Fibonacci series and the Golden Ratio

The *Fibonacci series* is determined by the iteration:

$$F_0 = 0, \; F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2}$$

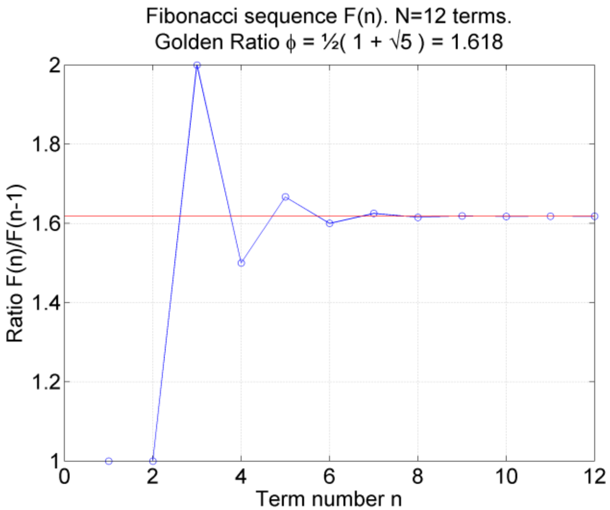
Define the ratio: $$R_n = \frac{F_n}{F_{n-1}}$$

Leonardo Fibonacci
1170-1250

```
n = 1,  F(1) = 1,    R(1)  = 1
n = 2,  F(2) = 1,    R(2)  = 1
n = 3,  F(3) = 2,    R(3)  = 2
n = 4,  F(4) = 3,    R(4)  = 1.5
n = 5,  F(5) = 5,    R(5)  = 1.6667
n = 6,  F(6) = 8,    R(6)  = 1.6
n = 7,  F(7) = 13,   R(7)  = 1.625
n = 8,  F(8) = 21,   R(8)  = 1.6154
n = 9,  F(9) = 34,   R(9)  = 1.619
n = 10, F(10) = 55,  R(10) = 1.6176
n = 11, F(11) = 89,  R(11) = 1.6182
n = 12, F(12) = 144, R(12) = 1.618
```

The ratio of sequential terms, R, converges towards the **Golden Ratio** ϕ.

$$\phi = \tfrac{1}{2}\left(1+\sqrt{5}\right) \approx 1.618$$



Binet's Formula can be used to determine the n^{th} term of the Fibonacci series:

$$F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}$$

Jacques Binet
1786-1856

$$\sqrt{5}\left(F_{n-1}+F_{n-2}\right) = \phi^{n-1} - (1-\phi)^{n-1} + \phi^{n-2} - (1-\phi)^{n-2}$$
$$= \phi^n\left(\phi^{-1}+\phi^{-2}\right) - (1-\phi)^n\left(\frac{1}{1-\phi}+\frac{1}{(1-\phi)^2}\right)$$

$$\phi^{-1} = \frac{2}{1+\sqrt{5}} = \frac{2-2\sqrt{5}}{1-5} = \tfrac{1}{2}\left(\sqrt{5}-1\right)$$
$$\therefore \phi^{-2} = \tfrac{1}{4}\left(\sqrt{5}-1\right)^2 = \tfrac{1}{4}\left(5-2\sqrt{5}+1\right) = \tfrac{1}{2}\left(3-\sqrt{5}\right)$$
$$\therefore \phi^{-1}+\phi^{-2} = \tfrac{1}{2}\left(\sqrt{5}-1+3-\sqrt{5}\right) = \tfrac{2}{2} = \boxed{1}$$

$$1-\phi = \tfrac{2}{2} - \tfrac{1}{2}\left(1+\sqrt{5}\right) = \tfrac{1}{2}\left(1-\sqrt{5}\right)$$
$$\frac{1}{1-\phi} = \frac{2}{1-\sqrt{5}} = \frac{2+2\sqrt{5}}{1-5} = -\tfrac{1}{2}\left(1+\sqrt{5}\right) = -\phi$$
$$\therefore \frac{1}{(1-\phi)^2} = \tfrac{1}{4}\left(1+5+2\sqrt{5}\right) = \tfrac{1}{2}\left(3+\sqrt{5}\right)$$
$$\therefore \frac{1}{1-\phi} + \frac{1}{(1-\phi)^2} = \tfrac{1}{2}\left(-1-\sqrt{5}+3+\sqrt{5}\right) = \tfrac{2}{2} = \boxed{1}$$

