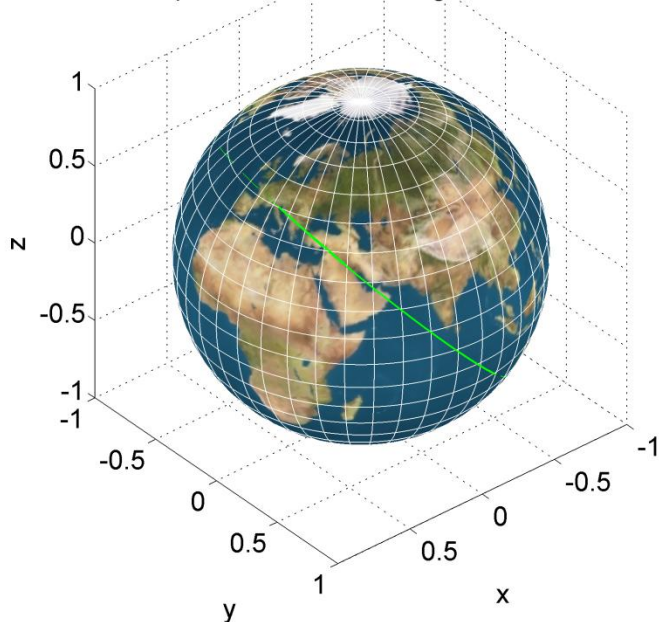


# Navigating the sphere

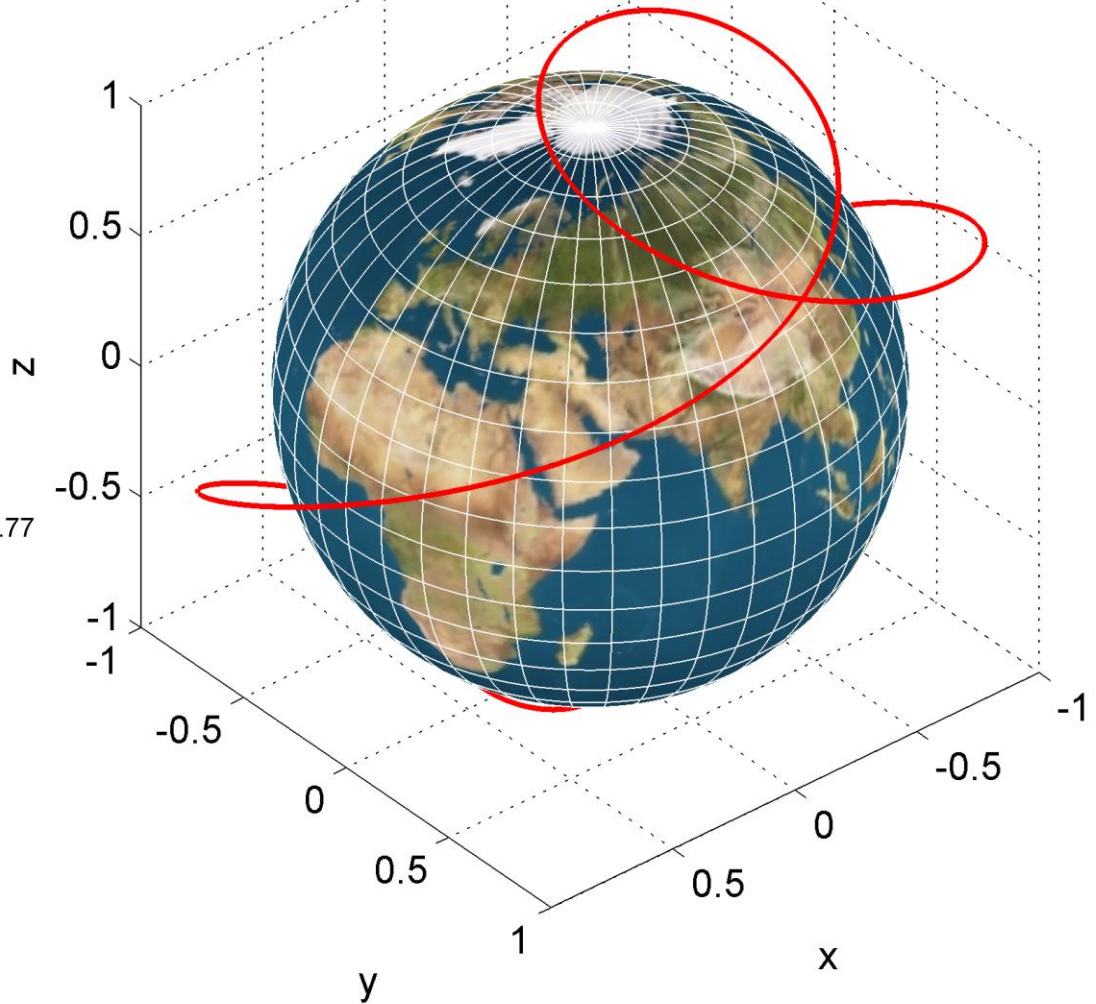
Andy French

February 2017

latA = 45°, longA = 12°, latB = -19°, longB = 99°, arc length = 1.77  
Equivalent Earth arc length = 11259km



$$\begin{aligned}r(t) &= 1.3 + 0.1 \cdot \sin(t) \\ \text{lat}(t) &= t + 0.1 \cdot \cos(3 \cdot t) \\ \text{long}(t) &= 2 \cdot t\end{aligned}$$



## Motivation

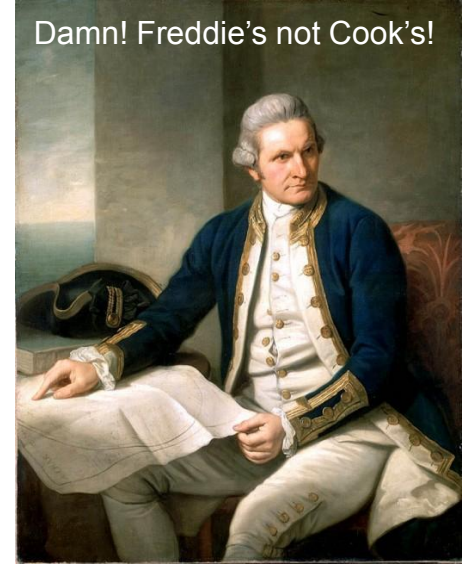
CSM is working on aspects of **maritime navigation** with his Div, and we got chatting on the way to Freddie's ....

Since to a very good approximation the Earth is spherical\* it occurred to me there are some classic mathematical problems to solve, and all involve:

- Visualizing trajectories defined in **spherical polar coordinates**
- Recipes for defining **circles** drawn on the **surface of a sphere**
- Transformations between **Earth centred Cartesian**  $x, y, z$  coordinates and a Cartesian '**East, North, Up**' system where  $x$  and  $y$  are in the *tangent plane* to a given latitude  $\lambda$  and longitude  $\phi$ .

