NUCLEAR FISSION & FUSION

1/ $^{92}_{36}Kr$

(i) $M_{Kr}c^2 = 36M_pc^2 + (92-36)M_nc^2 - B$

$M_{Kr} = 36M_p + 56M_n - B/c^2$

$M_{Kr} = 36 \times 1.6726 \times 10^{-27} + 56 \times 1.6749 \times 10^{-27} - \dfrac{764.77 \times 10^6 \times 1.602 \times 10^{-19}}{(2.998 \times 10^8)^2}$ (kg)

$= \boxed{1.526 \times 10^{-25}\ kg}$

$= \boxed{91.927u}$ (ie just a bit less than 92 as expected).

[$1u = 1.6605 \times 10^{-27}$ kg]

(ii) $^{235}_{92}U + ^{1}_{0}n \rightarrow ^{141}_{56}Ba + ^{92}_{36}Kr + 3 \times ^{1}_{0}n$

$\Delta E = B_{Ba} + B_{Kr} - B_U$

$\therefore B_U = B_{Ba} + B_{Kr} - \Delta E$

$B_U = (1145.36 + 764.77 - 173.28)$ MeV

$\boxed{B_U = 1736.85\ MeV}$

(iii) Assume mass of Uranium 235 is 235u (any mass
a) defect is negligible for this specific energy calculation).

$\therefore$ Energy released / kg of U-235 is $\dfrac{173.28 \times 10^6 \times 1.602 \times 10^{-19}\ J}{235 \times 1.6605 \times 10^{-27}\ kg}$

$= \boxed{7.11 \times 10^{13}\ J/kg}$

b) $\dfrac{\text{Energy density of U-235}}{\text{Energy density of natural gas}} \simeq \dfrac{7.11\times10^{13}}{4.9\times10^{7}} \simeq \boxed{1.45\times10^{6}}$

ie about 1.5 <u>million</u> times more energy dense!

c) $\dfrac{10^{19}\,J}{7.11\times10^{13}} \div 1000$ = tonnes of U-235 to correspond to 10^{19} J/year consumed by UK.

$= \boxed{140.6 \text{ tonnes}}$

{ ie a relatively small fleet of trucks, or a train, could easily transport this mass. Uranium has a density of $19.1\,g/cm^3 = 19{,}100\,kg/m^3$. So this is a volume of $140.6\times10^{3}/19100\ m^3 = \boxed{7.36\ m^3}$

ie a cube $(1.44\,m)^3$! }

(iv) $\underbrace{{}^{2}_{1}H}_{D} + \underbrace{{}^{3}_{1}H}_{T} \rightarrow {}^{4}_{2}He + {}^{1}_{0}n$ fusion reaction releases $\Delta E = 17.59$ MeV

$\Delta E = B_{He} - B_D - B_T \quad \therefore \quad B_T = B_{He} - B_D - \Delta E$

$B_{He} = 27.27$ MeV $\qquad B_D = M_p c^2 + M_n c^2 - 2.0141\,u/c^2$

$B_D = (938.3 + 939.6 - 2.0141 \times 931.5)$ MeV

$\Rightarrow B_T = 27.27 - 1.7659 - 17.59$ (MeV)

$\boxed{B_T = 7.91 \text{ MeV}}$

{ NUBASE yields 7.9707 MeV } Difference results from higher precision of M_p, M_n etc. }

v) Deuterium – Tritium fusion reaction produces 17.59 MeV for $\simeq 2u$ of deuterium and $3u$ of tritium.

[You can be more accurate if you use the binding energies of D and T calculated above → these give 2.0141u and 3.016u respectively].

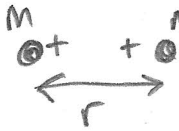
So energy density (in J/kg) is: $\dfrac{17.59 \times 10^6 \times 1.602 \times 10^{-19}\ J}{(2.0141 + 3.016) \times 1.6605 \times 10^{-27}\ kg}$

$= \boxed{3.37 \times 10^{14}\ J/kg}$

So in terms of tonnes to supply 10^{19} J to UK for a year:

⇒ 29.6 tonnes of fuel. (11.9 tonnes of deuterium and 17.7 tonnes of tritium).

vi) a)



