Navigating the sphere

Andy French

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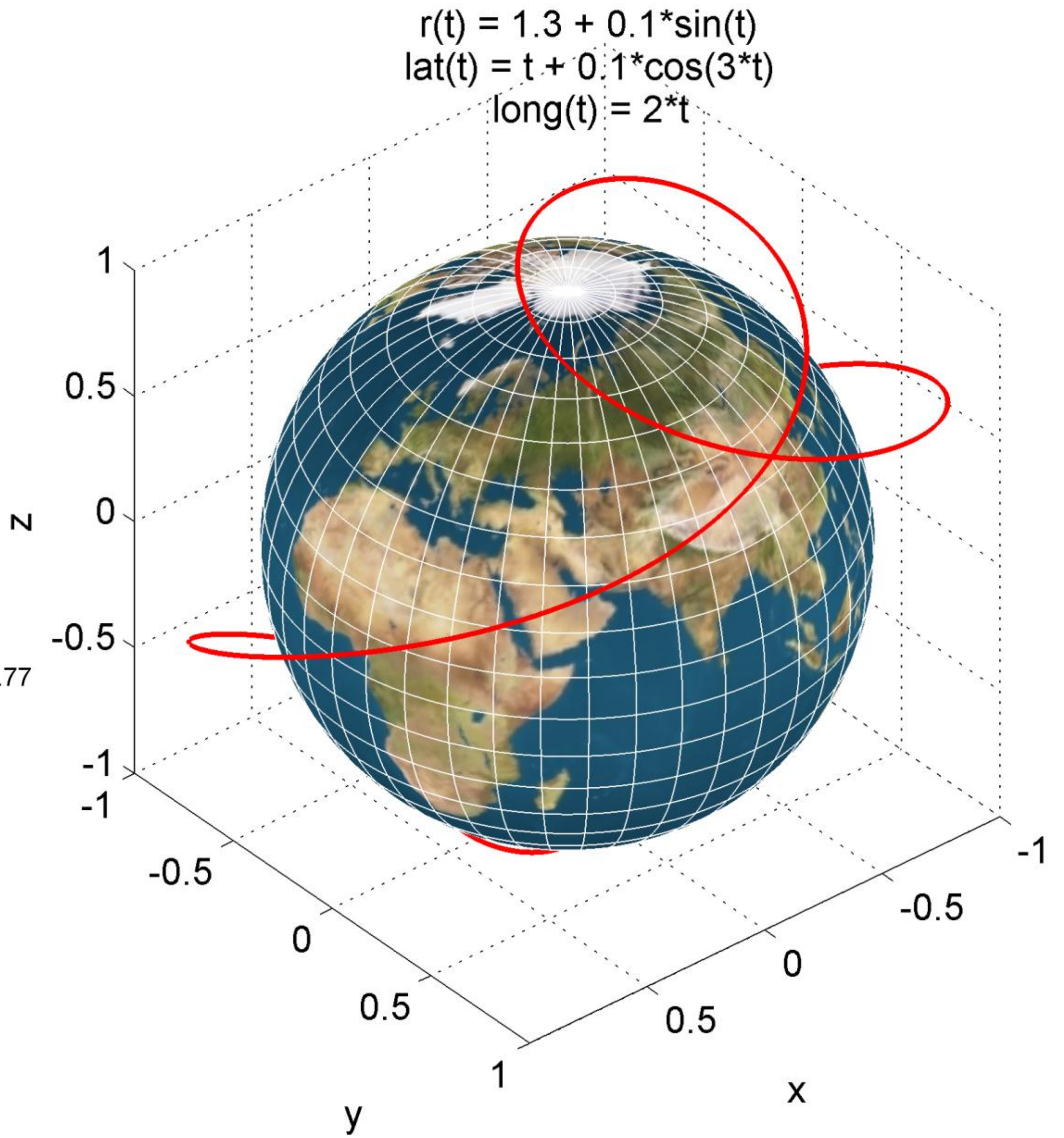
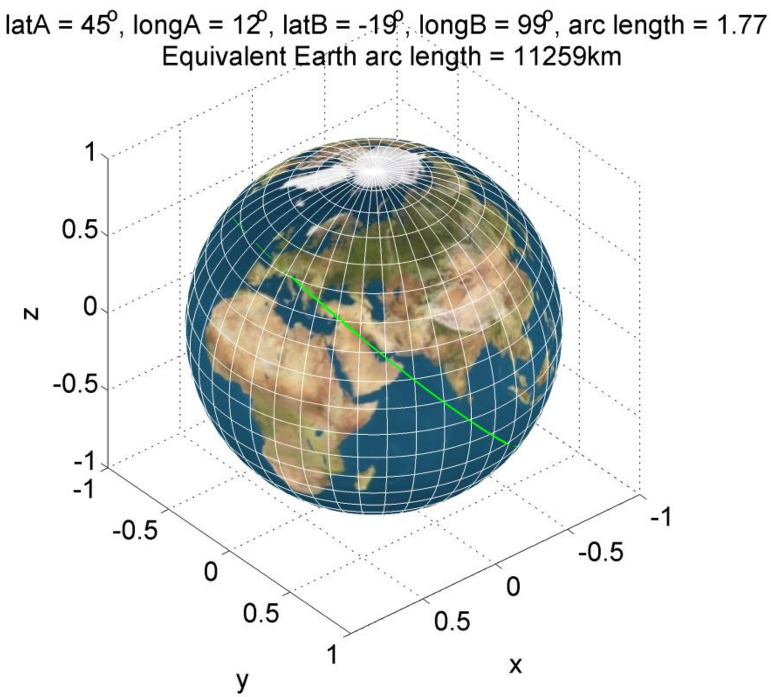




Motivation

CSM is working on aspects of **maritime navigation** with his Div, and we got chatting on the way to Freddie's ....

Since to a very good approximation the Earth is spherical* it occurred to me there are some classic mathematical problems to solve, and all involve:

- Visualizing trajectories defined in **spherical polar coordinates**

- Recipes for defining **circles** drawn on the **surface of a sphere**

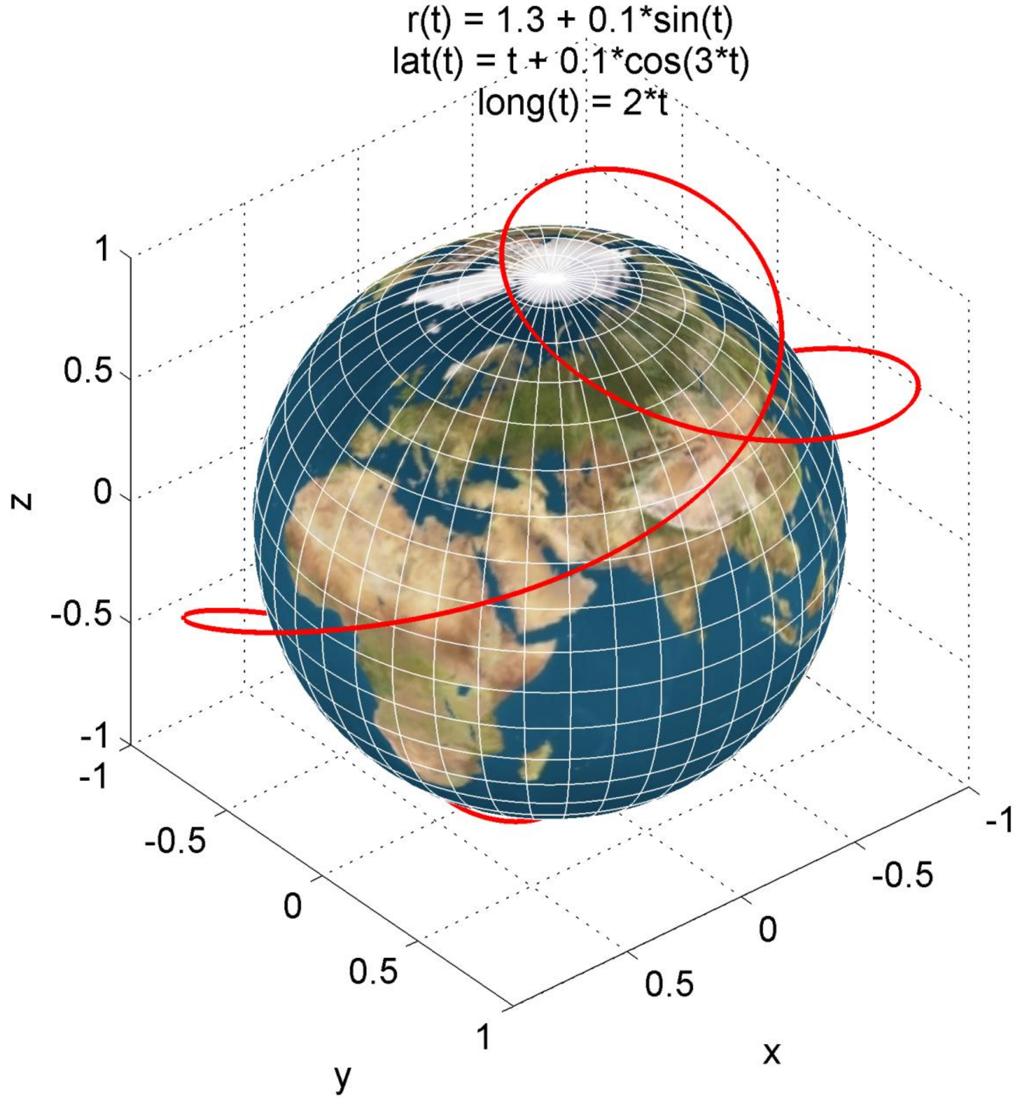
- Transformations between **Earth centred Cartesian** x,y,z coordinates and a Cartesian **'East, North, Up'** system where x and y are in the *tangent plane* to a given latitude λ and longitude ϕ.