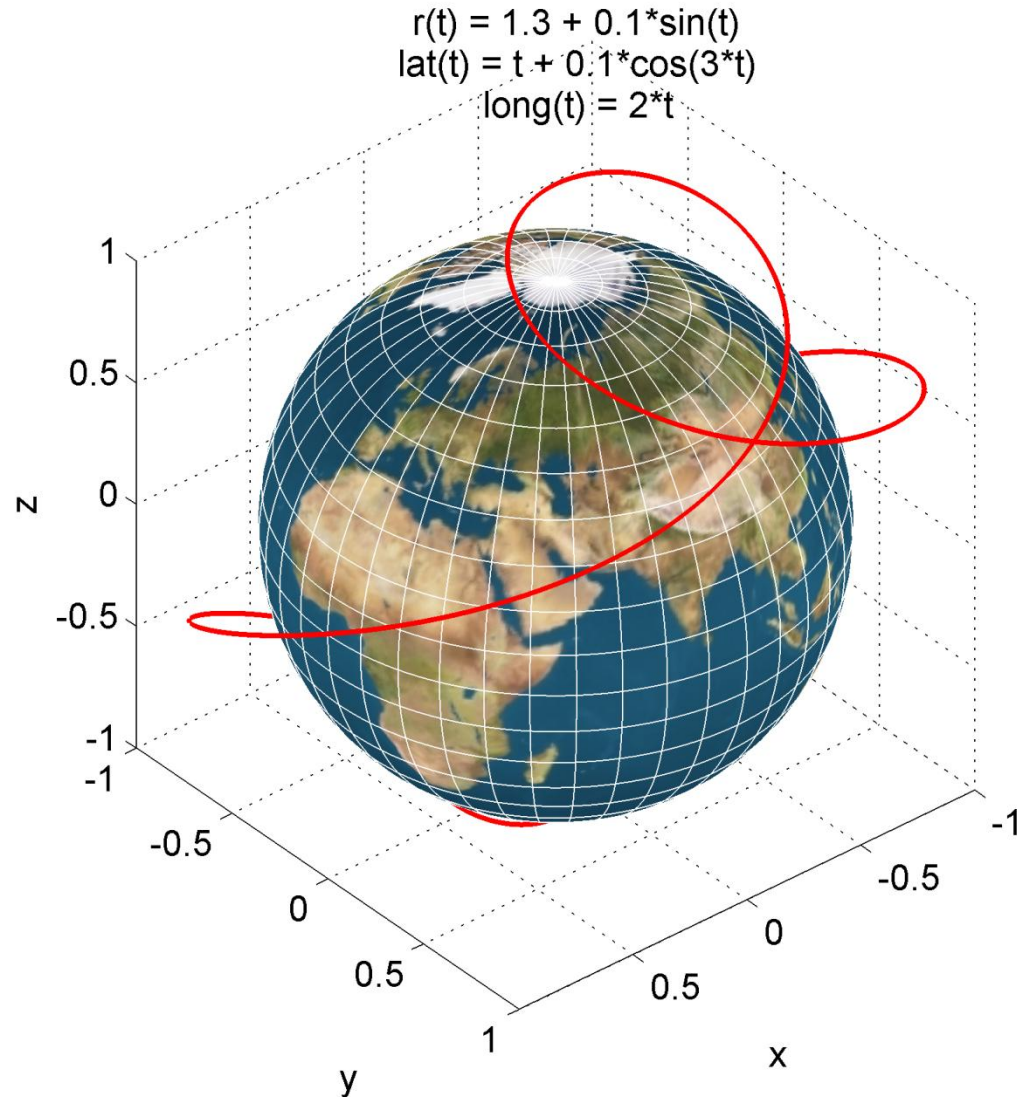
Due to the long history of Navigation there is *much estoterica* inherent in the textbook solutions to these problems. Can we do it systematically, using methods that a Pre-U student could readily understand?

\* The *polar radius* differs from the *equatorial radius* by a factor of  $\approx 1 - \frac{1}{228.257}$



## Three problems

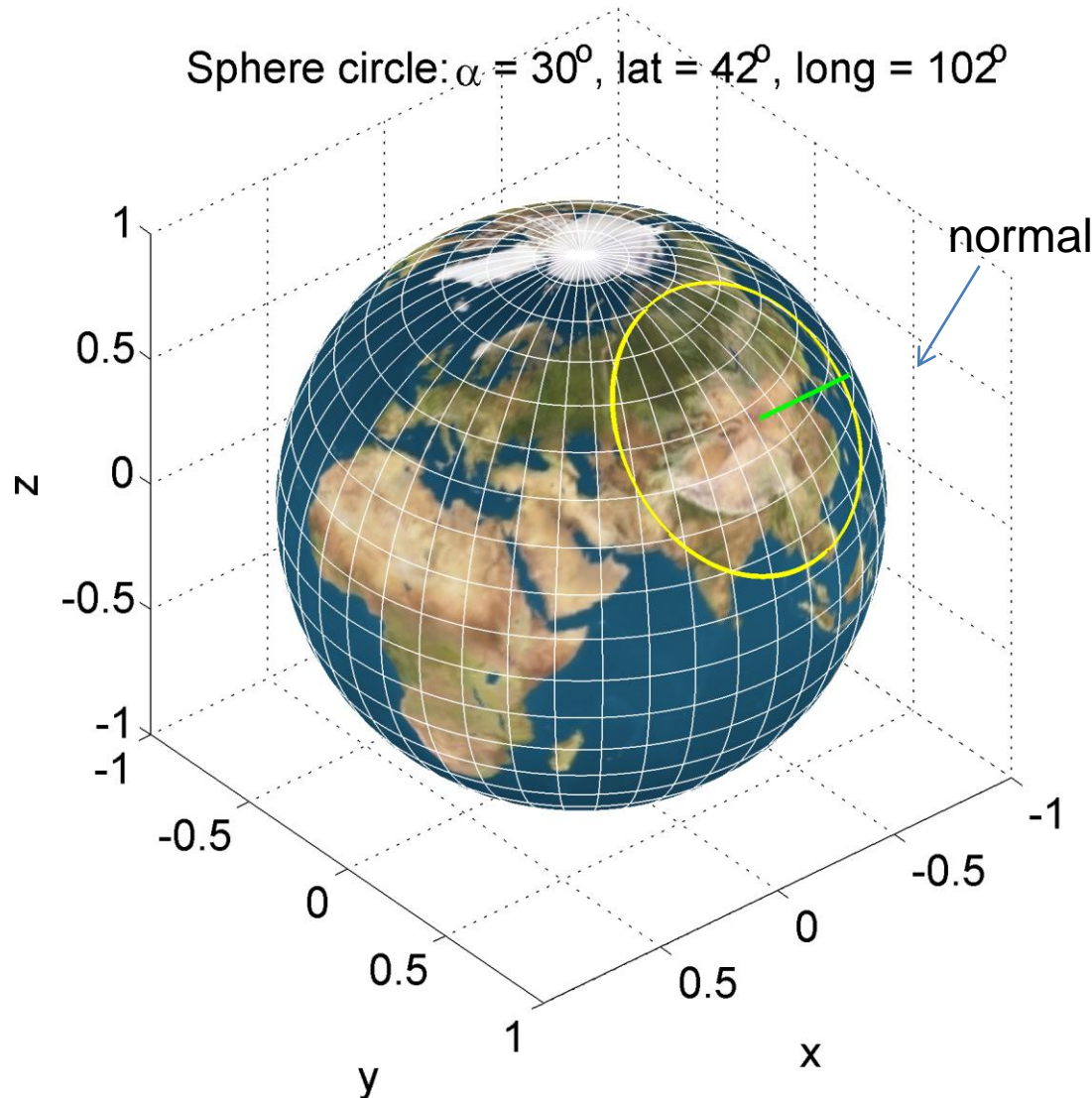
### 1. Visualize a curve defined **parametrically** in **spherical polar coordinates**



## Three problems ....

### 2. Plot a **circle** drawn on the **surface** of a sphere

This is the general problem with *special cases* being **Great Circles** and **lines of constant latitude**