Potential energy of two protons r apart is $\dfrac{e^2}{4\pi\varepsilon_0 r}$

if this equates to heat energy of the protons due to random, thermal motion at temperature T:

$$\frac{3}{2} k_B T \simeq \frac{e^2}{4\pi\varepsilon_0 r}$$

{ Calc. is 'order of magnitude' so 3/2 prefix of k_BT is largely irrelevant }

$$\Rightarrow \boxed{T \simeq \frac{e^2}{6\pi\varepsilon_0 k_B r}}$$

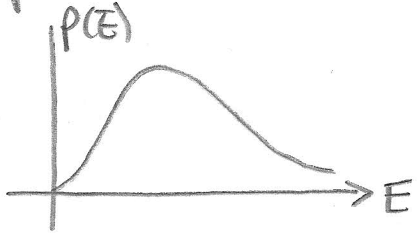
$$T \simeq \frac{(1.602 \times 10^{-19})^2}{6\pi \times 8.85 \times 10^{-12} \times 1.38 \times 10^{-23} \times 10^{-15}}$$

$$\simeq \boxed{1.1 \times 10^{10}\ K}$$

ie about 11 billion kelvin (!) to initiate 'thermal fusion' in a reactor on Earth.

b) The core temperature of the Sun $\simeq$ 15 million K.

Now energy of protons in the Sun will follow a distribution



$\{$ Maxwell – Boltzmann $\}$

$P(E) = \frac{2}{\sqrt{\pi}} (k_B T)^{-3/2} \sqrt{E}\, e^{-E/k_B T}$

where the average energy $\simeq \frac{3}{2} k_B \times 15 \times 10^6$ J

However, an increasingly small fraction will have much higher energies, even those 1000× times higher.

Since the Sun is MASSIVE, even a tiny fraction may be sufficient to initiate fusion.

[$L_\odot \simeq 3.828 \times 10^{26}$ J/s and the mass of the Sun is $M_\odot \simeq 1.99 \times 10^{30}$ kg. If $L_\odot$ derives from D + T fusion of 17.59 MeV, $L_\odot$ means $\frac{3.828 \times 10^{26} \text{ J/s}}{17.59 \times 10^6 \times 1.602 \times 10^{-19} \text{ J/reaction}}$

$= \boxed{1.358 \times 10^{38}}$ reactions a second.

Now there are $\simeq \frac{1.99 \times 10^{30}}{1.673 \times 10^{-27}}$ protons in the Sun ← mostly hydrogen.

and ∴ this number / 5 = possible D + T reactions*.

So fraction of possible reactions / s is: $\frac{1.358 \times 10^{38}}{\frac{1}{5} \times \frac{1.99 \times 10^{30}}{1.673 \times 10^{-27}}}$

$= \boxed{5.71 \times 10^{-19}}$ ie a very small fraction.]

* D and T are actually synthesized from ^1_1H in the "pp cycle" in our Sun. ie the D+T reaction is a simplification.

(vii) $^{239}_{94}Pu$ $Z = 94$ Even ; $A - Z = 145$ ODD } so no pairing term in SEMF.

$$\therefore B_{Pu} = 15.76 \times 239 - 17.81 \times 239^{2/3} - \frac{0.711 \times 94^2}{239^{1/3}} - \frac{23.702\,(239 - 2\times 94)^2}{239} \quad \text{(MeV)}$$

$$= \boxed{1810.45 \text{ MeV}}$$

{ ie $B/A \simeq 7.58$ MeV
NUBASE ⇒ 7.56 so close! }

(viii) Pu-239 fission:

$$\boxed{^{239}_{94}Pu + {}^{1}_{0}n \rightarrow {}^{134}_{54}Xe + {}^{103}_{40}Zr + 3 \times {}^{1}_{0}n}$$

$$B_{Zr} = 15.76 \times 103 - 17.81 \times 103^{2/3} - \frac{0.711 \times 40^2}{103^{1/3}} - \frac{23.702\,(103 - 2\times 40)^2}{103}$$

$$= \boxed{867.52 \text{ MeV}}$$

$^{103}_{40}Zr$ $Z = 40$ EVEN ; $A - Z = 63$ ODD } So no pairing term in SEMF

$$\Delta E = B_{Xe} + B_{Zr} - B_{Pu}$$

$$\Delta E = 8.4137 \times 134 + 867.52 - 1810.45 \quad \text{MeV}$$

$$= \boxed{184.51 \text{ MeV}}$$

So that is $\dfrac{184.51 \times 10^6 \times 1.602 \times 10^{-19}\text{ J}}{239 \times 1.6605 \times 10^{-27}\text{ kg}} = \boxed{7.45 \times 10^{13}\text{ J/kg}}$

(This is 4.8% better than U-235 fission).
↑ 7.11×10^{13} J/kg.

