$\mathbf{L} = \sum_i \mathbf{r}_i \times m_i \mathbf{v}_i$ Angular momentum of a system of particles with masses m_i, velocities $\mathbf{v}_i$, position vectors $\mathbf{r}_i$

$\boldsymbol{\tau} = \mathbf{r} \times \mathbf{f}$ Torque or turning moment due to a force $\mathbf{f}$ acting at displacement $\mathbf{r}$ (magnitude is "force $\times$ perpendicular distance")

$I = \sum_i m_i r_i^2$ Moment of inertia of a system of masses m_i, perpendicular distances r_i from a specified rotation axis

$L = I\omega$ Magnitude of angular momentum. Note in the absence of torque, angular momentum is conserved.

$\tau = I\dot{\omega} = \frac{dL}{dt}$ Torque is moment of inertia times angular acceleration $\dot{\omega} = \ddot{\theta}$, or the rate of change of angular momentum

Change of angular momentum (angular impulse) is $\Delta L = \int \tau dt$ $E = \frac{1}{2} I\omega^2$ Rotational kinetic energy

$I = mr^2$ Moment of inertia of a single particle, or a hoop of mass m and radius r rotating about hoop centre.

Cylinder $I = \frac{1}{2}mr^2$ Sphere $I = \frac{2}{5}mr^2$ Solid cone $I = \frac{3}{10}mr^2$ Parabolic cap $I = \frac{1}{3}mr^2$

$I_{z'} = I_z + md^2$ Parallel axis theorem (rotation axes separated by d)

$I_z = I_x + I_y$ Perpendicular axis theorem (for 2D laminae only. z rotation axis is out of the plane).

Rectangular plate of mass m of dimensions a, b : $I_x = \frac{1}{12}ma^2$, $I_y = \frac{1}{12}mb^2$, $I_z = \frac{1}{12}m\left(a^2 + b^2\right)$

Question1

(i) A vinyl record has a diameter of 12 inches and is 0.65mm thick. Vinyl has a density of 0.925g/cm^3. Calculate the moment of inertia (assuming it is a uniform disc), the angular momentum, and the rotational kinetic energy assuming that the disc rotates at 78RPM. 1inch = 2.54cm.

(ii) Assuming the Earth to be a uniform sphere of radius 6371km and mass $5.97 \times 10^{24}\,\text{kg}$, spinning once every 24 hours, calculate the (spin) angular momentum and rotational kinetic energy.

(iii) Compare the result in (ii) to (a) the orbital angular momentum of the Earth, Sun system, (b) the orbital angular momentum of the Moon about the Earth.

$1\text{AU} = 1.496 \times 10^{11}\,\text{m}$, Earth-moon separation = 394,402km, Lunar period $\approx$ 28 days, Moon mass = 7.35×10^{22} kg.

(iv) In the Bohr theory of the electron in the Hydrogen atom, the orbital angular momentum of the electron is quantized into integer n multiples of $\hbar = \frac{1}{2\pi} \times 6.63 \times 10^{-34}\,\text{kgm}^2\text{s}^{-1}$. Use this to calculate the ratio of the speed of light to the orbital speed of the electron. $c = 2.998 \times 10^8\,\text{ms}^{-1}$, $m_e = 9.109 \times 10^{-31}\,\text{kg}$, $r = 5.292 \times 10^{-11}\,\text{m}$, $n = 1$.

(v) Dinorwig ('Electric Mountain') is a pumped-storage hydroelectric power station in Llanberis, Wales. It has an electrical energy storage capacity of 33TJ. Assume all this energy is stored in solid iron cylinders of radius 5.0m and height 10.0m, which rotate at 10Hz. How many cylinders are needed? The density of iron is $7850\ \text{kgm}^{-3}$. Comment on the difficulty of storing energy via mechanical means.

(vi) During a festive radio show, a DJ accidentally drops a mince pie onto a 78RPM spinning vinyl disc, identical to the one in (i). The pie falls on a spot 2.1 inches from the centre of rotation and sticks there. If the disc was spinning freely without any torque just before the mince pie struck, determine the mass of the mince pie if the rotation rate drops to 60RPM.