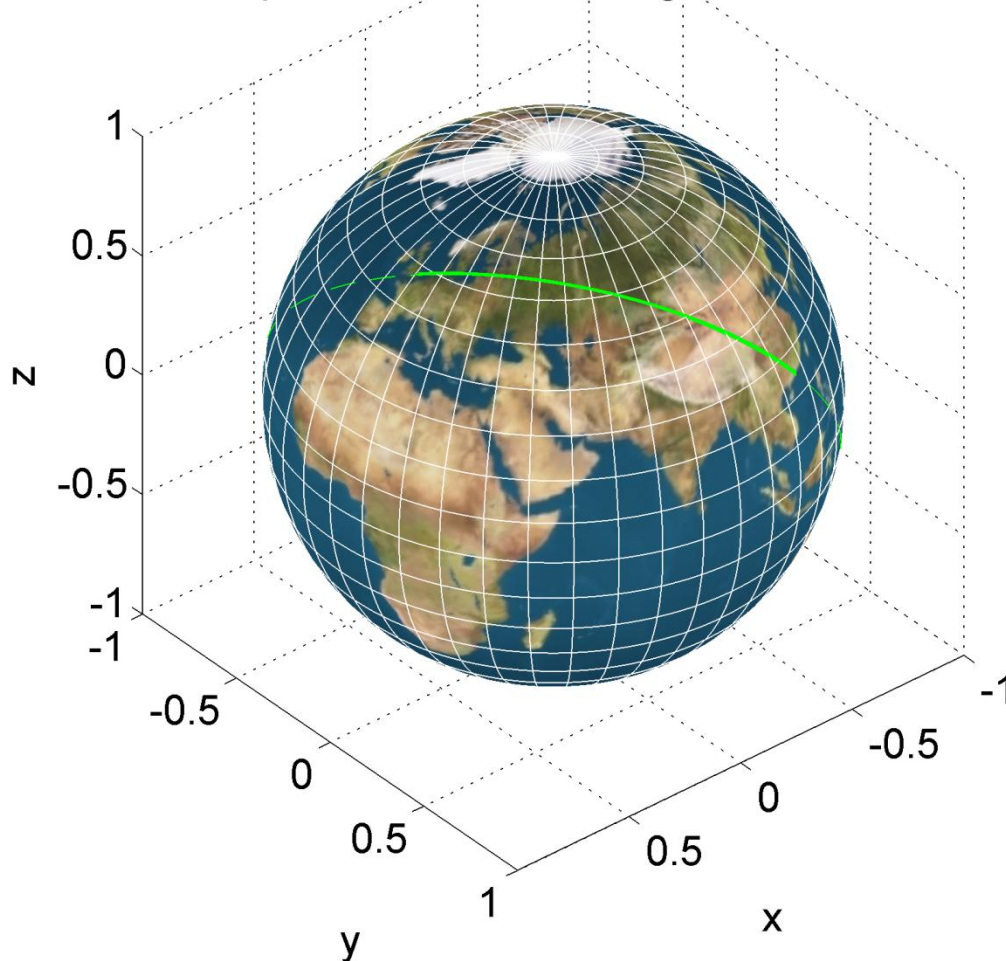


Define the circle via the **latitude** and **longitude** of a **normal** to the circle and the **angle**  $\alpha$  between the circumference, sphere centre and normal.

## Three problems ....

3. Define the **Great Circle arc**, which is the **shortest distance between two points on the surface of a sphere**. Very useful for aircraft or ship routings!

latA =  $52^\circ$ , longA =  $1^\circ$ , latB =  $22^\circ$ , longB =  $114^\circ$ , arc length = 1.5  
Equivalent Earth arc length = 9547.5km

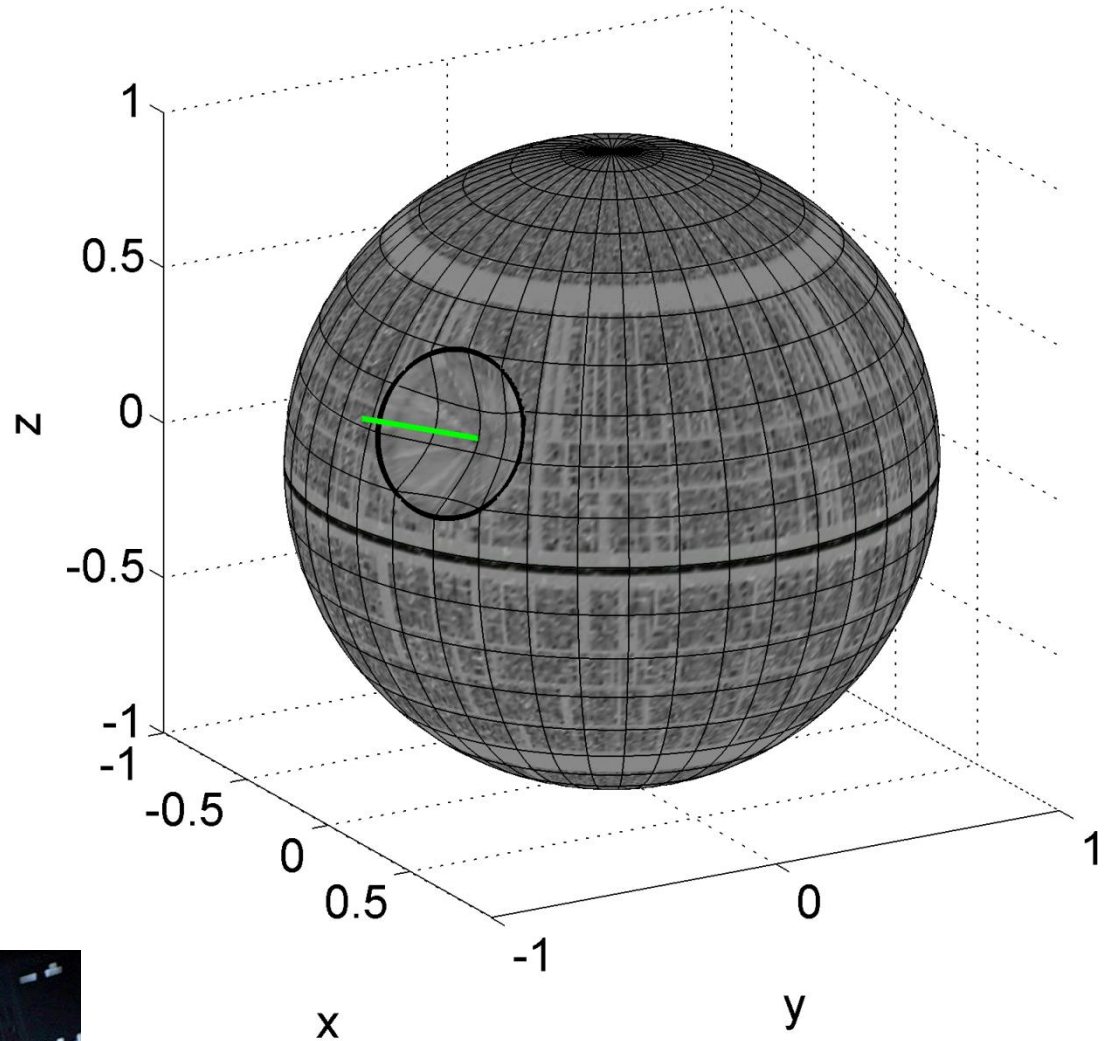


Example:  
London to Hong Kong

Accurate distance  
from Heathrow to Hong  
Kong Airport is 9,606km  
[Google Maps](#)

Plus an additional  
frivolous problem ....

Death Star:  $\alpha = 15^\circ$ , lat =  $21.2^\circ$ , long =  $-63^\circ$ , k = 0.2

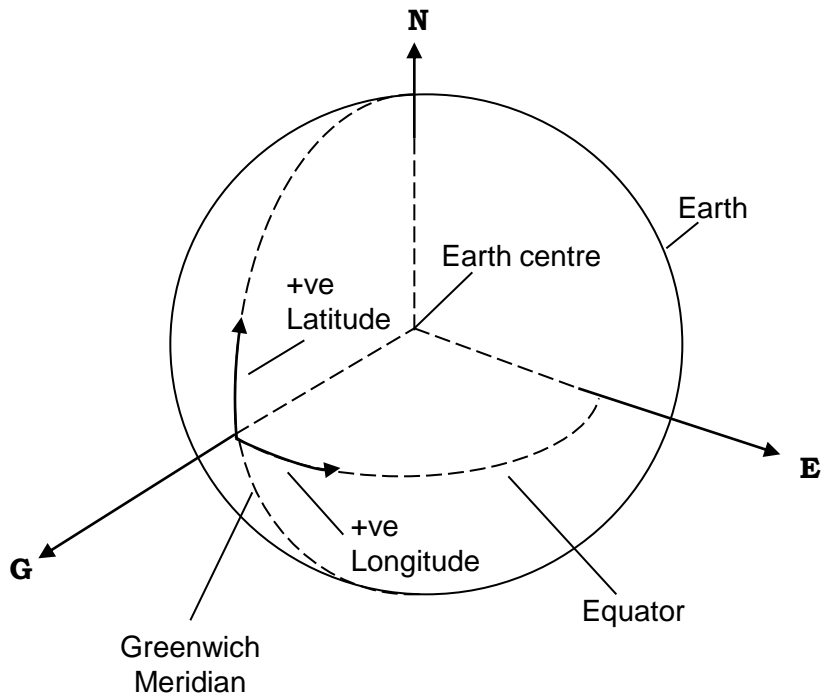


Is the Force strong  
enough to give me a  
parabolic indent in  
my Death Star?