(ix) Aneutronic fusion example: $\boxed{{}^2_1H + {}^6_3Li \rightarrow 2 \times {}^4_2He}$

a) $\Delta E = 2B_{He} - B_D - B_{Li}$

using SEMF calculator {I made a spreadsheet!} A = 6, Z = 3

$$\Delta E = 2 \times \underbrace{27.27}_{B_{He} \text{ from (iv)}} - \underbrace{1.7659}_{B_D \text{ from (iv)}} - 23.3624 \quad \text{(MeV)}$$

$\boxed{\Delta E = 29.41 \text{ MeV}}$

This is somewhat higher (31%) than the 'actual' value of $\boxed{22.4 \text{ MeV}}$.

B_{Li} (since Z, A small) is not a great estimate using the SEMF. It is (periodictable.com) 31.994 MeV.
This yields $\Delta E = 20.78$ MeV ....
it also yields $B_D = 2.2240$ MeV !
whereas NUBASE has $B_D = 1.7135$ MeV .....

{careful with internet calculations!}

$\boxed{\text{NUBASE: } B_{Li} = 30.4608 \text{ MeV}}$

$\Rightarrow \boxed{\Delta E = 22.3 \text{ MeV}}$ CLOSE ENOUGH!

b) ${}^2_1H + {}^6_3Li$ is likely to suffer much more coulomb repulsion than ${}^2_1H + {}^3_1H$ due to ${}^6_3Li$ having 3× the number of protons as ${}^3_1H$. Although the shell structure of the nucleus will make the exact calculation complicated, we might expect the potential energy to be 3× higher and ∴ the temperature for fusion to be 3× higher.

(x) If one ignores the pairing term, and if $\boxed{A = 2Z}$ $\Rightarrow$ asymmetry term vanishes too:

SEMF is: $B/A = a_V - a_S A^{-\frac{1}{3}} - a_C Z^2 A^{-\frac{4}{3}}$

So if $A = 2Z$: $\boxed{B/A = a_V - a_S 2^{-\frac{1}{3}} Z^{-\frac{1}{3}} - a_C 2^{-\frac{4}{3}} Z^{\frac{2}{3}}}$

$$\frac{d(B/A)}{dZ} = \frac{1}{3} a_S 2^{-\frac{1}{3}} Z^{-\frac{4}{3}} - \frac{2}{3} a_C 2^{-\frac{4}{3}} Z^{-\frac{1}{3}}$$

$\therefore$ maxima of B/A when $\frac{d(B/A)}{dZ} = 0$

$$\Rightarrow \frac{1}{3} a_S 2^{-\frac{1}{3}} Z^{-\frac{4}{3}} = \frac{2}{3} a_C 2^{-\frac{4}{3}} Z^{-\frac{1}{3}}$$

$$\frac{a_S}{2a_C} \times 2 = Z$$

$$\therefore Z = a_S/a_C = \frac{17.81}{0.711} \simeq \boxed{25}$$

$^{55}_{25}Mn$ is the most stable isotope of $\boxed{\text{Manganese}}$.

{HOWEVER!}

$^{56}_{26}Fe$ is typically quoted as the most stable isotope i.e. with the largest B/A value. The highest is actually $\boxed{^{62}_{28}Ni}$ with $B/A = 8.7945$ MeV.

{actually most elements have $A > 2Z$}

Now in the SEMF, if $Z = a_S/a_C$ and $A = 2Z$

$$\left(\frac{B}{A}\right)_{max} = a_V - a_S 2^{-\frac{1}{3}}\left(\frac{a_S}{a_C}\right)^{-\frac{1}{3}} - a_C 2^{-\frac{4}{3}}\left(\frac{a_S}{a_C}\right)^{\frac{2}{3}}$$

$$= 15.76 - 17.81 \times 2^{-\frac{1}{3}}\left(\frac{17.81}{0.711}\right)^{-\frac{1}{3}} - 0.711 \times 2^{-\frac{4}{3}}\left(\frac{17.81}{0.711}\right)^{\frac{2}{3}}$$

$$= \boxed{8.51 \text{ MeV}}$$

(Q3) → See spreadsheet.

[This also includes the SEMF calculator].

$^{56}_{26}Fe$ has $B/A = 8.77$ MeV

(if ignore pair factor this is 8.74 MeV).