Therefore:

$$\sqrt{5}\left(F_{n-1}+F_{n-2}\right) = \phi^n(1) - (1-\phi)^n(1) = \sqrt{5}F_n$$
$$\therefore \boxed{F_n = F_{n-1}+F_{n-2}}$$

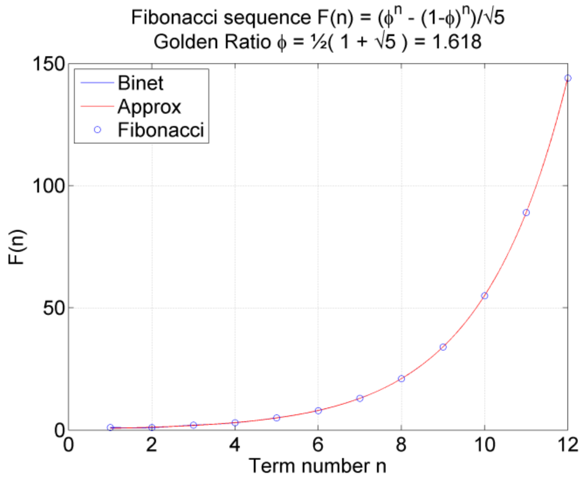
i.e. **Binet's formula** is consistent with the iterative definition of the Fibonacci series.

$$R_n = \frac{F_n}{F_{n-1}} = \frac{\phi^n - (1-\phi)^n}{\phi^{n-1} - (1-\phi)^{n-1}}$$
$$= \frac{\phi^{n-1}}{\phi^{n-1}} \frac{\phi\left(1-\left(\frac{1}{\phi}-1\right)^n\right)}{\left(1-\left(\frac{1}{\phi}-1\right)^{n-1}\right)} = \frac{\phi\left(1-\left(\frac{1}{\phi}-1\right)^n\right)}{\left(1-\left(\frac{1}{\phi}-1\right)^{n-1}\right)}$$
$$\therefore \boxed{\lim_{n\to\infty}\left(\frac{F_n}{F_{n-1}}\right) \to \phi}$$

The ratio of sequential terms, R, converges towards the **Golden Ratio** ϕ.

$$1-\phi = \tfrac{1}{2}\left(1-\sqrt{5}\right) \approx -0.618 \quad \text{so:}$$
$$(1-\phi)^n \to 0 \;\text{ as }\; n \to \infty$$

For large n: $$F_n \to \frac{\phi^n}{\sqrt{5}}$$



Binet's Formula can be used to determine the n^{th} term of the Fibonacci series:

Jacques Binet
1786-1856

$$F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}}$$

$$\phi = \tfrac{1}{2}\left(1+\sqrt{5}\right) \approx 1.618$$

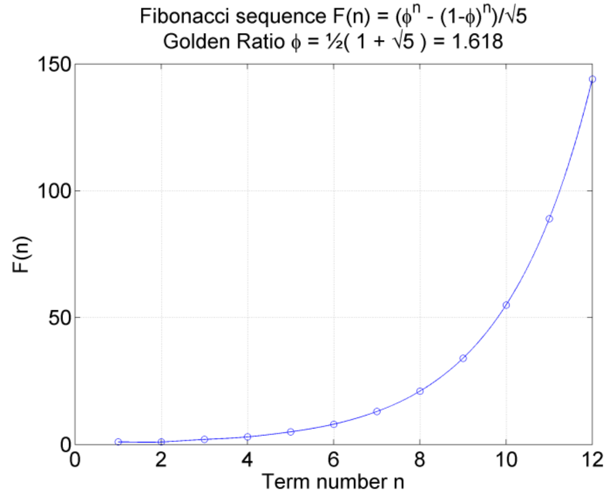
Golden Ratio

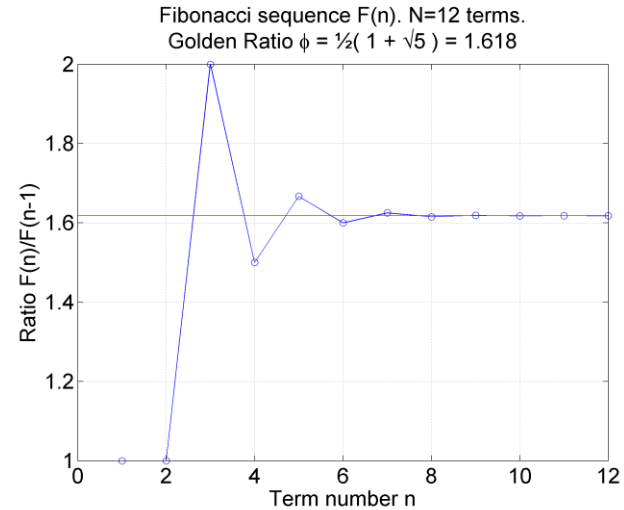
$$\phi^{-1} = \frac{2}{1+\sqrt{5}} = \frac{2-2\sqrt{5}}{1-5} = \tfrac{1}{2}\left(\sqrt{5}-1\right)$$

$$1-\phi = \tfrac{2}{2} - \tfrac{1}{2} - \tfrac{1}{2}\sqrt{5} = -\tfrac{1}{2}\left(\sqrt{5}-1\right) = -\tfrac{1}{\phi}$$

$$\therefore \phi - 1 = \tfrac{1}{\phi}$$

$$\therefore F_n = \frac{\phi^n - (1-\phi)^n}{\sqrt{5}} = \frac{\phi^n - \left(-\frac{1}{\phi}\right)^n}{\sqrt{5}}$$





Sum of Fibonacci series

$$S_N = \sum_{n=0}^{N-1} F_n + F_N = \tfrac{1}{\sqrt{5}}\left(1+\phi+\phi^2+\ldots+\phi^{N-1}\right) + \tfrac{1}{\sqrt{5}}\phi^N$$

$$-\tfrac{1}{\sqrt{5}}\left(1+\left(-\tfrac{1}{\phi}\right)+\left(-\tfrac{1}{\phi}\right)^2+\ldots+\left(-\tfrac{1}{\phi}\right)^{N-1}\right) - \tfrac{1}{\sqrt{5}}\left(-\tfrac{1}{\phi}\right)^N$$

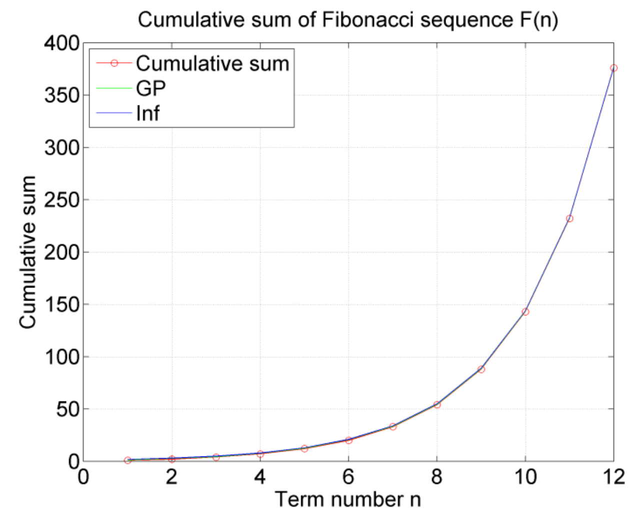
$$= \tfrac{1}{\sqrt{5}}\frac{\phi^N-1}{\phi-1} - \tfrac{1}{\sqrt{5}}\frac{\left(-\frac{1}{\phi}\right)^N-1}{-\frac{1}{\phi}-1} + \tfrac{1}{\sqrt{5}}\left(\phi^N - \left(-\tfrac{1}{\phi}\right)^N\right)$$