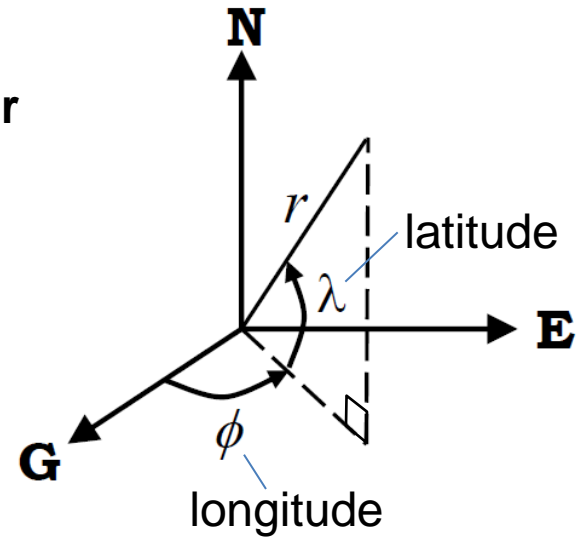


Don't forget to distort  
the lines of latitude and  
longitude too...





## Spherical polar coordinates



$$|\mathbf{G}| = |\mathbf{E}| = |\mathbf{N}| = 1$$

$$\mathbf{G} \cdot \mathbf{E} = \mathbf{G} \cdot \mathbf{N} = \mathbf{E} \cdot \mathbf{N} = 0$$

$$\mathbf{G} \times \mathbf{E} = \mathbf{N}$$

$$\mathbf{E} \times \mathbf{N} = \mathbf{G}$$

$$\mathbf{N} \times \mathbf{G} = \mathbf{E}$$



Geocentric “Greenwich, East, North” unit vectors – a right handed set.

Position vector

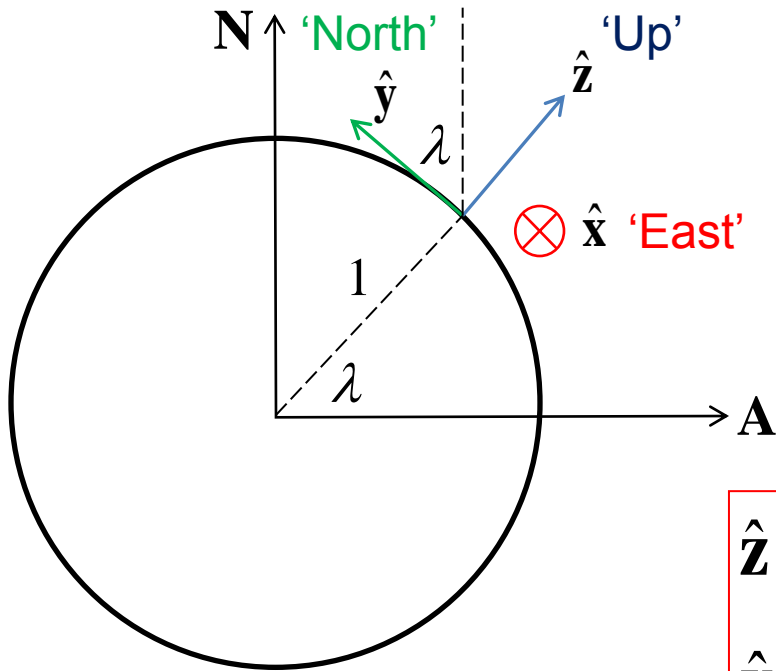
$$\mathbf{r} = X\mathbf{G} + Y\mathbf{E} + Z\mathbf{N}$$

$$X = r \cos \lambda \cos \phi$$

$$Y = r \cos \lambda \sin \phi$$

$$Z = r \sin \lambda$$

For simplicity, we will use a unit sphere of radius  $r = 1$



Also define a Cartesian coordinate system based upon a **local tangent plane** to a point on a unit sphere characterized by  $(\lambda, \phi)$

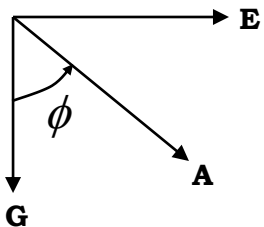
$$\hat{\mathbf{z}} = (\mathbf{G} \cos \phi + \mathbf{E} \sin \phi) \cos \lambda + \mathbf{N} \sin \lambda$$

$$\hat{\mathbf{y}} = -(\mathbf{G} \cos \phi + \mathbf{E} \sin \phi) \sin \lambda + \mathbf{N} \cos \lambda$$

$$\hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{z}}$$

$$\hat{\mathbf{x}} = \mathbf{E} \cos \phi - \mathbf{G} \sin \phi$$

$$\mathbf{A} = \mathbf{G} \cos \phi + \mathbf{E} \sin \phi$$



$$\mathbf{r} = X\mathbf{G} + Y\mathbf{E} + Z\mathbf{N} \quad \text{Position vector}$$

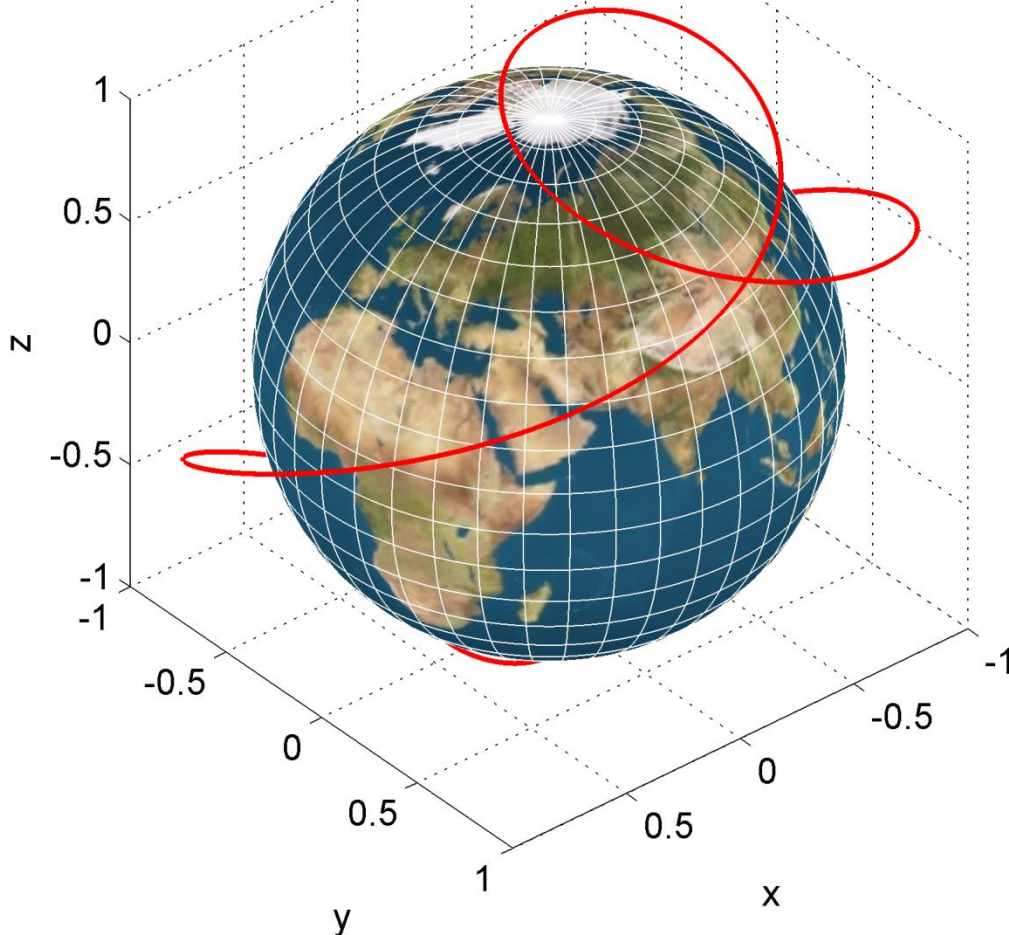
i.e. based upon a fixed  $(\lambda, \phi)$  of the  $x, y, z$  system origin