$^{62}_{28}Ni$ has $B/A = \boxed{8.79 \text{ MeV}}$

2/ $^{239}_{94}Pu + ^{1}_{0}n \rightarrow ^{240}_{94}Pu + \gamma$ (27% of the time as Pu-239 fission)

For $^{240}_{94}Pu$: $B = 1817.17$ MeV
(NUBASE yields 1765.41 MeV)

$^{239}_{94}Pu$: $B = 1810.45$ MeV
(NUBASE yields 1758.88 MeV)

∴ If all binding energy change results in energy of γ ray (a single photon)

$$E_\gamma = \Delta B = 6.72 \text{ MeV}$$

[NUBASE: $E_\gamma = 6.53$ MeV]

So since $E_\gamma = \frac{hc}{\lambda} \Rightarrow \boxed{\lambda = \frac{hc}{E_\gamma}}$

The smallest λ is when E_γ = all the ΔB

So using SEMF: $\lambda_{min} = \frac{6.63 \times 10^{-34} \times 2.998 \times 10^{8}}{6.72 \times 10^{6} \times 1.602 \times 10^{-19}}$ (m)

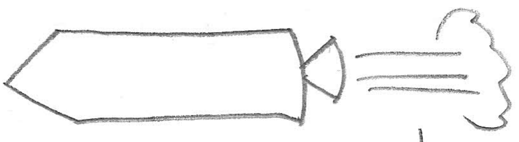
$$= \boxed{1.85 \times 10^{-13} \text{ m}}$$

Note a ready reckoner for γ rays of energy E/MeV

$$\boxed{\lambda = 1.241\times10^{-12}\,\text{m}\,/\,(E/\text{MeV})}$$

3/ See spreadsheet.

4/ Fusion engine!



$^{2}_{1}H + {}^{3}_{1}H \rightarrow {}^{4}_{2}He + {}^{1}_{0}n$ is the proposed reaction.

$\Delta E = 17.59\,\text{MeV}$ per reaction.

Assume this is the KE of the $^{4}_{2}He$ nuclei, which are ionized ($^{4}_{2}He^{2+}$) and fired out the back of ship

Relativistic calculation for speed of $^{4}_{2}He$ ions v

$$\left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}} mc^2 = \Delta E + mc^2 \qquad [\text{"}E=\gamma mc^2\text{"}]$$

$$\therefore \left(\frac{\Delta E + mc^2}{mc^2}\right)^{-2} = 1 - \frac{v^2}{c^2} \qquad \therefore v = c\sqrt{1 - \frac{mc^2}{\Delta E + mc^2}}$$

$$\boxed{v = c\sqrt{1 - \frac{1}{1+\Delta E/mc^2}}}$$

So $$\Delta E/mc^2 \simeq \frac{17.59\times10^{6}\times1.602\times10^{-19}\,\text{J}}{4.00\times1.6605\times10^{-27}\,\text{kg}\times(2.998\times10^{8}\,\text{m/s})^2} \simeq \boxed{4.72\times10^{-3}}$$

NUBASE : $M_{He} = 4.0026\,u$

SEMF : $M_{He} = 4.0014\,u$

So actually a classical calculation is probably ok, ish...

$v = 0.069c$ ie

$$\boxed{v = 2.055 \times 10^{7}} \text{ m/s}$$

Classical calc: $\frac{1}{2}Mv^2 = \Delta E$

$$\therefore v = \sqrt{2\Delta E/M}$$

$$v = \sqrt{\frac{2 \times 17.59 \times 10^{6} \times 1.602 \times 10^{-19}}{4 \times 1.6605 \times 10^{-27}}}$$

$$\boxed{v = 2.913 \times 10^{7} \text{ m/s}}$$

(ie about 42% too high).

Now if 1 kg of fuel is used, this is

$$\frac{1.00 \text{ kg}}{5 \times 1.6605 \times 10^{-27}} = 1.204 \text{ reactions}$$

So if M kg/s are used # reactions is $\frac{M}{5u}$

Total momentum transferred from rocket is $Mv \times \frac{4}{5}$ per second which is the thrust of the rocket. ($\frac{M}{5u} \times$ mass of ${}^{4}_{2}\text{He}$)
↑
4u

So thrust $\boxed{F = \frac{M}{\text{kg/s}} \times \frac{4}{5} \times 2.055 \times 10^{7} \text{ N}}$

Now the AJ10 rocket engine used in the Apollo missions has a thrust of ≈ 43.7 kN. $\therefore$ $\boxed{M \simeq 2.66 \text{ g/s}}$ is needed to supply equivalent thrust for a fusion engine!