Due to the long history of Navigation there is *much estoterica* inherent in the textbook solutions to these problems. Can we do it systematically, using methods that a Pre-U student could readily understand?

* The *polar radius* differs from the *equatorial radius* by a factor of $\approx 1 - \frac{1}{228.257}$

Three problems

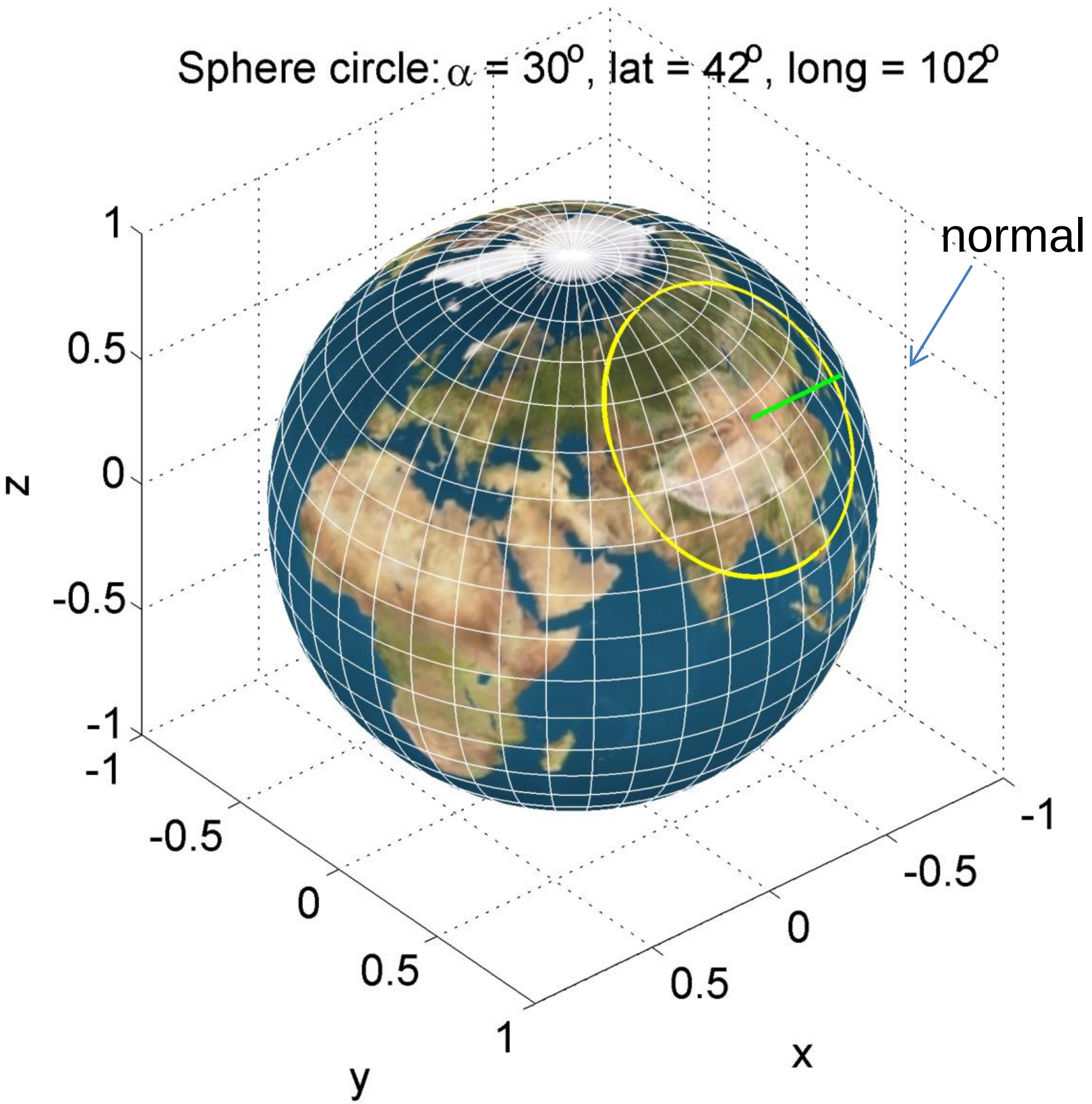
1. **Visualize** a curve defined **parametrically** in **spherical polar coordinates**



Three problems ....

2. Plot a **circle** drawn on the **surface** of a sphere

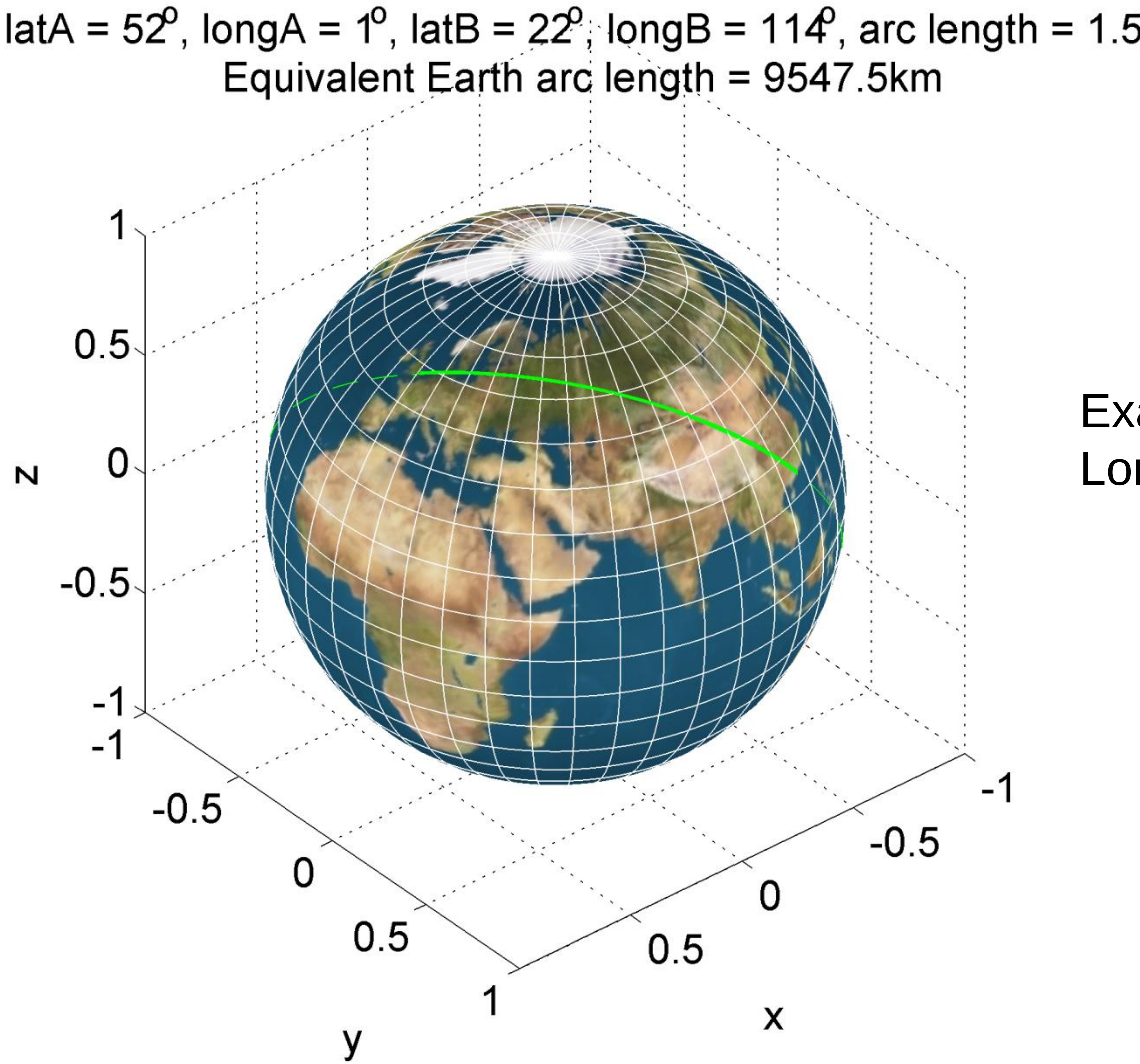
This is the general problem with *special cases* being **Great Circles** and **lines of constant latitude**



Define the circle via the **latitude** and **longitude** of a **normal** to the circle and the **angle** α between the circumference, sphere centre and normal.