$$\mathbf{r} = \cos \lambda \cos \phi \mathbf{G} + \cos \lambda \sin \phi \mathbf{E} + \sin \lambda \mathbf{N} + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$



## Parametric trajectory plot MATLAB demo `spolartraj.m`

$$\begin{aligned}r(t) &= 1.3 + 0.1 \cdot \sin(t) \\ \text{lat}(t) &= t + 0.1 \cdot \cos(3 \cdot t) \\ \text{long}(t) &= 2 \cdot t\end{aligned}$$



Position vector

$$\mathbf{r} = X\mathbf{G} + Y\mathbf{E} + Z\mathbf{N}$$

$$X = r \cos \lambda \cos \phi$$

$$Y = r \cos \lambda \sin \phi$$

$$Z = r \sin \lambda$$

For example:

$$0 \leq t \leq 4\pi$$

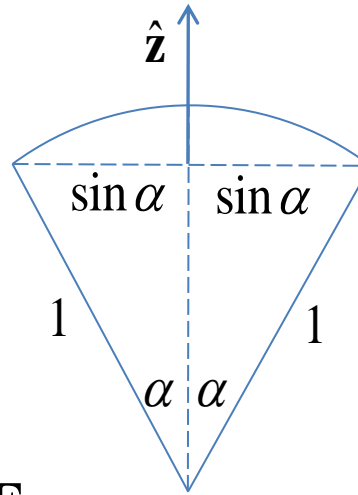
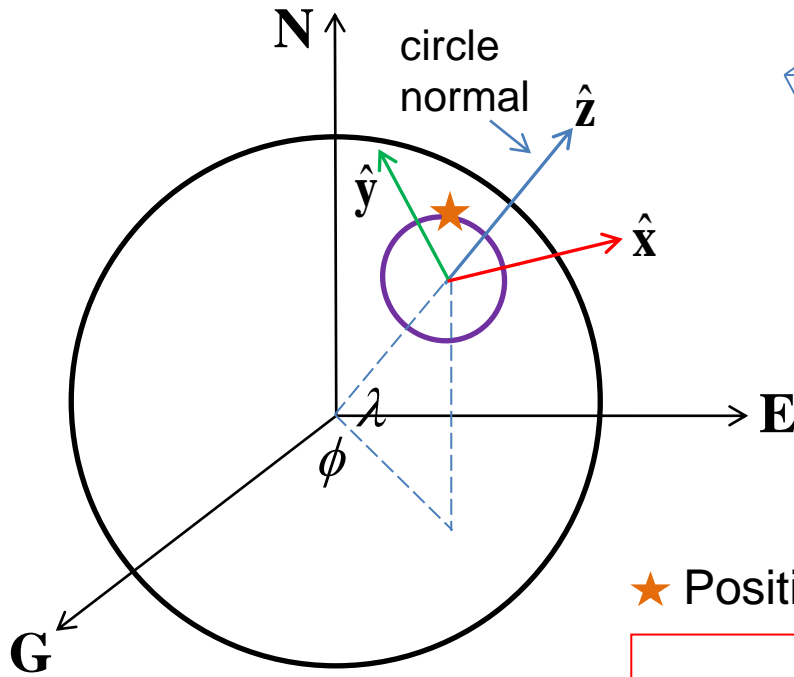
$$r = 1.3 + 0.1 \sin t$$

$$\lambda = t + 0.1 \cos 3t$$

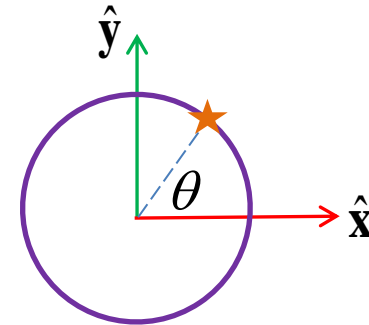
$$\phi = 2t$$

MATLAB: `plot3(x,y,z)`

# Circle on a unit sphere



The radius of the circle is  $\sin \alpha$



★ Position vector (from origin of sphere\*)

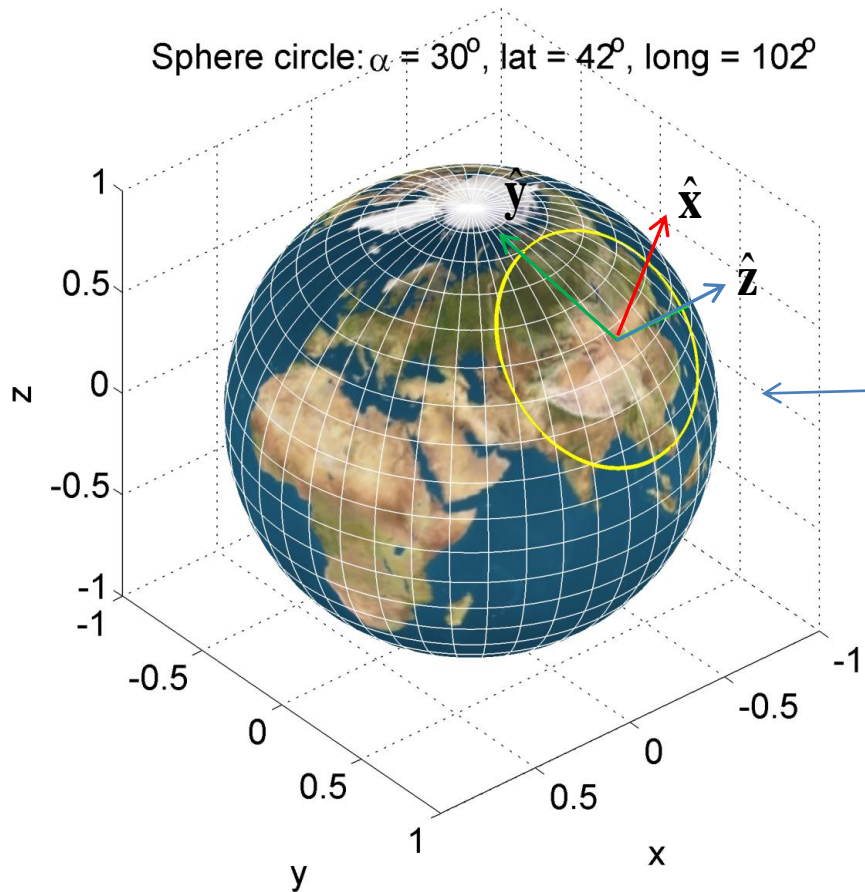
$$\mathbf{r} = \sin \alpha (\hat{\mathbf{x}} \cos \theta + \hat{\mathbf{y}} \sin \theta) + \hat{\mathbf{z}} \cos \alpha$$

$$\begin{aligned} \hat{\mathbf{z}} &= (\mathbf{G} \cos \phi + \mathbf{E} \sin \phi) \cos \lambda + \mathbf{N} \sin \lambda \\ \hat{\mathbf{y}} &= -(\mathbf{G} \cos \phi + \mathbf{E} \sin \phi) \sin \lambda + \mathbf{N} \cos \lambda \\ \hat{\mathbf{x}} &= \mathbf{E} \cos \phi - \mathbf{G} \sin \phi \end{aligned}$$

To plot, define  $\theta$  in (for example) 300 linearly spaced steps between 0 and  $2\pi$ . Then work out X,Y,Z coordinates.