Geometric progression: $$a + ar + ar^2 + \ldots + ar^{n-1} = a\frac{r^n - 1}{r-1}$$

$$= \tfrac{1}{\sqrt{5}}\left(\frac{\phi^N-1}{\phi-1} - \frac{\left(-\frac{1}{\phi}\right)^N-1}{1-\phi-1}\right) + \tfrac{1}{\sqrt{5}}\left(\phi^N - \left(-\tfrac{1}{\phi}\right)^N\right) = \tfrac{1}{\sqrt{5}}\left(\frac{\phi^N-1}{\phi-1} - \frac{\left(-\frac{1}{\phi}\right)^N-1}{\phi}\right) + \tfrac{1}{\sqrt{5}}\left(\phi^N - \left(-\tfrac{1}{\phi}\right)^N\right)$$

$$\therefore S_{N\to\infty} \to \tfrac{1}{\sqrt{5}}\left(\frac{\phi^N}{\phi-1} + \frac{1}{\phi}\right) + \tfrac{1}{\sqrt{5}}\phi^N \to \tfrac{1}{\sqrt{5}}\phi^N\left(\frac{1}{\phi-1} + 1 + \phi^{-N-1}\right) \to \tfrac{\phi}{(\phi-1)\sqrt{5}}\phi^N$$

$$\therefore S_{N\to\infty} = \tfrac{1}{\sqrt{5}}\phi^{N+2} \approx 1.1708\phi^N$$



Using asymptotic result:

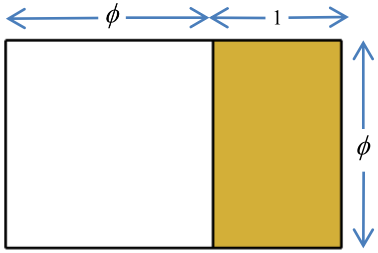
$$F_n \to \frac{\phi^n}{\sqrt{5}}$$

$$S_N \approx \sum_{n=0}^{N-1} F_n + F_N = \tfrac{1}{\sqrt{5}}\left(1+\phi+\phi^2+\ldots+\phi^{N-1}\right) + \tfrac{1}{\sqrt{5}}\phi^N = \tfrac{1}{\sqrt{5}}\left(\frac{\phi^N-1}{\phi-1}\right) + \tfrac{1}{\sqrt{5}}\phi^N = \tfrac{1}{\sqrt{5}}\phi^N\left(\tfrac{1}{\phi-1}+1\right) - \tfrac{1}{\sqrt{5}}\frac{1}{\phi-1}$$

$$= \tfrac{1}{\sqrt{5}}\phi^N(\phi+1) - \tfrac{1}{\sqrt{5}}\phi \quad \therefore S_{N\to\infty} \to \tfrac{\phi+1}{\sqrt{5}}\phi^N \approx 1.1708\phi^N$$

Note: $\phi + 1 = \phi^2$

Golden Rectangle and Golden Ratio



$$\frac{\phi}{1} = \frac{1+\phi}{\phi} =$$

$$\phi^2 = 1+\phi$$

$$\phi^2 - \phi - 1 = 0$$

$$\left(\phi - \tfrac{1}{2}\right)^2 - \tfrac{1}{4} - 1 = 0$$

$$\phi = \tfrac{1}{2} \pm \tfrac{\sqrt{5}}{2}$$

If we assert that $\phi > 1$ we can take the positive root

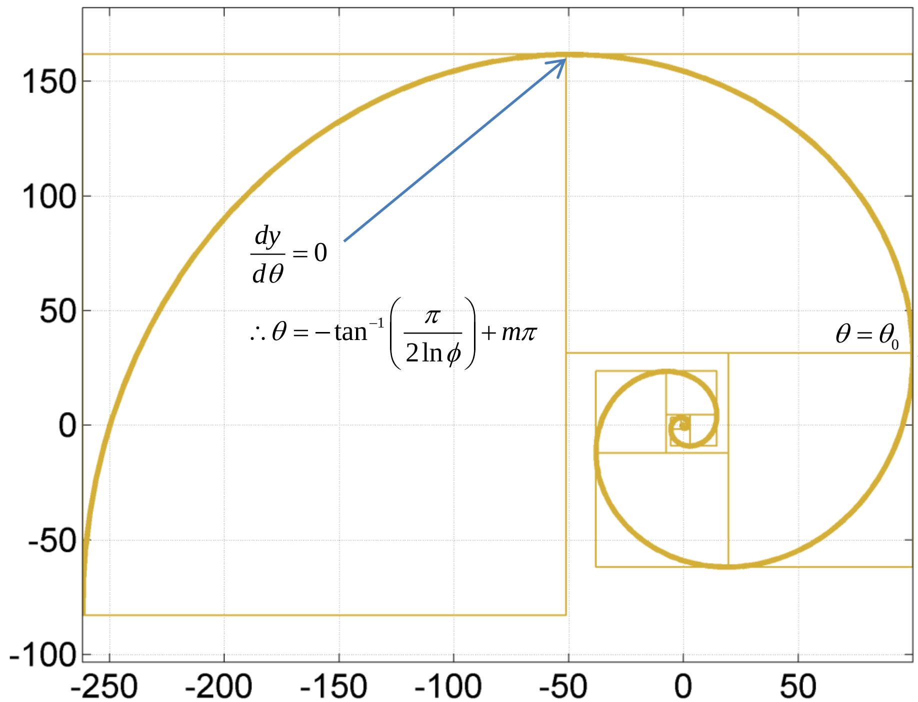
$$\phi = \tfrac{1}{2}\left(1+\sqrt{5}\right)$$