Three problems ....

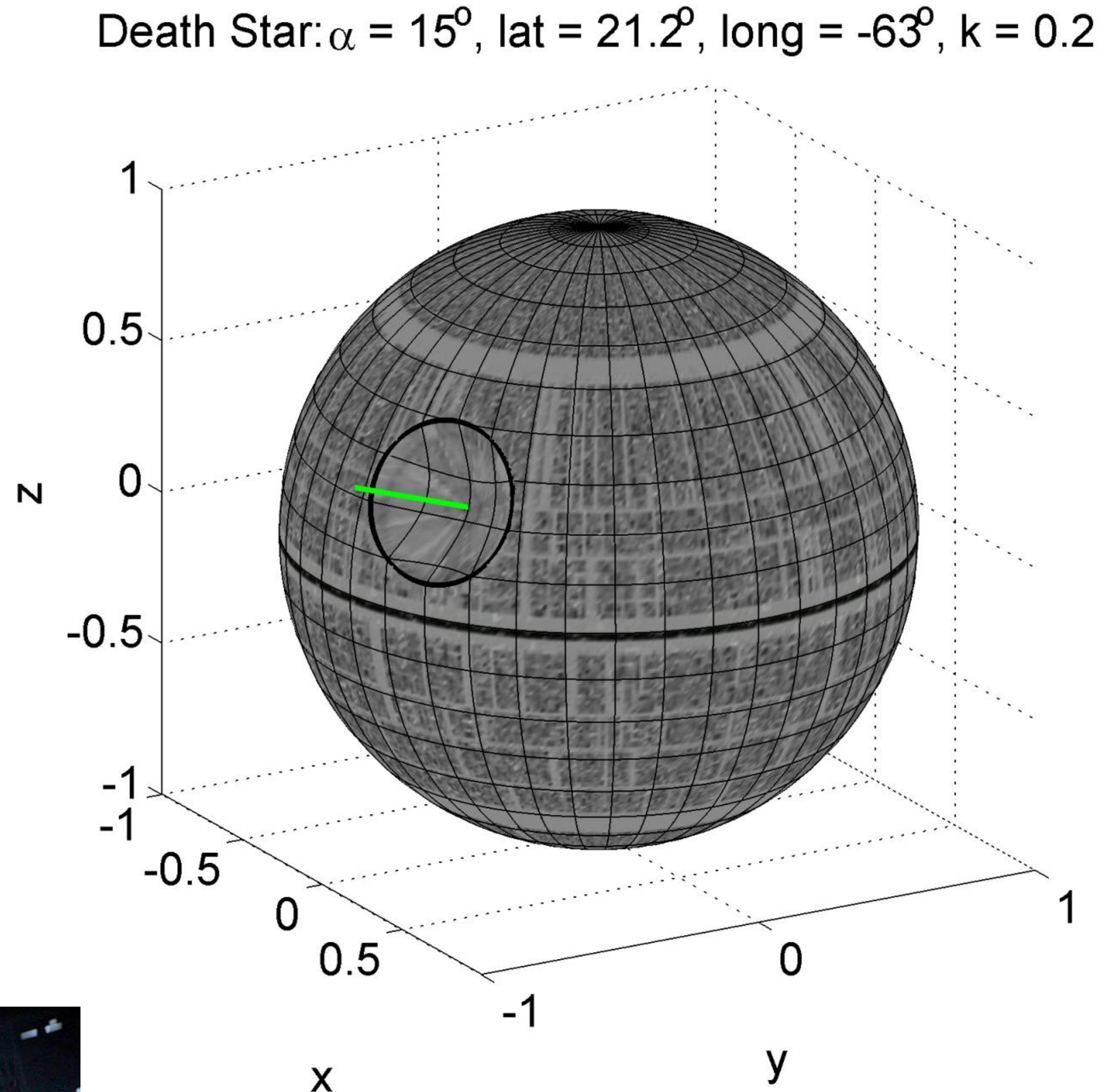
3. Define the **Great Circle arc**, which is the **shortest distance between two points on the surface of a sphere**. Very useful for aircraft or ship routings!



Example:
London to Hong Kong

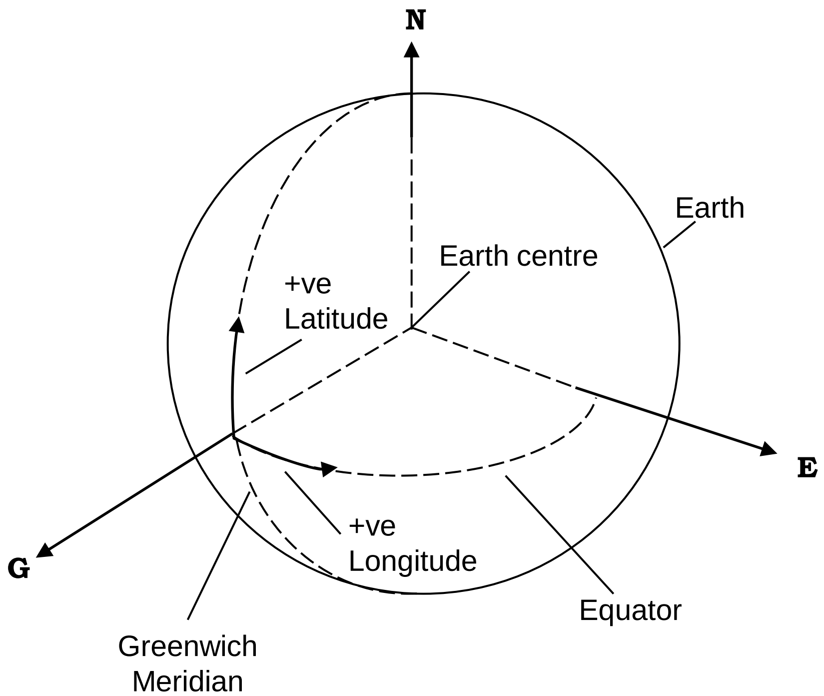
Accurate distance from Heathrow to Hong Kong Airport is 9,606km
[Google Maps]()

Plus an additional frivolous problem ....

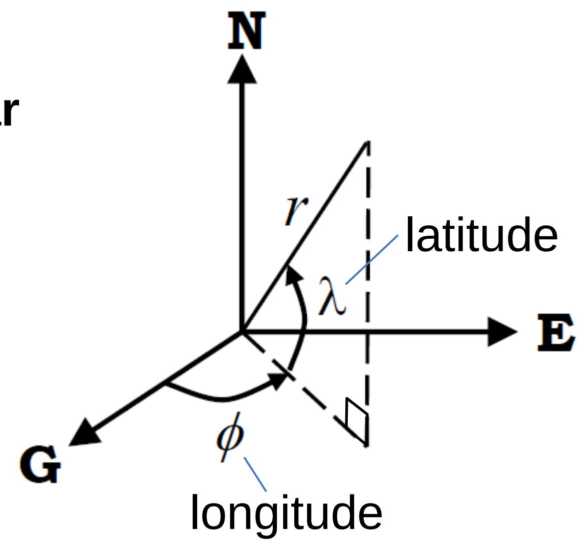


Is the Force strong enough to give me a parabolic indent in my Death Star?

Don't forget to distort the lines of latitude and longitude too...



Spherical polar coordinates



Position vector

$$\mathbf{r} = X\mathbf{G} + Y\mathbf{E} + Z\mathbf{N}$$

$$X = r\cos\lambda\cos\phi$$

$$Y = r\cos\lambda\sin\phi$$

$$Z = r\sin\lambda$$

$$|\mathbf{G}| = |\mathbf{E}| = |\mathbf{N}| = 1$$

$$\mathbf{G}\cdot\mathbf{E} = \mathbf{G}\cdot\mathbf{N} = \mathbf{E}\cdot\mathbf{N} = 0$$