\*i.e. *not* surface in this particular case. This is OK since  $\hat{\mathbf{z}}$  is a radial vector.

## MATLAB demo spherecircle.m



*Special cases:*

### Lines of Longitude

$$\alpha = \frac{1}{2} \pi$$

$$\lambda = 0$$

### Lines of Latitude

$$\lambda = \pm \frac{1}{2} \pi$$

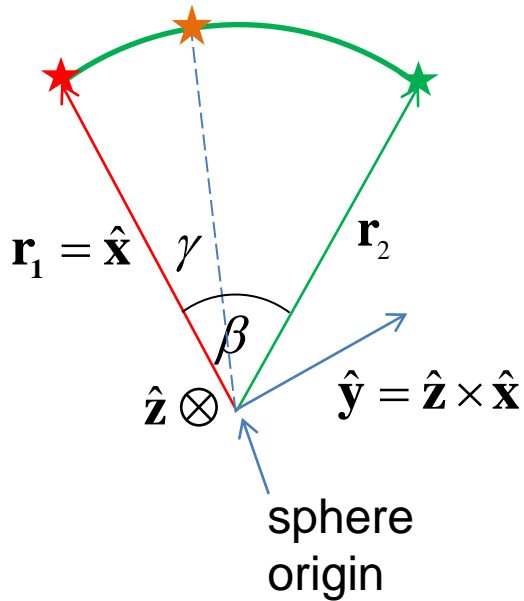
$$0 \leq \alpha \leq \frac{1}{2} \pi$$

### Any Great Circle

$$\alpha = \frac{1}{2} \pi$$

$$\mathbf{r} = \sin \alpha (\hat{\mathbf{x}} \cos \theta + \hat{\mathbf{y}} \sin \theta) + \hat{\mathbf{z}} \cos \alpha$$

# Great circle arc between two points on a unit sphere



Using **radians**, *arc length* is  $\beta$

unity  
↓ ↓

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = |\mathbf{r}_1| |\mathbf{r}_2| \cos \beta = \cos \beta$$

$$\therefore \beta = \cos^{-1} \mathbf{r}_1 \cdot \mathbf{r}_2$$

Unless the points are on a *diameter*, we can form a unique **normal vector** to the great circle between the two points.

$$\hat{\mathbf{z}} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{|\mathbf{r}_1 \times \mathbf{r}_2|}$$

Set  $\hat{\mathbf{x}} = \mathbf{r}_1 \quad \therefore \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{x}}$  to form a right handed set of basis vectors

Position vector ★

$$\mathbf{r} = \hat{\mathbf{x}} \cos \gamma + \hat{\mathbf{y}} \sin \gamma$$

$$0 \leq \gamma \leq \beta$$

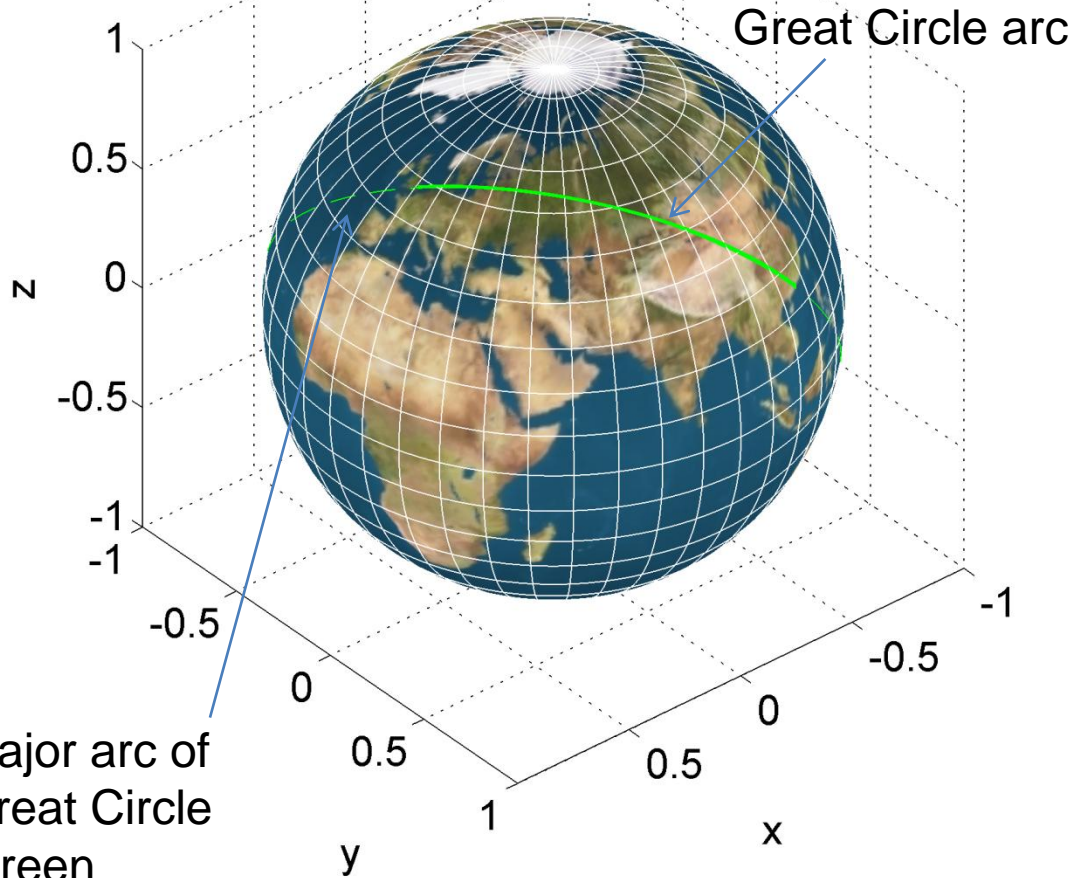
If the points *are* on a diameter, *any* Great Circle passing through them will suffice!

$$\beta = \pi$$

Choose  $\beta$  to be the *minor arc* i.e. the smaller of  $\beta$  and  $2\pi - \beta$

# MATLAB demo greatcircle.m

latA = 52°, longA = 1°, latB = 22°, longB = 114°, arc length = 1.5  
Equivalent Earth arc length = 9547.5km



Use the arrow keys to modify the latitude and longitude of position 1 and a, z and s, d keys for position 2

$$\mathbf{r} = X\mathbf{G} + Y\mathbf{E} + Z\mathbf{N}$$

$$X = \cos \lambda \cos \phi$$

$$Y = \cos \lambda \sin \phi$$

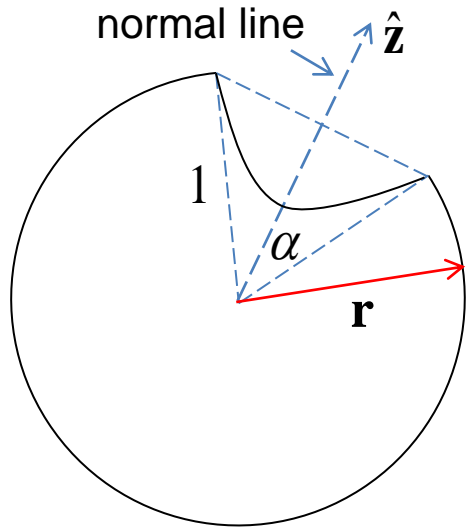
$$Z = \sin \lambda$$

Major arc of Great Circle (green dashed line)