(vii) The SpaceX *Dragon* orbiter has a mass of about 3,000kg and a radius of about 2.0m. Modeling this as a solid parabolic cap, calculate the energy expended by side thrusters in order to cancel out a roll of 30RPM.

(viii) If the roll reduction in (vii) is performed by a diametrically opposing pair of axial thrusters, calculate the thrust from each if the roll reduction takes 5.0s. Also calculate the total angle (in degrees) rotated by the orbiter in this time.

(ix) I can spin a sphere at 5Hz. If I flip a coin instead, and it has the same mass, radius and rotational kinetic energy, at what frequency does it turn?

(x) In a 'clutch' system, two spinning cylinders are pushed together, whereupon they rotate with a common angular speed Ω. Consider two cylinders of masses m and $2m$, which are rotating at angular speeds 3ω and ω respectively, in the same initial direction. Both cylinders have the same radius r. Determine Ω, and also the energy dissipated by the system ΔE in terms of m, r, ω. Show that the fraction of energy lost is $\frac{8}{33}$.

Question 2

A potter has a turntable of diameter 30cm, with a mass of 8.0kg. It is spun up to an angular speed of 160RPM and then left to freely rotate without torque. A blob of sticky clay of mass 2.0kg is dropped 5cm from the centre of the wheel, and it is on the point of slipping. Calculate the coefficient of friction between the clay and the wheel.

Question 3

A cylinder of radius $r\sqrt{3}$ and mass $4m$ is rotating freely at angular speed Ω. An initially stationary second cylinder of radius r and mass m is brought into contact with the larger cylinder until they touch. As a result of the collision, the total loss of energy is fraction ε of the original rotational kinetic energy of the larger cylinder. If both cylinders are allowed to freely rotate and maintain slip-free contact, calculate an expression for (a) the angular speed ω of the smaller cylinder, (b) the speed loss $\Delta\Omega$ of the larger cylinder, (c) the net angular impulse that must be provided to the system.

Question 4

Coopers Hill, with a 1 in 2 gradient, is the site of the annual Gloucestershire Cheese Rolling event. A 4kg Double Gloucester cheese (in a rigid cylindrical casing), of radius 18cm is allowed to roll from the top of the hill from rest. The hill is 200 yards (183m). If the cheese rolls to the bottom of the hill without slipping or hitting a bump, determine (a) the speed of the cheese in km/h and (b) the minimum coefficient of friction between the cheese and the hillside. How would the results differ if the cheese was the same mass, but spherical?

Question 5

A sphere of radius k is rolled from rest from the top of a ramp of height h. When the sphere reaches the bottom of the ramp it rises up a 'loop the loop' track consisting of a vertical circle of radius r. Assuming that the sphere does not slip during the rolling, show that in order to complete a 'loop the loop' without losing contact with the track, $h > \frac{27}{10}(r-k)$.

Question 6 ** HINT: Resolve hinge force $F = |\mathbf{F}|$ into radial and tangential components: $\mathbf{F} = S\hat{\mathbf{r}} - R\hat{\boldsymbol{\theta}}$. **

A rectangular square metal door of side a and mass m is initially fixed in an upright position above an opening. It is hinged horizontally along the lower edge of the square. The door is allowed to fall shut after being given a small horizontal impulse of magnitude mu. Determine an expression for the magnitude of the hinge force F as a multiple of the weight of the door. If $u = \frac{1}{2}\sqrt{ga}$, show that: $F/mg = \sqrt{\left(\frac{5}{2}\cos\theta - 2\right)^2 + \frac{1}{16}\sin^2\theta}$. Use a computer to plot this vs θ.

Question 7

A uniform spherical star of initial mass M and radius R is initially spinning at angular speed Ω. Nuclear fusion processes cause the star to radiate, which effectively result in a loss of mass. The rate of loss of mass is always proportional to the surface area of the star, and the star always remains the same density. If the radiated mass can be assumed to convey zero angular momentum, show that if the remaining lifetime of the star is T, and t is time since the star was radius R, $r = R\left(1-\frac{t}{T}\right)^a$, $m = M\left(1-\frac{t}{T}\right)^b$, $\omega = \Omega\left(1-\frac{t}{T}\right)^c$, where powers a, b, c are integers to be found. Sketch r, m, ω vs t.