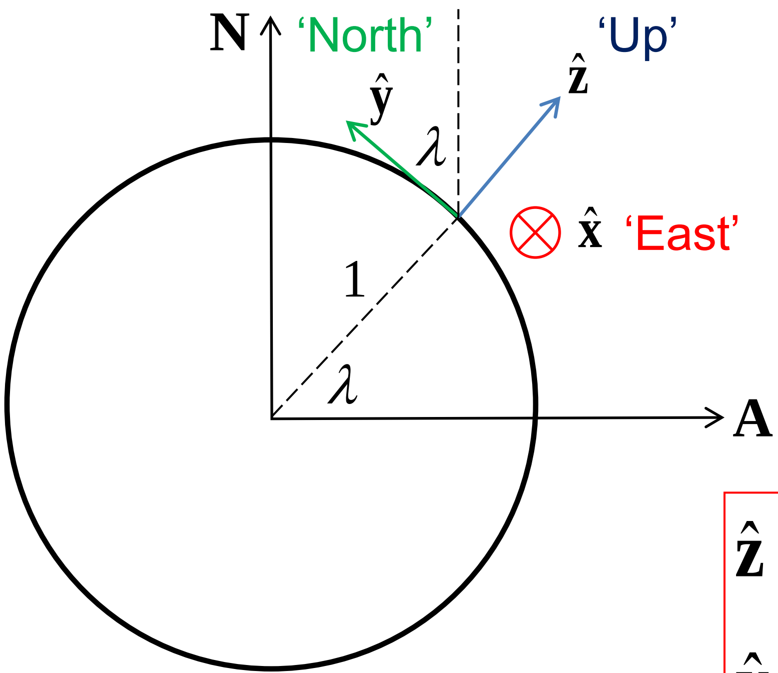
$$\mathbf{G}\times\mathbf{E} = \mathbf{N}$$

$$\mathbf{E}\times\mathbf{N} = \mathbf{G}$$

$$\mathbf{N}\times\mathbf{G} = \mathbf{E}$$

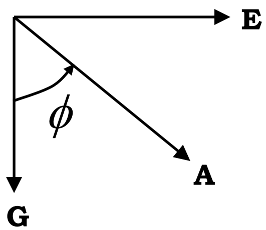
For simplicity, we will use a unit sphere of radius $r = 1$

Geocentric “Greenwich, East, North” **unit vectors** – a *right handed set*.



Also define a Cartesian coordinate system based upon a **local tangent plane** to a point on a unit sphere characterized by (λ, ϕ)

$$\mathbf{A} = \mathbf{G}\cos\phi + \mathbf{E}\sin\phi$$



$$\hat{\mathbf{z}} = \left(\mathbf{G}\cos\phi + \mathbf{E}\sin\phi\right)\cos\lambda + \mathbf{N}\sin\lambda$$

$$\hat{\mathbf{y}} = -\left(\mathbf{G}\cos\phi + \mathbf{E}\sin\phi\right)\sin\lambda + \mathbf{N}\cos\lambda$$

$$\hat{\mathbf{x}} = \hat{\mathbf{y}} \times \hat{\mathbf{z}}$$

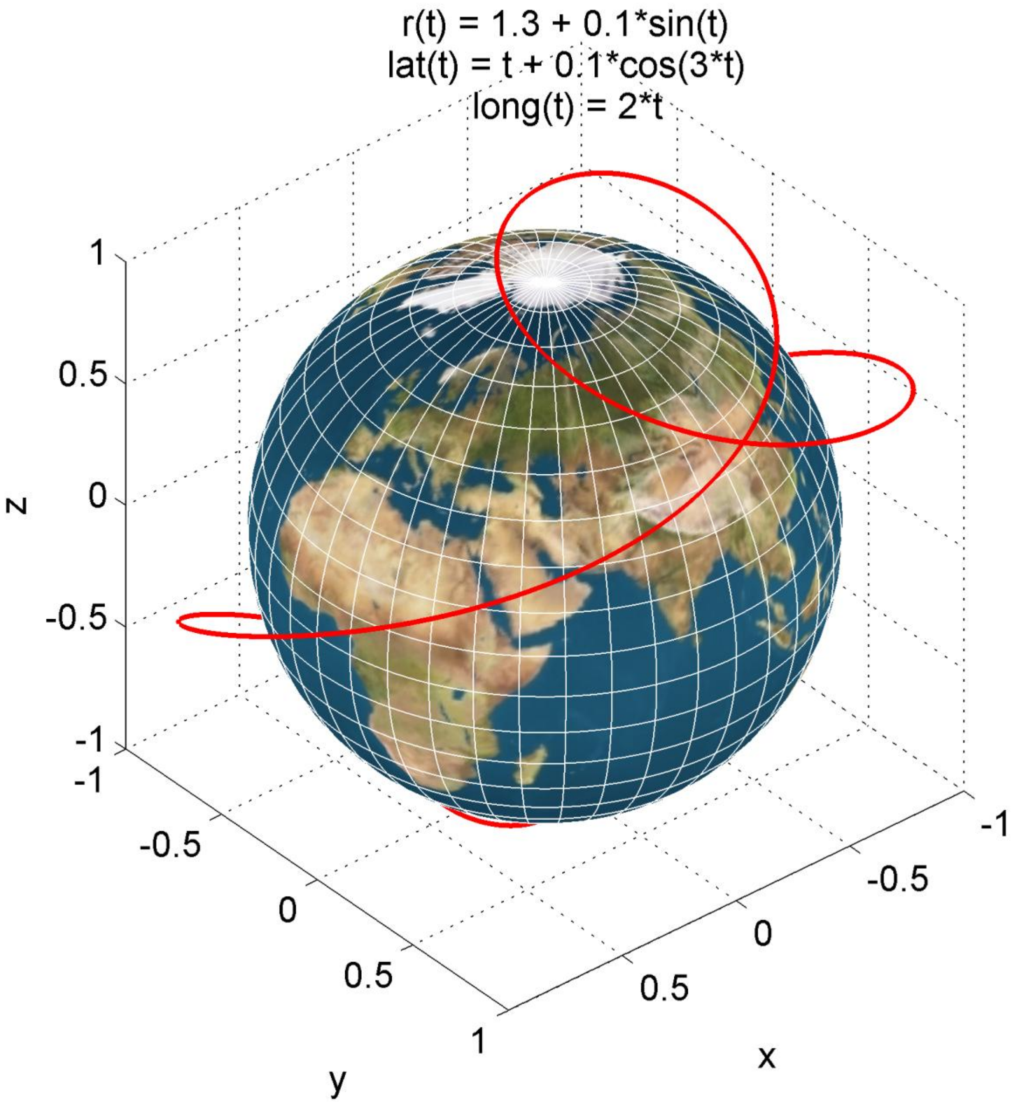
$$\hat{\mathbf{x}} = \mathbf{E}\cos\phi - \mathbf{G}\sin\phi$$

$$\mathbf{r} = X\mathbf{G} + Y\mathbf{E} + Z\mathbf{N}$$ Position vector

i.e. based upon a fixed (λ, ϕ) of the x,y,z system origin

$$\mathbf{r} = \cos\lambda\cos\phi\mathbf{G} + \cos\lambda\sin\phi\mathbf{E} + \sin\lambda\mathbf{N} + x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

Parametric trajectory plot
MATLAB demo `spolartraj.m`



Position vector

$$\mathbf{r} = X\mathbf{G} + Y\mathbf{E} + Z\mathbf{N}$$

$$X = r\cos\lambda\cos\phi$$

$$Y = r\cos\lambda\sin\phi$$

$$Z = r\sin\lambda$$

For example:

$$0 \le t \le 4\pi$$

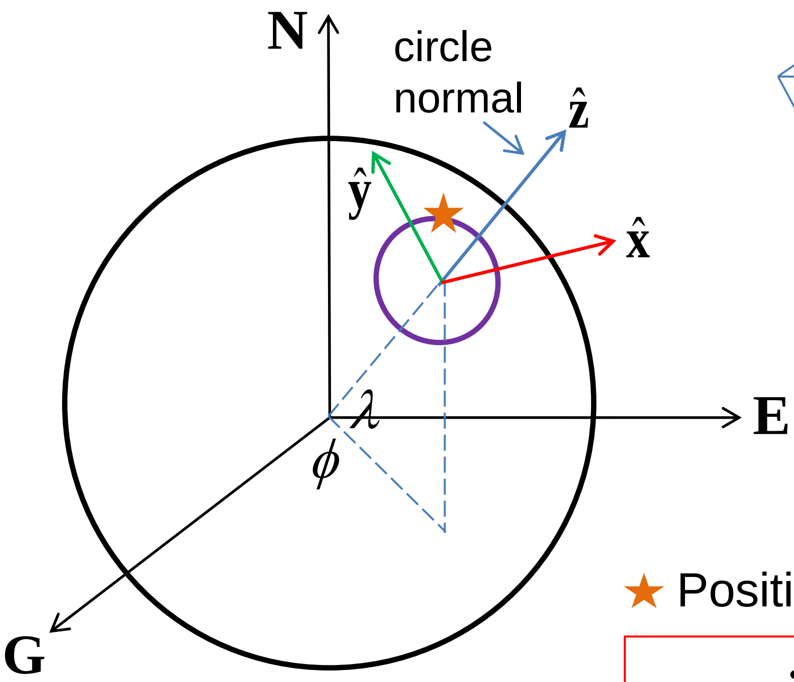
$$r = 1.3 + 0.1\sin t$$

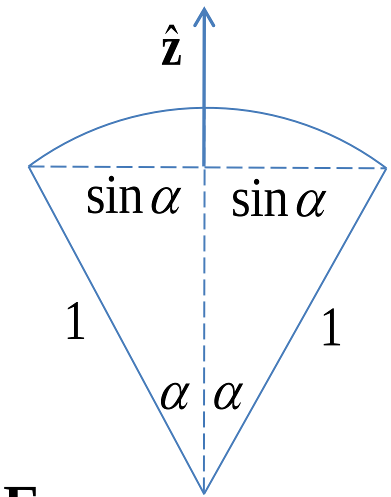
$$\lambda = t + 0.1\cos 3t$$

$$\phi = 2t$$

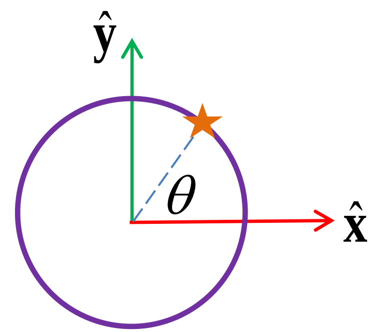
MATLAB: `plot3(x,y,z)`

Circle on a unit sphere





The radius of the circle is $\sin\alpha$



★ Position vector (from origin of sphere*)