MATLAB: `plot3(x,y,z)`

To plot we need X,Y,Z in geocentric Cartesians

# Parabolic indent (or cap!) on a sphere (“Death Star Problem”)

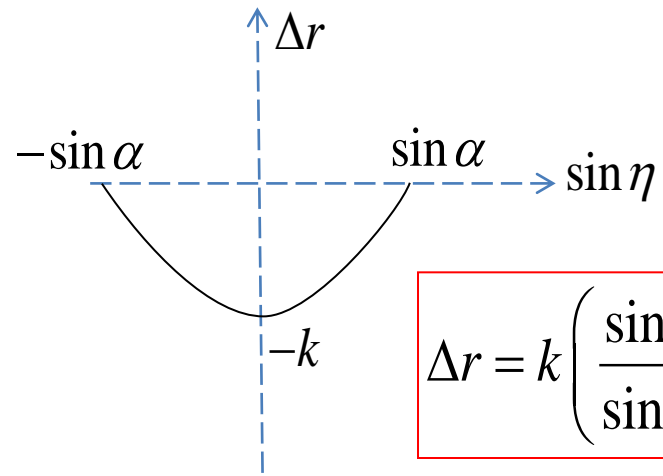


$$\mathbf{r} = X\mathbf{G} + Y\mathbf{E} + Z\mathbf{N}$$

$$X = \cos \lambda \cos \phi$$

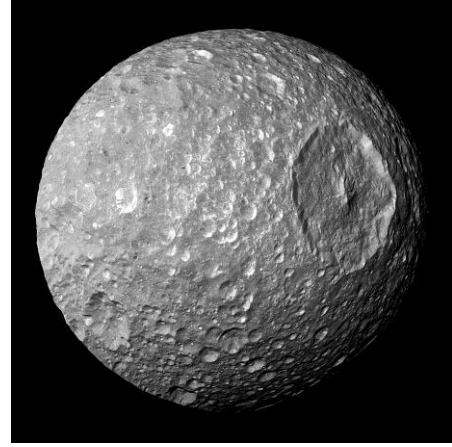
$$Y = \cos \lambda \sin \phi$$

$$Z = \sin \lambda$$



$$\Delta r = k \left( \frac{\sin^2 \eta}{\sin^2 \alpha} - 1 \right)$$

Define **parabolic** indent as a function of the sine of an angle  $\eta$  from normal line



*Mimas* – a Moon of Saturn  
(Diameter 396km)



The *Death Star* from *Star Wars*  
(Diameter 120km)

Find coordinates on (unit)sphere surface which satisfy:

$$\eta = \cos^{-1} \mathbf{r} \cdot \hat{\mathbf{z}}$$

$$|\eta| \leq \alpha$$

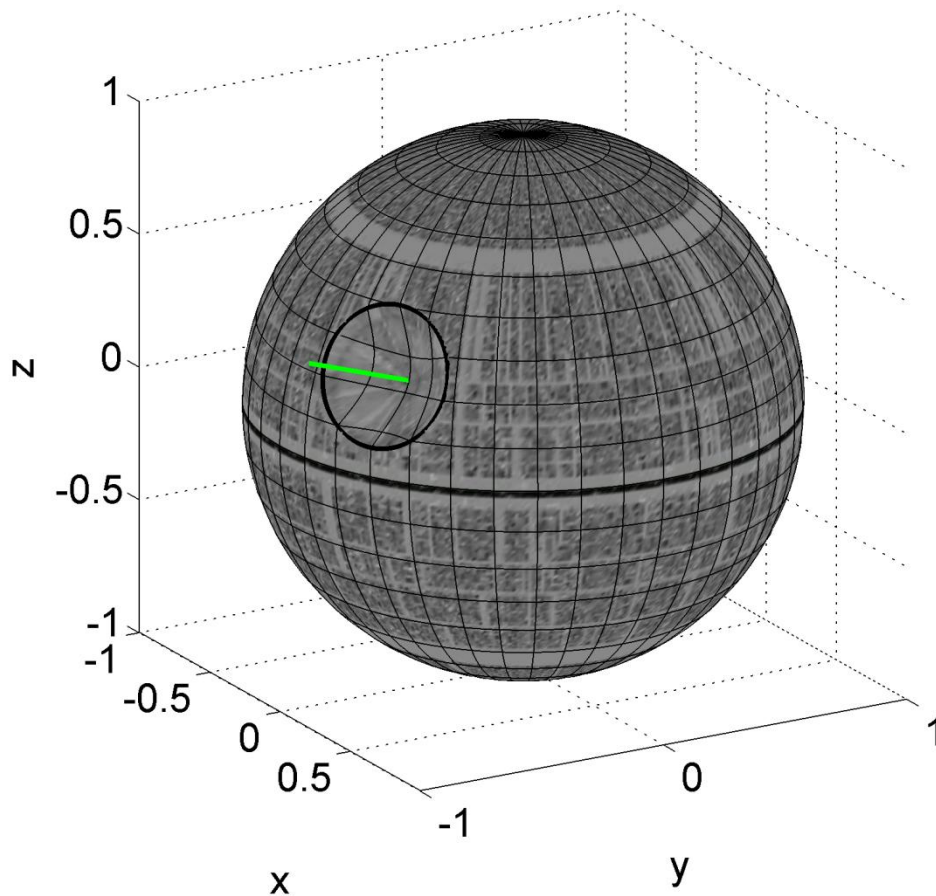
For these points, scale  $X, Y, Z$  by

$$1 + \Delta r$$

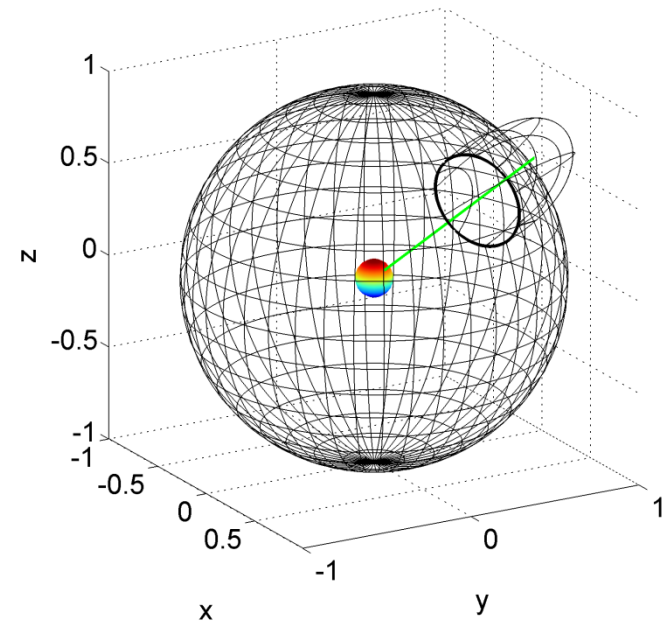
## MATLAB demo deathstar.m

Use arrow keys to move the indent around in latitude and longitude

Death Star:  $\alpha = 15^\circ$ , lat =  $21.2^\circ$ , long =  $-63^\circ$ ,  $k = 0.2$



Death Star:  $\alpha = 15^\circ$ , lat =  $-219^\circ$ , long =  $195^\circ$ ,  $k = -0.75$

