\frac{1}{3}Ma^2}$$

mass / unit area is M/a^2



$I_{z'} = I_z + Md^2$

[Note



$I = \frac{1}{12}Ma^2$

so by // axis theorem

our I is $\frac{1}{12}Ma^2 + M\left(\frac{a}{2}\right)^2 = \frac{1}{12}Ma^2 + \frac{Ma^2}{4}$

$$= \frac{Ma^2 + 3Ma^2}{12} = \frac{1}{3}Ma^2 \,]$$

If impulse Mu is given, then centre of mass of door moves initially with velocity u horizontally

$\therefore$ when $\theta = 0$, $\frac{a}{2}\omega_0 = u \Rightarrow \boxed{\omega_0 = \frac{2u}{a}}$

Using torque $= I\dot{\omega}$ (about hinge)

$$\Rightarrow \quad mg\sin\theta \times \frac{a}{2} = \frac{1}{3}ma^2\dot{\omega}$$

$$\Rightarrow \quad \boxed{\dot{\omega} = \frac{3}{2}\frac{g\sin\theta}{a}}$$

conservation of energy: $\frac{1}{2}\left(\frac{1}{3}ma^2\right)\omega_0^2 = \frac{1}{2}\left(\frac{1}{3}ma^2\right)\omega^2 - mg\underbrace{\frac{a}{2}(1-\cos\theta)}_{\text{Drop of C.O.M.}}$

$\left[\omega_0^2 = \frac{4u^2}{a^2}\right]$

$$\Rightarrow \quad \frac{4}{6}ma^2\frac{u^2}{a^2} = \frac{1}{6}ma^2\omega^2 - \frac{mga}{2}(1-\cos\theta)$$

$$4u^2 = a^2\omega^2 - 3ga(1-\cos\theta)$$

$$\boxed{\omega^2 = \frac{4u^2 + 3ga(1-\cos\theta)}{a^2}}$$

Newton II: $\parallel \hat{r}$: $-m\frac{a}{2}\omega^2 = S - mg\cos\theta$

$\parallel \hat{\theta}$: $m\frac{a}{2}\dot{\omega} = mg\sin\theta - R$

So

$$S = mg\cos\theta - \frac{ma}{2}\omega^2$$

$$S = mg\cos\theta - \frac{ma}{2}\left(\frac{4u^2 + 3ga(1-\cos\theta)}{a^2}\right)$$

$$S = mg\cos\theta\left(1 + \frac{3}{2}\right) - \frac{ma}{2a^2}(4u^2 + 3ga)$$

$$\boxed{S = \frac{5}{2}mg\cos\theta - \frac{1}{2}mg\left(\frac{4u^2}{ag} + 3\right)}$$

$$R = mg\sin\theta - \frac{ma}{2}\dot{\omega} = mg\sin\theta - \frac{ma}{2}\left(\frac{3}{2}\frac{g\sin\theta}{a}\right)$$

$\therefore \quad R = mg\sin\theta\left(1 - \tfrac{3}{4}\right)$

$\therefore \quad \boxed{R = \tfrac{1}{4} mg\sin\theta}$

So $\quad F = \sqrt{S^2 + R^2}$

$$\therefore \quad \boxed{\frac{F}{mg} = \sqrt{\left[\frac{5}{2}\cos\theta - \frac{1}{2}\left(\frac{4u^2}{ga} + 3\right)\right]^2 + \left[\frac{1}{4}\sin\theta\right]^2}}$$

check it makes sense! $\quad \theta = 0, \ \cos\theta = 1, \ \sin\theta = 0$

$$\Rightarrow \frac{F}{mg} = \frac{5}{2} - \frac{1}{2}\left(\frac{4u^2}{ga} + 3\right)$$

$$= \boxed{1 - \frac{2u^2}{ga}}$$

Newton II radially inwards when $\theta = 0$:

$$\frac{mu^2}{a/2} = mg - F$$

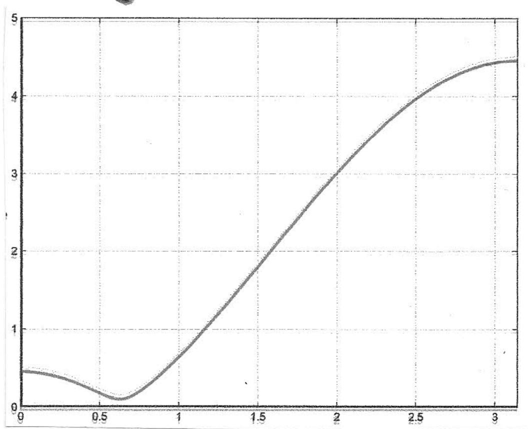
$$\therefore \quad F = mg - \frac{2mu^2}{a}$$

$$\therefore \quad \frac{F}{mg} = 1 - 2u^2/ga \ \checkmark$$

let $\quad \dfrac{4u^2}{ga} = 1 \quad \Rightarrow \quad \boxed{u = \tfrac{1}{2}\sqrt{ga}}$

so $\quad \boxed{\dfrac{F}{mg} = \sqrt{\left[\frac{5}{2}\cos\theta - 2\right]^2 + \dfrac{\sin^2\theta}{16}}}$

