Arithmetic tricks using 'semi-algebra'

A royal road to mastery and appreciation of algebraic techniques is to **start with small positive integers and their powers.** The idea is to use algebraic ideas (factorizing, expanding brackets , difference of two squares...) to simplify otherwise laborious *sans calculator* arithmetic.

Why sans calculator? Well the point is to force you to learn these techniques! Think of it as an intellectual puzzle or a game.

Difference of two squares

$$\boxed{a^2 - b^2 = (a+b)(a-b)}$$

Unless you have memorized the squares, a difference between two squares can sometimes be easier to calculate in factorized form, especially when $a + b$ or $a - b$ is a power of ten.

$$17^2 - 7^2 = (17+7)(17-7) = 24 \times 10 = \boxed{240}$$

$$943^2 - 57^2 = (943+57)(943-57) = 1{,}000 \times 886 = \boxed{886{,}000}$$

$$38.4^2 - 11.6^2 = \left(\tfrac{384}{10}\right)^2 - \left(\tfrac{116}{10}\right)^2 = \tfrac{1}{100}\left(384^2 - 116^2\right)$$

$$= \tfrac{1}{100}(384+116)(384-116) = \tfrac{1}{100} \times 500 \times 268 = \frac{2680}{2} = \boxed{1{,}340}$$

How to set problems like this:

$$a + b = 10^n$$

$$\therefore \boxed{b = 10^n - a}$$

$$\therefore a - b = 2a - 10^n$$

$$\therefore a^2 - b^2 = (a+b)(a-b) = 10^n\left(2a - 10^n\right)$$

Choose integers n and a where a is any integer smaller than 10^n

$$a = 83,\ n = 2,\quad b = 10^2 - 83 = \boxed{17}$$

$$\therefore 83^2 - 17^2 = 100(2 \times 83 - 100) = 6{,}600$$

$$\therefore 83^2 - 17^2 = (83+17)(83-17) = 100 \times 66 = \boxed{6{,}600}$$

$$a = 666,\ n = 3,\quad b = 10^3 - 666 = \boxed{334}$$

$$\therefore 666^2 - 334^2 = (1000)(332) = \boxed{332{,}000}$$

X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	X
20	20	40	60	80	100	120	140	160	180	200	220	240	260	280	300	320	340	360	380	400	20
19	19	38	57	76	95	114	133	152	171	190	209	228	247	266	285	304	323	342	361	380	19
18	18	36	54	72	90	108	126	144	162	180	198	216	234	252	270	288	306	324	342	360	18
17	17	34	51	68	85	102	119	136	153	170	187	204	221	238	255	272	289	306	323	340	17
16	16	32	48	64	80	96	112	128	144	160	176	192	208	224	240	256	272	288	304	320	16
15	15	30	45	60	75	90	105	120	135	150	165	180	195	210	225	240	255	270	285	300	15
14	14	28	42	56	70	84	98	112	126	140	154	168	182	196	210	224	238	252	266	280	14
13	13	26	39	52	65	78	91	104	117	130	143	156	169	182	195	208	221	234	247	260	13
12	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228	240	12
11	11	22	33	44	55	66	77	88	99	110	121	132	143	154	165	176	187	198	209	220	11
10	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200	10
9	9	18	27	36	45	54	63	72	81	90	99	108	117	126	135	144	153	162	171	180	9
8	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128	136	144	152	160	8
7	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105	112	119	126	133	140	7
6	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114	120	6
5	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100	5
4	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80	4
3	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60	3
2	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	2
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	1
X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	X

Perfect squares

$$\boxed{(a+b)^2 = a^2 + 2ab + b^2}$$

$$176^2 + 2\times176\times24 + 24^2 = (176+24)^2 = 200^2 = \boxed{40{,}000}$$

$$87^2 - \underbrace{74}_{2\times37}\times87 + 37^2 = (87-37)^2 = 50^2 = \boxed{2{,}500}$$

Perfect cubes

$$\boxed{(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3}$$

$$1331 + 363\times9 + 33\times81 + 729$$
$$= 11^3 + 3\times11^2\times9 + 3\times11\times9^2 + 9^3$$
$$= (11+9)^3 = 20^3 = \boxed{8{,}000}$$

$$\begin{array}{c}
1 \\
1 \quad 1 \\
\boxed{1 \quad 2 \quad 1} \\
\boxed{1 \quad 3 \quad 3 \quad 1} \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1
\end{array}$$

Pascal's Triangle yields the *binomial coefficients*

Difference of two cubes

$$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3$$
$$\therefore (a-b)^3 + 3a^2b - 3ab^2 = a^3 - b^3$$
$$\therefore (a-b)^3 + 3ab(a-b) = a^3 - b^3$$
$$\therefore (a-b)\left((a-b)^2 + 3ab\right) = a^3 - b^3$$
$$\therefore \boxed{(a-b)(a^2 + ab + b^2) = a^3 - b^3}$$

This is a useful *identity* to remember

$$\frac{123^3 - 23^3}{123^2 + 123\times23 + 23^2} = 123 - 23 = \boxed{100}$$

$$123^2 = (120+3)^2 = 120^2 + 2\times120\times3 + 9 = 14{,}400 + 720 + 9 = \boxed{15{,}129}$$

$$567^2 = (600-33)^2 = 600^2 - 2\times600\times33 + 33^2$$

Think what sum or difference of two numbers makes our next steps easier!

$$33^2 = (30+3)^2 = 30^2 + 2\times30\times3 + 3^2 = 900 + 180 + 9 = \boxed{1{,}089}$$

$$\therefore 567^2 = 360{,}000 - 39{,}600 + 1{,}089 = \boxed{321{,}489}$$

Note knowledge of the perfect square formula shortcuts the 'grid multiplication' idea, increasing fluency. Being able to perform calculations faster also can make the exercise more fun, and builds confidence.

Grid multiplication breaks down the multiplication calculation into easy steps if you know your multiplication tables up to 100 and can multiply by ten.

	500	60	7
500	250,000	30,000	3,500
60	30,000	3,600	420
7	3,500	420	49

+	
	250,000
	30,000
	3,500
	30,000
	3,600
	420
	3,500
	420
	49
	321,489

Importantly it visualizes all the 'cross terms' in the calculation (i.e. the off-diagonals)

However, there is much more to write down and then sum.

If you can spot an arithmetic shortcut – use it! You can always go back to a 'slow but sure' method like grid multiplication if you can't spot a shortcut.

You can clearly see that:

$$567^2 \neq 500^2 + 60^2 + 7^2$$