Notes on quadratic equations

The expanded form of a quadratic equation

An equation is quadratic in variable x if it has the form

$$y = ax^2 + bx + c \tag{1}$$

where *coefficients* a, b, c are fixed numbers (or *constants*).

When $x = 0$, $y = c$ i.e. the y intercept of the quadratic is $(0, c)$

Note also the shape of the quadratic:

$$\begin{aligned} \text{If } a &> 0 \text{ the quadratic will be a smile } \cup & (2) \\ \text{If } a &< 0 \text{ the quadratic will be a frown } \cap & (3) \end{aligned}$$

The first derivative of a quadratic equation and its stationary point

$$\begin{aligned} y &= ax^2 + bx + c & (4) \\ \frac{dy}{dx} &= 2ax + b & (5) \end{aligned}$$

The stationary point of a quadratic equation (there is only one) is when the gradient $\frac{dy}{dx} = 0$.

$$\begin{aligned} 0 &= 2ax + b & (6) \\ &\Rightarrow \; x = -\frac{b}{2a} & (7) \end{aligned}$$

Substituting this value of x into the original quadratic gives the y coordinate of the stationary point

$$\begin{aligned} y &= ax^2 + bx + c & (8) \\ y &= a\left(-\frac{b}{2a}\right)^2 + b\left(-\frac{b}{2a}\right) + c & (9) \\ &= \frac{b^2}{4a} - \frac{b^2}{2a} + c & (10) \\ &= -\frac{b^2}{4a} + c & (11) \end{aligned}$$

Hence the coordinate (x_s, y_s) of the stationary point of the quadratic is

$$(x_s, y_s) = \left(-\frac{b}{2a}, -\frac{b^2}{4a} + c\right) \tag{12}$$

Getting the stationary point via completing the square

$$\begin{aligned} y &= ax^2 + bx + c && (13)\\ &= a\left\{x^2 + \frac{bx}{a}\right\} + c && (14)\\ &= a\left\{\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2}\right\} + c && (15)\\ \therefore\ y &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c && (16) \end{aligned}$$

The stationary point is when y is maximized ("frown", $a < 0$) or minimized ("smile", $a > 0$). Since $\left(x + \frac{b}{2a}\right)^2$ is always positive, this must occur when $x + \frac{b}{2a} = 0$ i.e. when $x = -\frac{b}{2a}$.

From the completed square form, we can simply read off the coordinates of the stationary point, (or the *vertex*) of the quadratic.

$$\begin{aligned} y &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c && (17)\\ (x_s, y_s) &= \left(-\frac{b}{2a}, -\frac{b^2}{4a} + c\right) && (18) \end{aligned}$$

Finding roots of the quadratic using the quadratic formula

The completed square form of the quadratic allows us to find x values that make $y = 0$. i.e. the *roots* of the quadratic, or where it crosses the x axis.

$$\begin{aligned} y &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c && (19)\\ 0 &= a\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a} + c && (20)\\ \frac{b^2}{4a} - c &= a\left(x + \frac{b}{2a}\right)^2 && (21)\\ \frac{b^2}{4a^2} - \frac{c}{a} &= \left(x + \frac{b}{2a}\right)^2 && (22)\\ \frac{b^2}{4a^2} - \frac{4ac}{4a^2} &= \left(x + \frac{b}{2a}\right)^2 && (23)\\ \frac{b^2 - 4ac}{4a^2} &= \left(x + \frac{b}{2a}\right)^2 && (24)\\ \frac{\pm\sqrt{b^2 - 4ac}}{2a} &= x + \frac{b}{2a} && (25) \end{aligned}$$

$$\therefore x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \qquad (26)$$

The discriminant $\Delta = b^2 - 4ac$

Note the discriminant

$$\Delta = b^2 - 4ac \tag{27}$$

tells us about the quadratic.

- If $b^2 - 4ac > 0$ the quadratic will have *two roots*
  i.e. it crosses the x axis twice at $x_\pm = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$

- If $b^2 - 4ac = 0$ the quadratic will have a *single root*
  i.e. it crosses the x axis once at $x = \frac{-b}{2a}$. Note this is also the vertex!

- If $b^2 - 4ac < 0$ the quadratic will have *no real roots*
  This means the quadratic is entirely above or below the x axis.