$$\mathbf{r} = \sin\alpha\left(\hat{\mathbf{x}}\cos\theta + \hat{\mathbf{y}}\sin\theta\right) + \hat{\mathbf{z}}\cos\alpha$$

$$\hat{\mathbf{z}} = \left(\mathbf{G}\cos\phi + \mathbf{E}\sin\phi\right)\cos\lambda + \mathbf{N}\sin\lambda$$

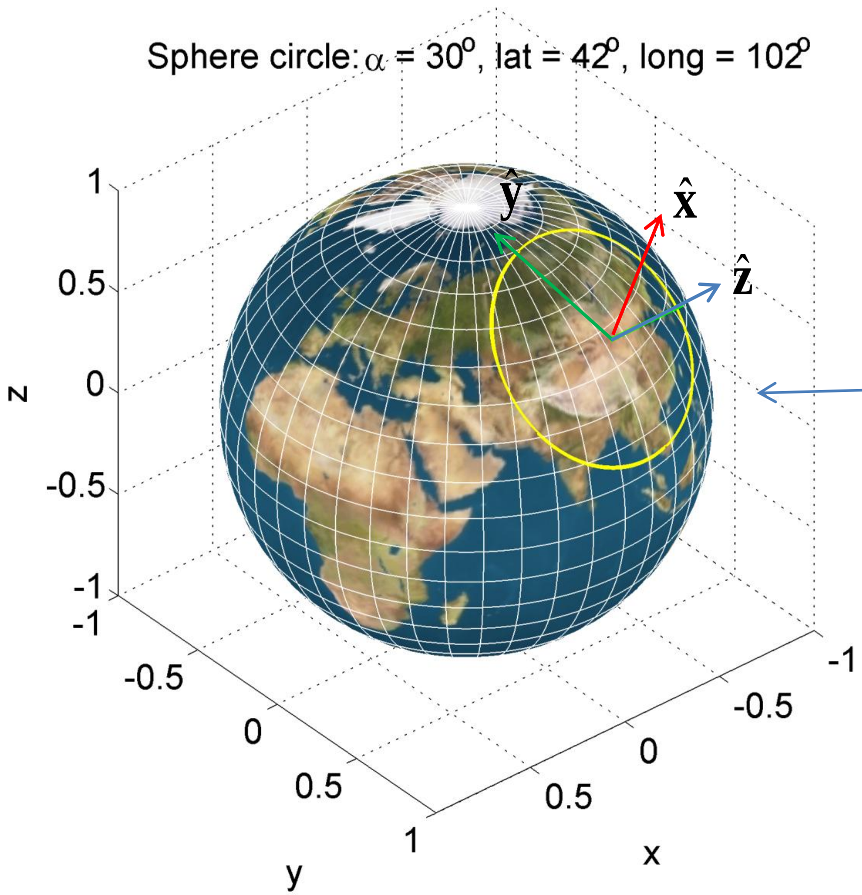
$$\hat{\mathbf{y}} = -\left(\mathbf{G}\cos\phi + \mathbf{E}\sin\phi\right)\sin\lambda + \mathbf{N}\cos\lambda$$

$$\hat{\mathbf{x}} = \mathbf{E}\cos\phi - \mathbf{G}\sin\phi$$

To plot, define θ in (for example) 300 linearly spaced steps between 0 and 2π. Then work out X,Y,Z coordinates.

*i.e. *not* surface in this particular case. This is OK since $\hat{\mathbf{z}}$ is a radial vector.

MATLAB demo `spherecircle.m`



Same code **`function`** used to plot all the circles!

Special cases:

Lines of Longitude

$$\alpha = \tfrac{1}{2}\pi$$

$$\lambda = 0$$

Lines of Latitude

$$\lambda = \pm\tfrac{1}{2}\pi$$

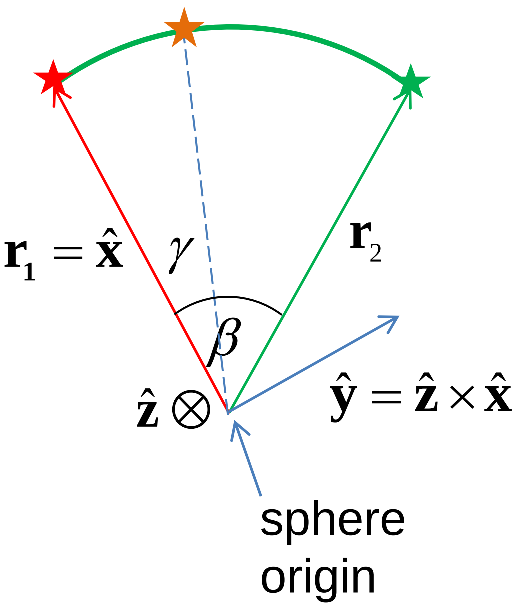
$$0 \le \alpha \le \tfrac{1}{2}\pi$$

Any Great Circle

$$\alpha = \tfrac{1}{2}\pi$$

$$\mathbf{r} = \sin\alpha\left(\hat{\mathbf{x}}\cos\theta + \hat{\mathbf{y}}\sin\theta\right) + \hat{\mathbf{z}}\cos\alpha$$

Great circle arc between two points on a unit sphere



Using **radians**, *arc length* is β

unity

$$\mathbf{r}_1 \cdot \mathbf{r}_2 = |\mathbf{r}_1||\mathbf{r}_2|\cos\beta = \cos\beta$$
$$\therefore \beta = \cos^{-1}\mathbf{r}_1 \cdot \mathbf{r}_2$$

Unless the points are on a *diameter,* we can form a unique **normal vector** to the great circle between the two points.

$$\hat{\mathbf{z}} = \frac{\mathbf{r}_1 \times \mathbf{r}_2}{|\mathbf{r}_1 \times \mathbf{r}_2|}$$

Set $\hat{\mathbf{x}} = \mathbf{r}_1 \quad \therefore \hat{\mathbf{y}} = \hat{\mathbf{z}} \times \hat{\mathbf{x}}$ to form a right handed set of basis vectors