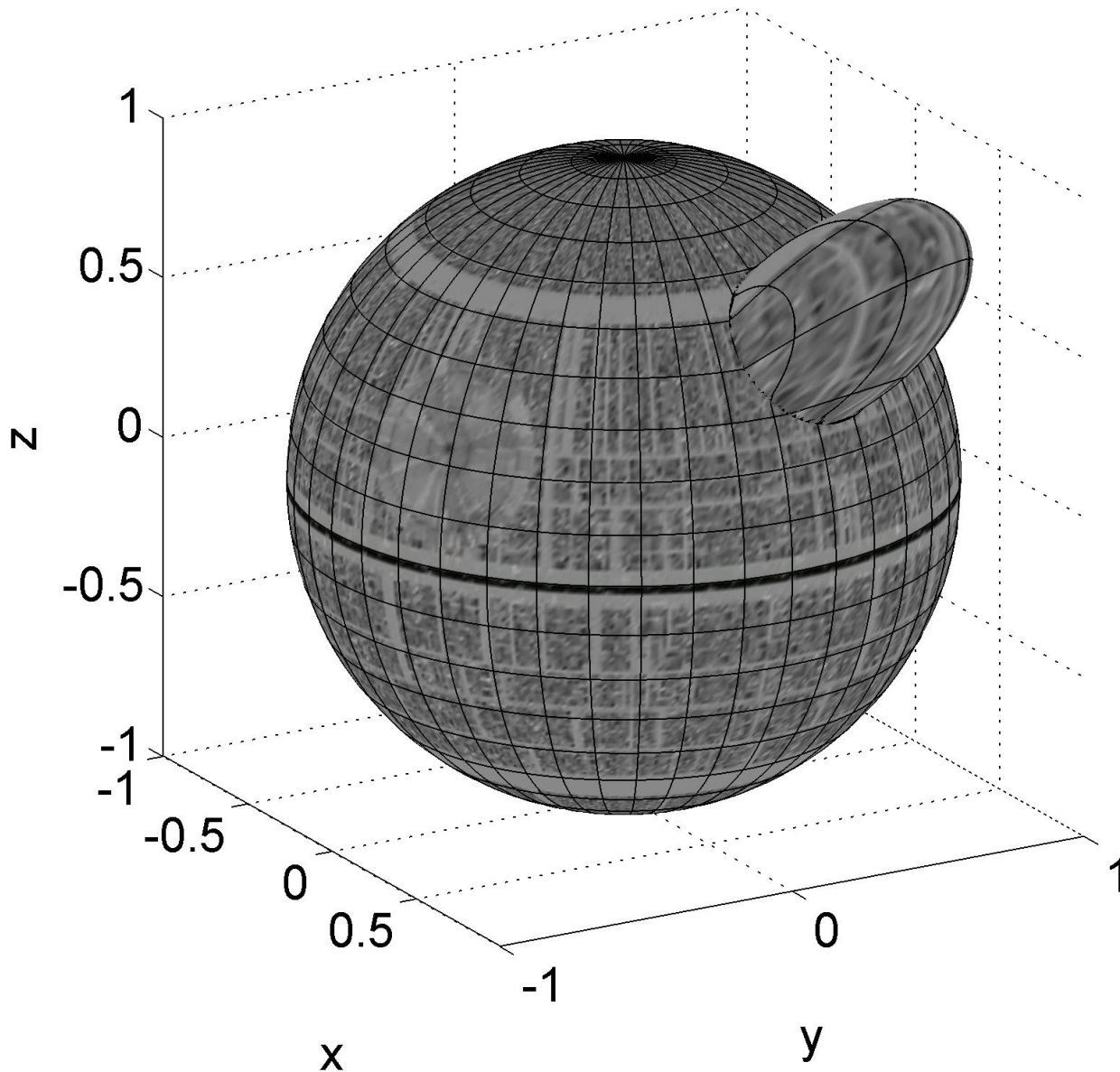


Press `h` to toggle wire frame mode  
Press `v` to return to textured surface

Change size of circle (i.e.  $\alpha$ )  
using `a` and `z` keys

Change  $k$  using `k` and `m` keys

Death Star:  $\alpha = 15^\circ$ , lat =  $-219^\circ$ , long =  $195^\circ$ ,  $k = -0.75$



Give the Death Star  
a 'nose' if  $k < 0$

Now that Disney has bought  
the rights to Star Wars...



Pinocchio has experienced the dark side of the Force ....





Where to find these resources ... [www.eclecticon.info](http://www.eclecticon.info)

The screenshot shows the homepage of [www.eclecticon.info](http://www.eclecticon.info). At the top, there is a navigation bar with menu items: Art, Books, Comedy, Films, Fitness, Gastronomy, and Maths. The 'Maths' menu item is highlighted in green, and a green arrow points from it to the 'Eclecticon' text in the central graphic. To the right of the navigation bar, it says 'First created July 2012' and 'Last updated Dec 2016'. The central graphic features the text 'Dr French's Eclecticon' in a decorative font, with a colorful, multi-layered fractal-like shape in the center. The page is surrounded by various logos and icons, including BBC Radio 1-6, Planet Rock, Wikipedia, WolframAlpha, the Guardian Datablog, Google, Directgov, Isle of Wight, and others. At the bottom, there is a footer with more menu items: Mountaineering, Music, Philosophy, Photography, Physics, Programming, and Writing. A circular logo for 'Lectures Dr French's Eclecticon' is also visible in the bottom right corner.



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Books

Comedy

Films

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Maths

First created July 2012



Last updated Jan 2017

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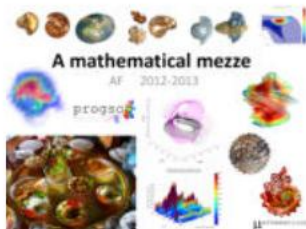
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★★★★★  
FANTASTIC  
WORK!  
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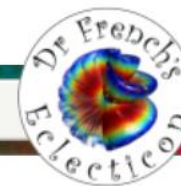
Philosophy

Photography

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Last updated Jan 2017

# Geometry

[Areas of basic shapes](#)

[Arcs, sectors, radians](#)

[Argand diagram complex loci](#)

▶ [Cartesian equation of a circle](#)

▶ [Circle theorems](#)

[Lines and angles](#)

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[Transformations using matrices: Rotations](#)

[Transformations using matrices: Not about the origin](#)

[Transformations using matrices: Invariant lines](#)

[Vectors](#)

[Vector equations of lines and planes](#)

[Volumes of basic solids](#)

[Geometric transformations using matrices](#)

[IGCSE trigonometry proofs](#)

[First 86 Pythagorean triples](#)

[Special triangles](#)

[Basic geometry notes](#)

[Chimborazo](#)

[Proof of circle theorems](#)

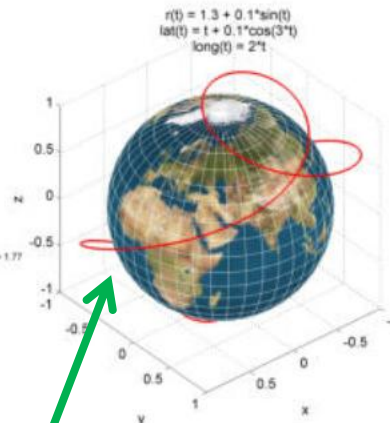
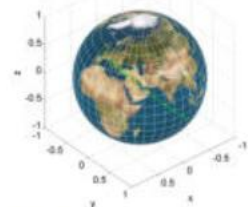
MATLAB  
code

## ▶ Navigating the sphere

Andy French

February 2017

lat = 40°, long = 12°, lat0 = 10°, long0 = 90°, arc length = 1.77  
Equivalent Earth arc length = 11250km



Presentation

[All construction  
presentations](#)



Construction of a regular  
pentagon

Dr Andrew French

Construction of a regular  
hexagon and dodecagon

Dr Andrew French



Constructing a Manx window

Dr Andrew French



Construction\* of a regular  
heptagon

Dr Andrew French



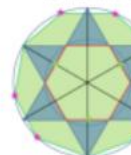
Approximately regular N-gon  
construction scheme

Dr Andrew French, Adrian Ahmed



Construction\* of a nonagon

Dr Andrew French



Mountaineering

Music

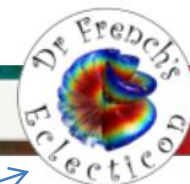
Philosophy

Photography

Physics

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Also on **Lectures (3)** – Click on the icon here