Also, if the discriminant is a square number, we should be able to to *factorize* the quadratic i.e. write it in the form

$$y = (Ax + B)(Cx + D) \tag{28}$$

where A, B, C, D are integers.

Note the roots in this case are $-\frac{B}{A}$ and $-\frac{D}{C}$ corresponding to $Ax + B = 0$ or $Cx + D = 0$.

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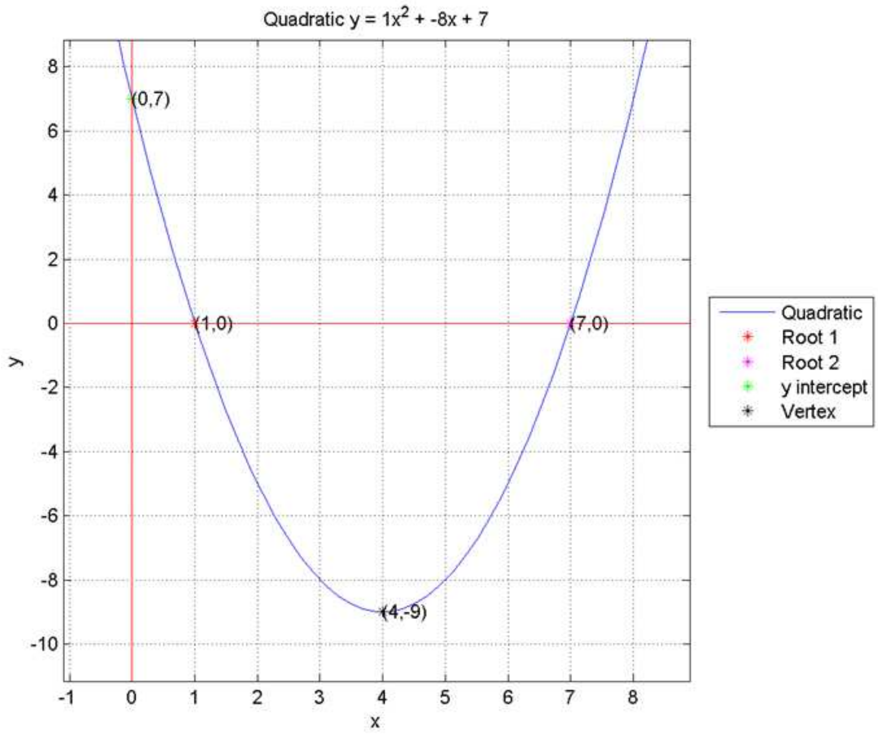

Figure 1: Quadratic equation $y = x^2 - 8x + 7 = (x-4)^2 - 9 = (x-1)(x-7)$

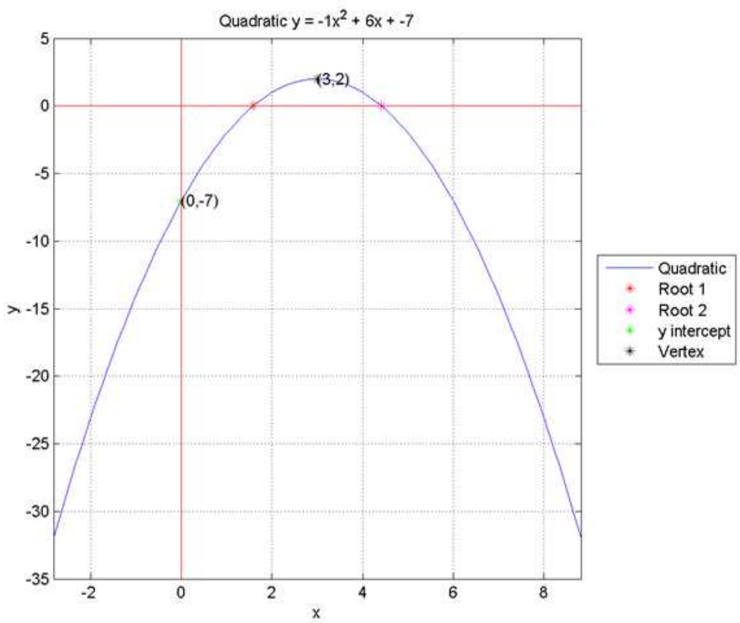

Figure 2: Quadratic equation $y = -x^2 + 6x - 7 = -(x-3)^2 + 2$