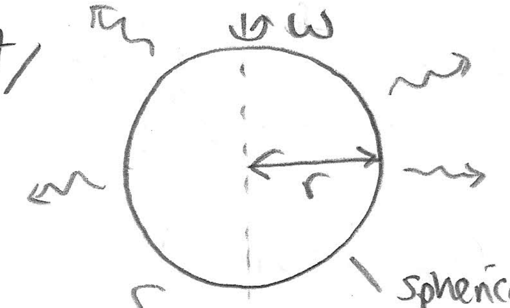
$\uparrow F/mg$



$\longrightarrow \theta/\text{rad}$

* Blank! *

7/



Assume star is always density ρ

$$\therefore \quad \rho = \frac{M}{\frac{4\pi}{3}R^3}$$

$$m = \tfrac{4}{3}\pi r^3 \rho$$

$$\therefore \quad m = \tfrac{4}{3}\pi r^3 \frac{M}{\frac{4\pi}{3}R^3}$$

$$\therefore \quad \boxed{m = M\left(\frac{r}{R}\right)^3}$$

$m(0) = M$
$\omega(0) = \Omega$
$r(0) = R$

Assume radiation process (which results in mass loss) conserves angular momentum

$$\therefore \quad \tfrac{2}{5} m r^2 \omega = \tfrac{2}{5} M R^2 \Omega$$

$$\Rightarrow \quad \boxed{m r^2 \omega = M R^2 \Omega}$$

Now rate of loss of mass $\propto$ surface area

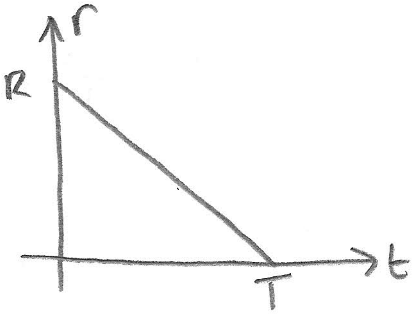
$$\therefore \quad \frac{dm}{dt} = -k \times 4\pi r^2$$

$$\therefore \quad \frac{d}{dt}\left(\frac{M r^3}{R^3}\right) = -k \times 4\pi r^2$$

$$\frac{3M}{R^3} r^2 \frac{dr}{dt} = -k \times 4\pi r^2$$

$$\frac{dr}{dt} = -4\pi k R^3 / 3M$$

$$\therefore \quad r = R - \frac{4\pi k R^3}{3M} t$$



Define T to be lifetime of star

$$\Rightarrow \quad \boxed{r(t) = R\left(1 - t/T\right)}$$

Note $k = \dfrac{\left|\frac{dM}{dt}\right|_{t=0}}{4\pi R^2}$ { could estimate this experimentally? }

and $\dfrac{4\pi k R^3 T}{3M} = R$ (if $r=0$)

$\Rightarrow$ $\boxed{T = \dfrac{3M}{4\pi R^2 k}}$ Hence find.

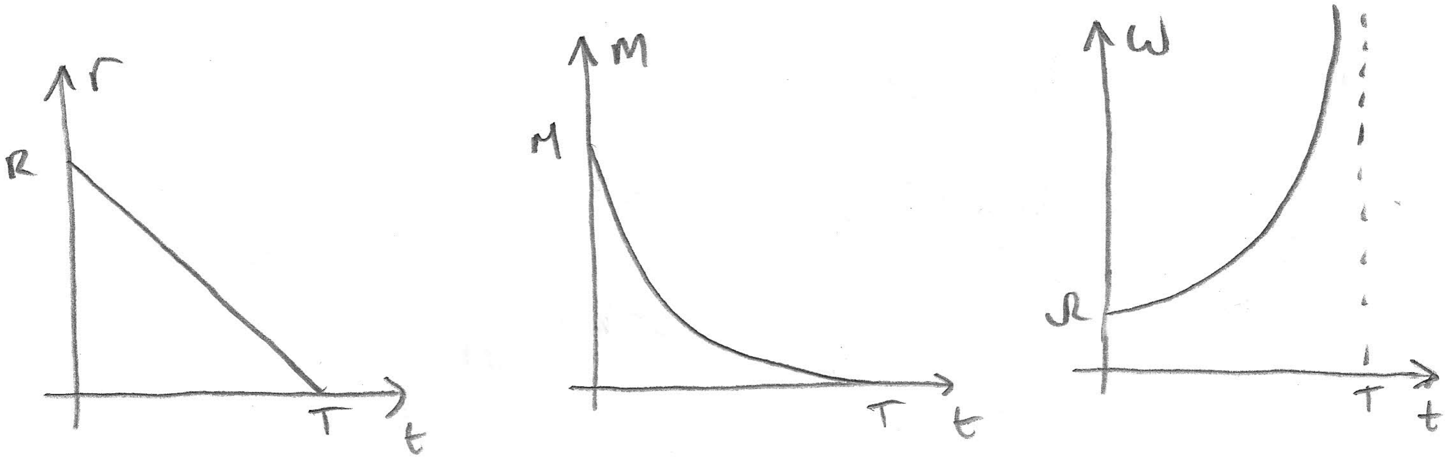
So if $r(t) = R(1 - t/T)$

$M(t) = M\left(\frac{r}{R}\right)^3 \Rightarrow \boxed{M(t) = M\left(1 - t/T\right)^3}$

Now $m r^2 \omega = M R^2 \Omega$

$\Rightarrow$ $\omega = \dfrac{M R^2 \Omega}{M\left(1-t/T\right)^3 R^2 \left(1 - t/T\right)^2}$

$\therefore$ $\boxed{\omega = \Omega\left(1 - t/T\right)^{-5}}$



ie predict rotation rate to speed up significantly as star loses mass. Perhaps explains high rotation rates of pulsars etc?