Position vector ★

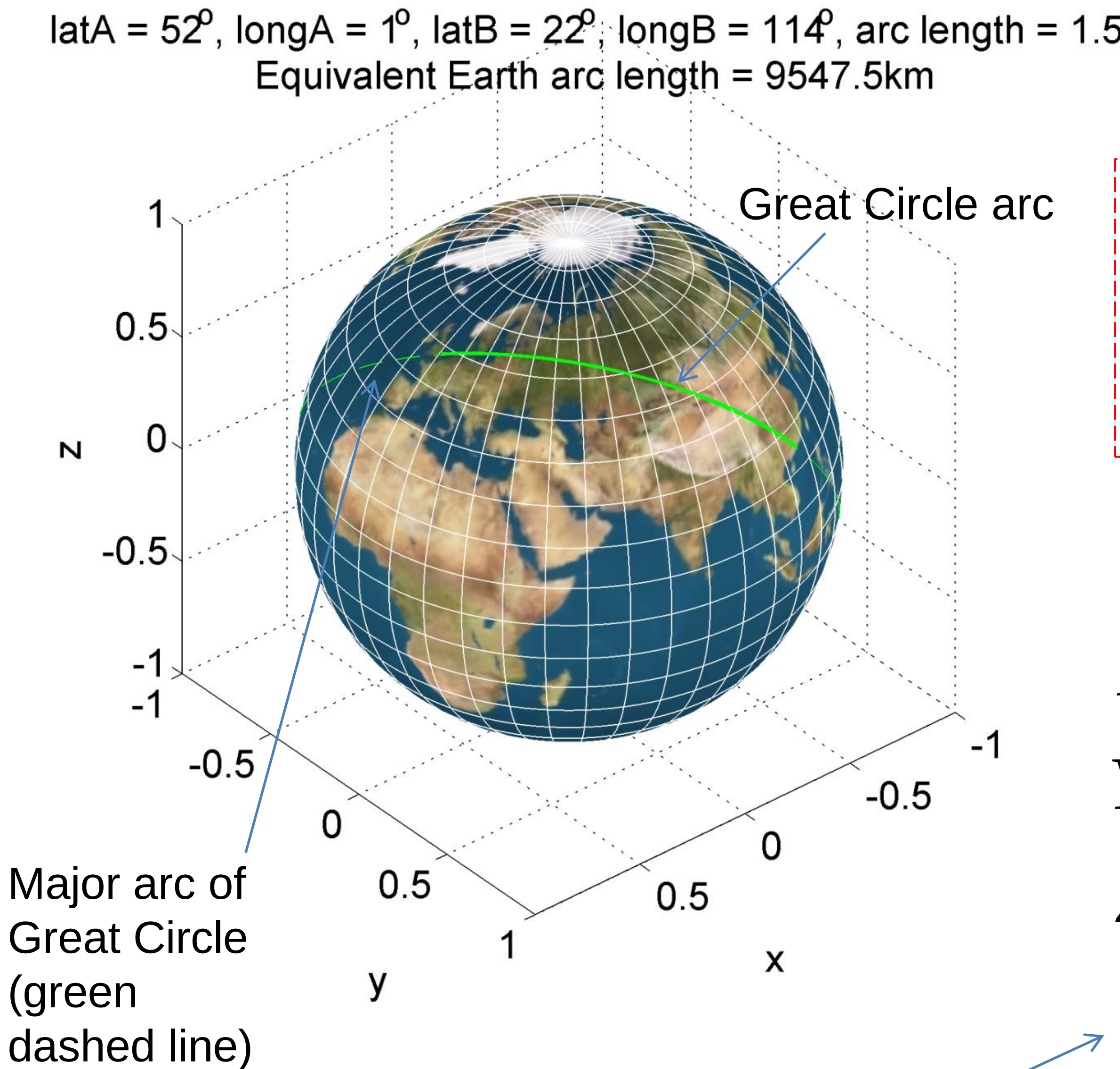
$$\mathbf{r} = \hat{\mathbf{x}}\cos\gamma + \hat{\mathbf{y}}\sin\gamma$$
$$0 \le \gamma \le \beta$$

If the points *are* on a diameter, *any* Great Circle passing through them will suffice!

$$\beta = \pi$$

Choose β to be the *minor arc* i.e. the smaller of β and 2π - β

MATLAB demo `greatcircle.m`



Use the `arrow keys` to modify the latitude and longitude of position 1 and `a,z` and `s,d` keys for position 2

$$\mathbf{r} = X\mathbf{G} + Y\mathbf{E} + Z\mathbf{N}$$

$$X = \cos\lambda\cos\phi$$

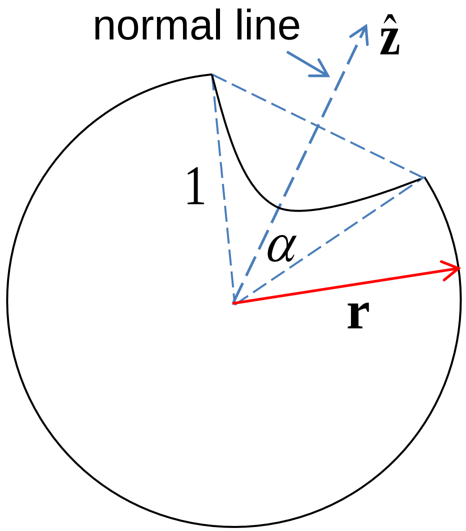
$$Y = \cos\lambda\sin\phi$$

$$Z = \sin\lambda$$

To plot we need X,Y,Z in geocentric Cartesians

MATLAB: `plot3(x,y,z)`

Parabolic indent (or cap!) on a sphere ("Death Star Problem")

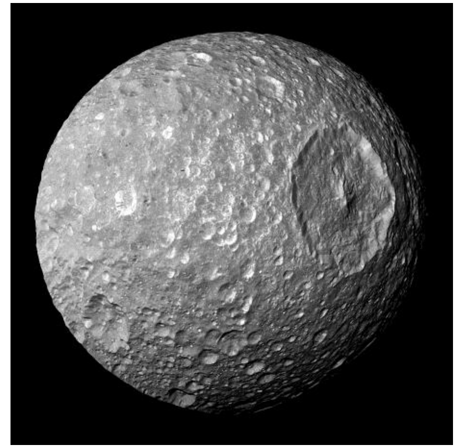


$$\mathbf{r} = X\mathbf{G} + Y\mathbf{E} + Z\mathbf{N}$$

$$X = \cos\lambda\cos\phi$$

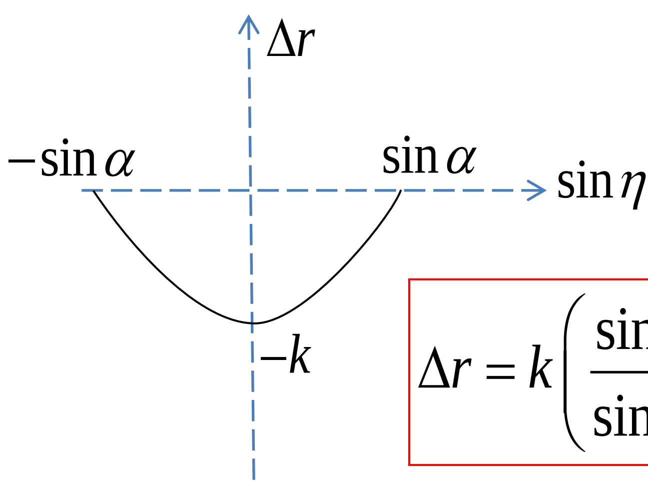
$$Y = \cos\lambda\sin\phi$$

$$Z = \sin\lambda$$



Mimas – a Moon of Saturn
(Diameter 396km)

The *Death Star* from *Star Wars* (Diameter 120km)



$$\Delta r = k\left(\frac{\sin^2\eta}{\sin^2\alpha} - 1\right)$$

Define **parabolic** indent as a function of the sine of an angle η from normal line

Find coordinates on (unit)sphere surface which satisfy:

$$\eta = \cos^{-1}\mathbf{r}\cdot\hat{\mathbf{z}}$$

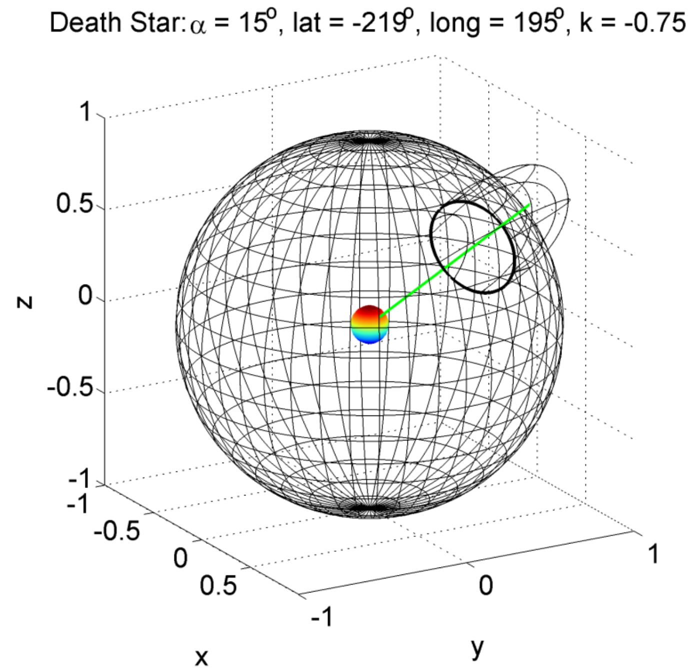
$$|\eta| \le \alpha$$

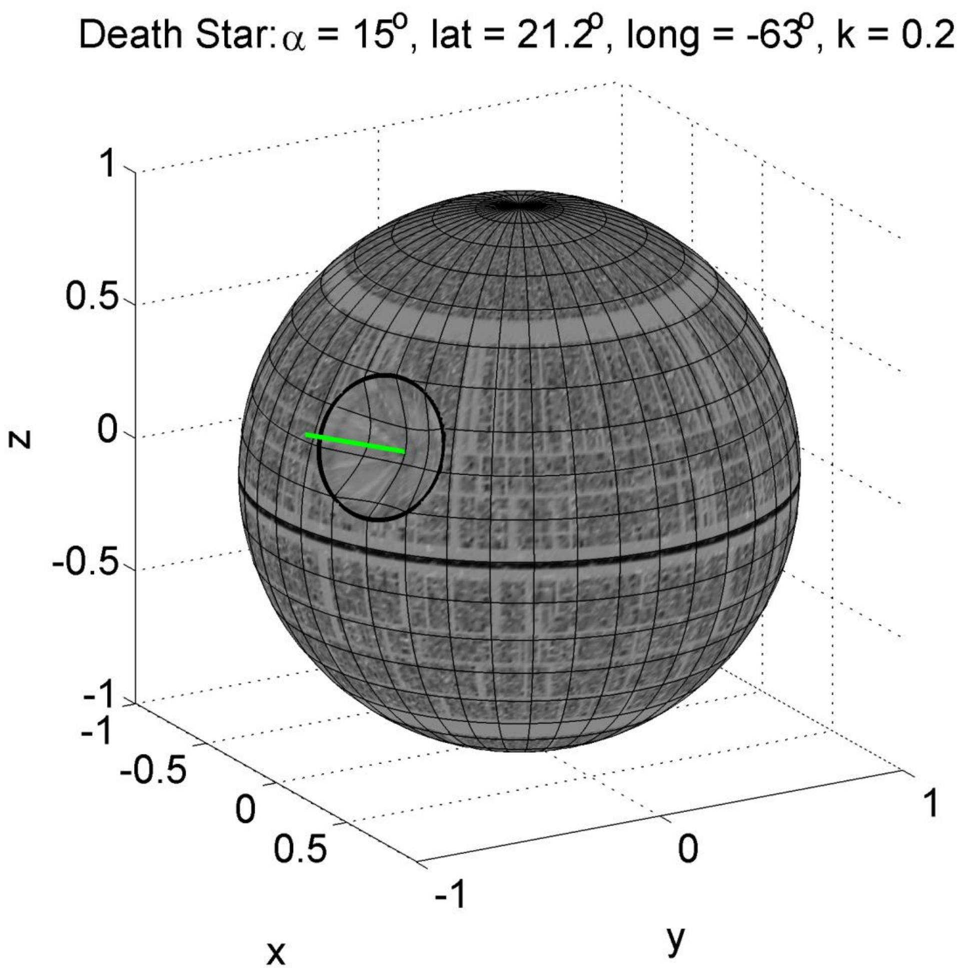
For these points, scale X,Y,Z by

$$1 + \Delta r$$

MATLAB demo `deathstar.m`

Use `arrow keys` to move the indent around in latitude and longitude



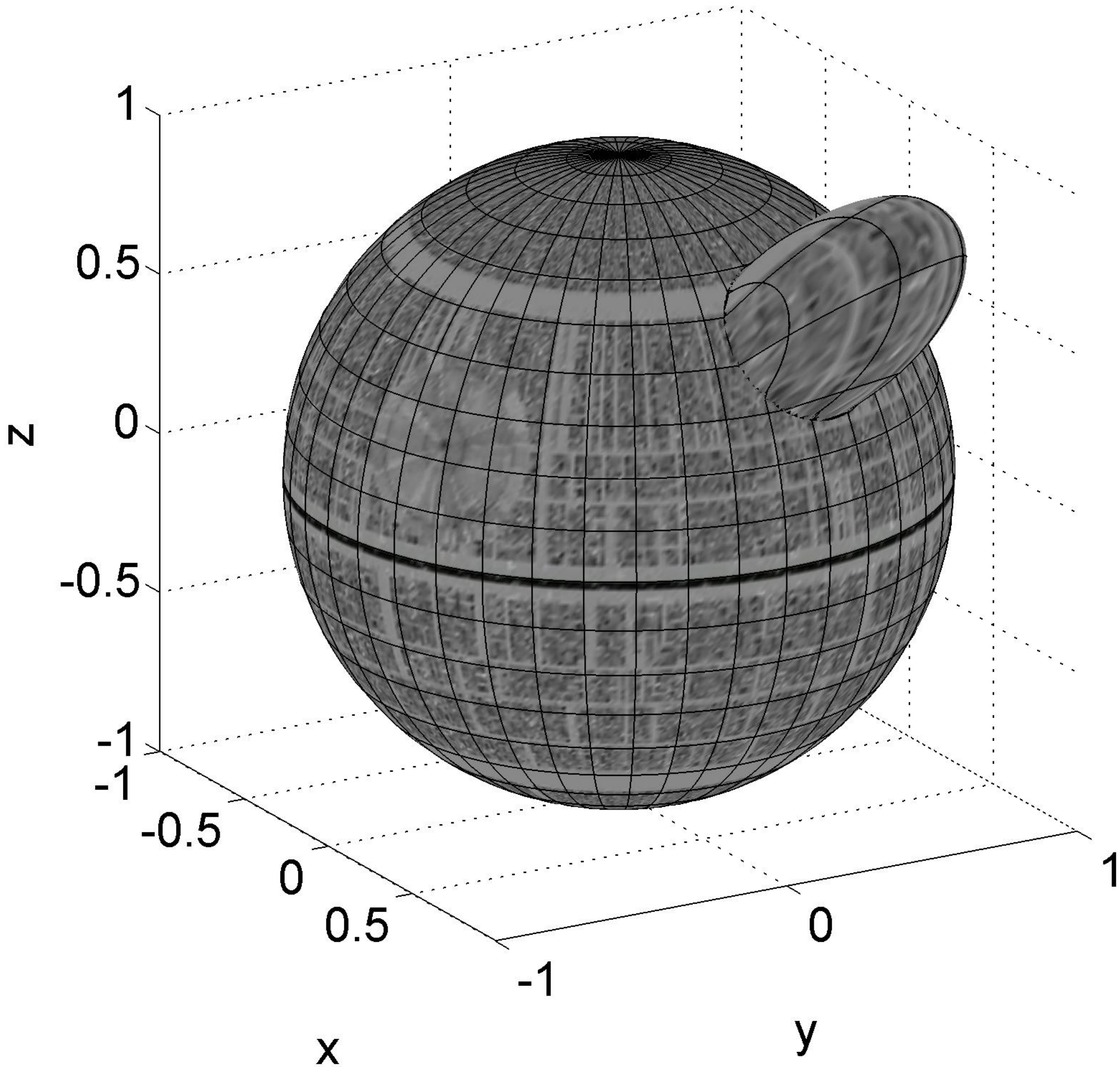


Press `h` to toggle wire frame mode
Press `v` to return to textured surface

Change size of circle (i.e. α) using `a` and `z` keys

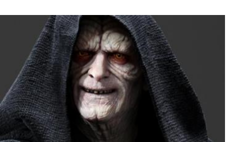
Change k using `k` and `m` keys

Death Star: $\alpha = 15^o$, lat = -219^o, long = 195^o, k = -0.75



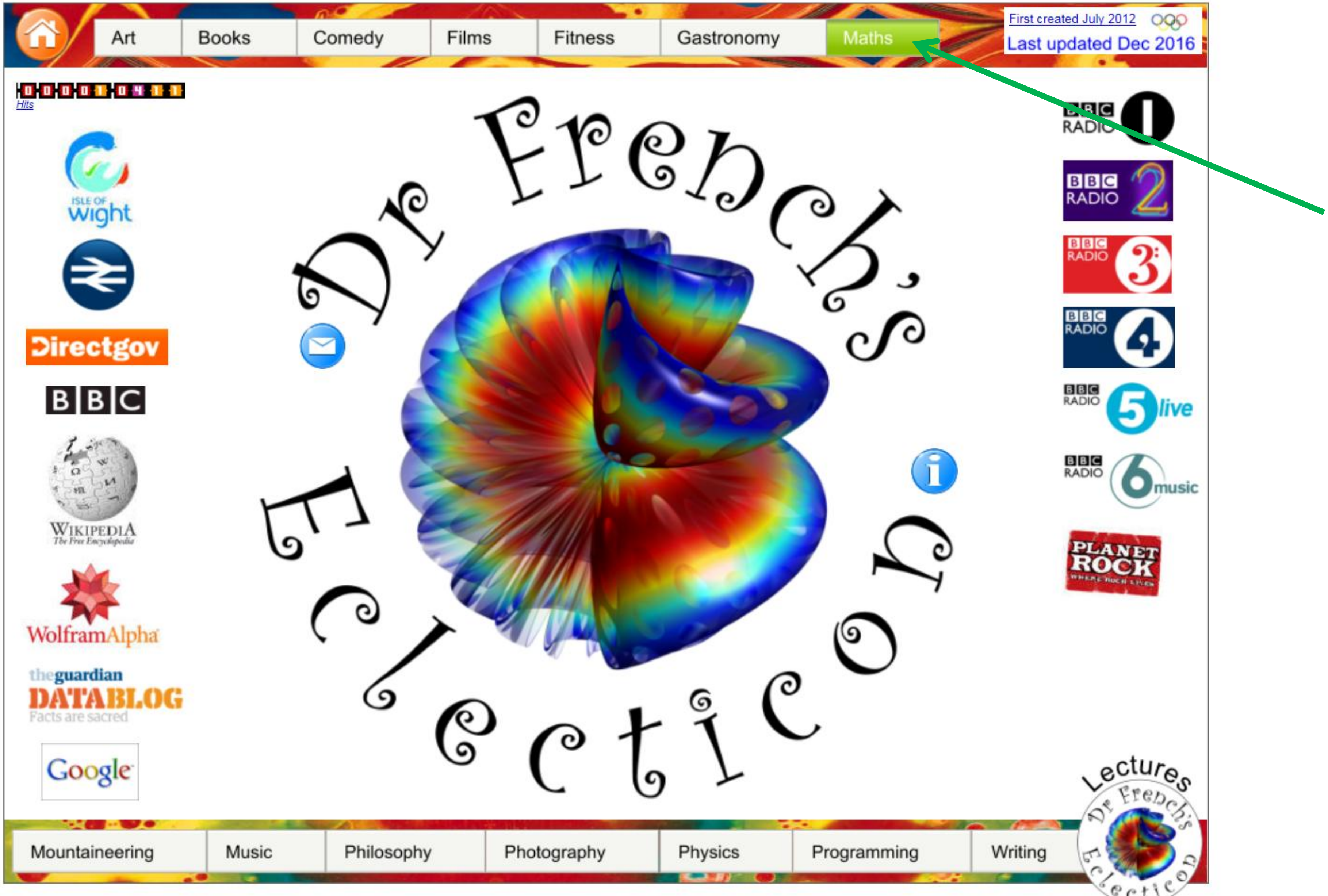
Give the Death Star a 'nose' if $k < 0$

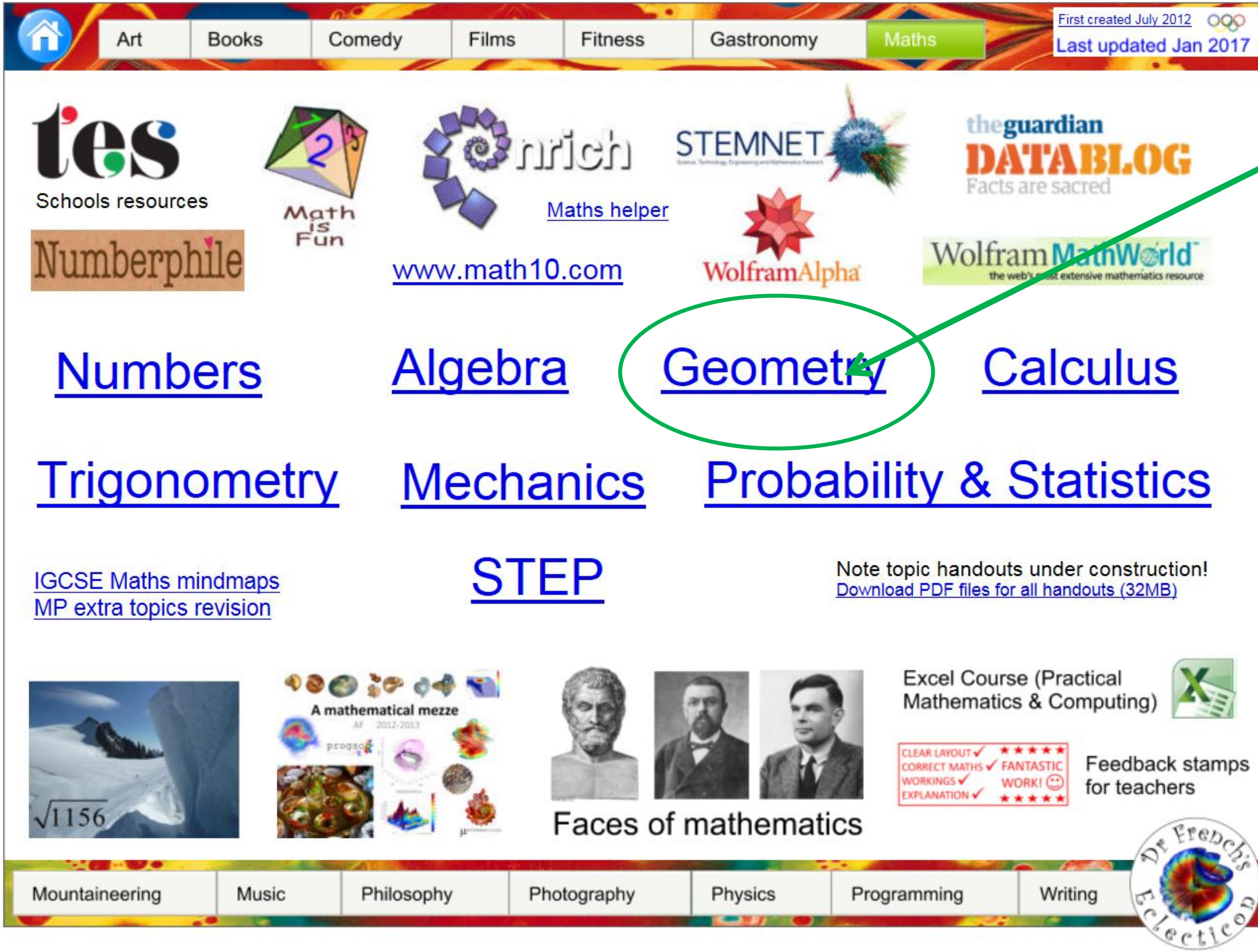
Now that Disney has bought the rights to Star Wars...



Pinocchio has experienced the dark side of the Force ....

Where to find these resources ... www.eclecticon.info





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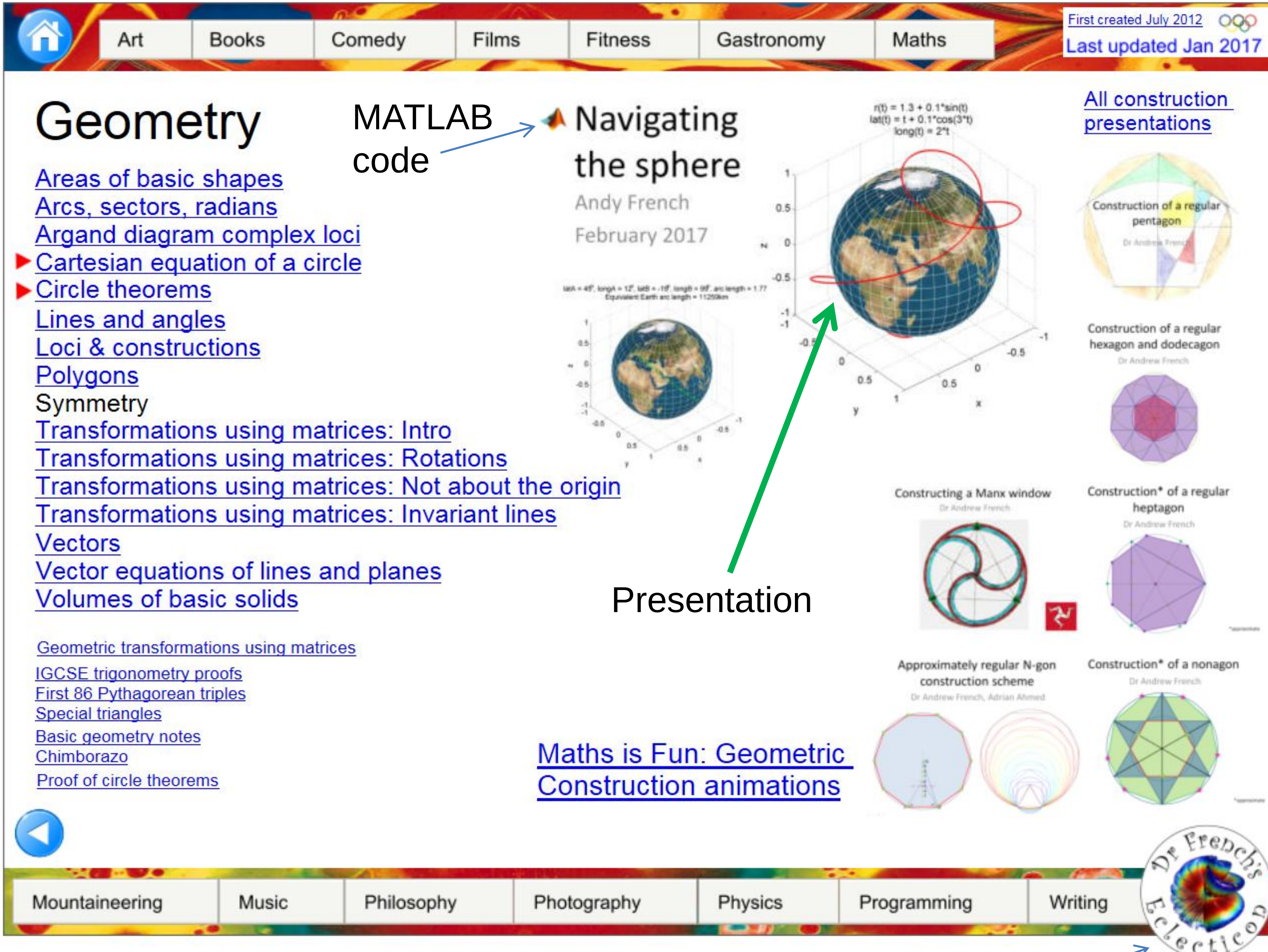

Also on **Lectures (3)** – Click on the icon here