$$r = a\phi^{\frac{2\theta}{\pi}} \; ; \; a = \frac{100}{\phi^{\frac{2\theta_0}{\pi}} \cos\theta_0} \; ; \; \theta_0 = \tan^{-1}\left(\frac{2\ln\phi}{\pi}\right) \approx 0.2973 = 17.03^{\circ}$$

$$x = r\cos\theta \; ; \; y = r\cos\theta$$

$$\theta = -16\pi \; ...... \; \theta_0 + \pi$$

Golden spiral: r = 95.488$\phi^{2\theta/\pi}$. ϕ = ½(1 + √5)



$$\frac{dx}{d\theta} = 0 \quad \therefore \theta = \tan^{-1}\left(\frac{2\ln\phi}{\pi}\right) + n\pi$$

$(100,0)$

$$r = a\phi^{\frac{2\theta}{\pi}}$$

$$a = \frac{100}{\phi^{\frac{2\theta_0}{\pi}} \cos\theta_0}$$

$$\theta_0 = \tan^{-1}\left(\frac{2\ln\phi}{\pi}\right)$$

$$x = 100, y = 0, \theta = \theta_0$$

Set scaling factor a such that golden spiral passes through (100,0).

Find the coordinates of the spiral turns

$$r = ae^{\frac{2\theta\ln\phi}{\pi}} \quad \therefore \frac{dr}{d\theta} = \frac{2\ln\phi}{\pi} r \quad \longleftarrow \quad \phi = e^{\ln\phi}$$

$$x = r\cos\theta \; ; \; y = -r\sin\theta$$

$$\frac{dx}{d\theta} = -r\sin\theta + \frac{2\ln\phi}{\pi} r\cos\theta$$

$$\frac{dy}{d\theta} = -r\cos\theta - \frac{2\ln\phi}{\pi} r\sin\theta$$

$$\frac{dx}{d\theta} = 0 \Rightarrow \tan\theta = \frac{2\ln\phi}{\pi} \quad \therefore \boxed{\theta = \tan^{-1}\left(\frac{2\ln\phi}{\pi}\right) + n\pi}$$

$$\frac{dy}{d\theta} = 0 \Rightarrow \tan\theta = -\frac{\pi}{2\ln\phi} \quad \therefore \boxed{\theta = -\tan^{-1}\left(\frac{\pi}{2\ln\phi}\right) + m\pi}$$

m,*n* are integers

$$\therefore \boxed{\theta = \theta_0 + \left(m - \tfrac{1}{2}\right)\pi}$$

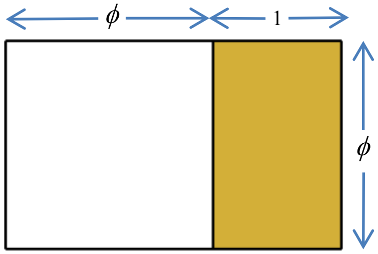
are angles at spiral turning points.

Note identity:

$$\tan^{-1}\alpha + \tan^{-1}\tfrac{1}{\alpha} = \tan^{-1}\left(\frac{\alpha + \frac{1}{\alpha}}{1 - \alpha\frac{1}{\alpha}}\right) = \tfrac{\pi}{2}$$

$$\therefore -\tan^{-1}\tfrac{1}{\alpha} = \tan^{-1}\alpha - \tfrac{\pi}{2}$$

Golden Rectangle and Golden Ratio



$$\frac{\phi}{1} = \frac{1+\phi}{\phi} =$$

$$\phi^2 = 1 + \phi$$

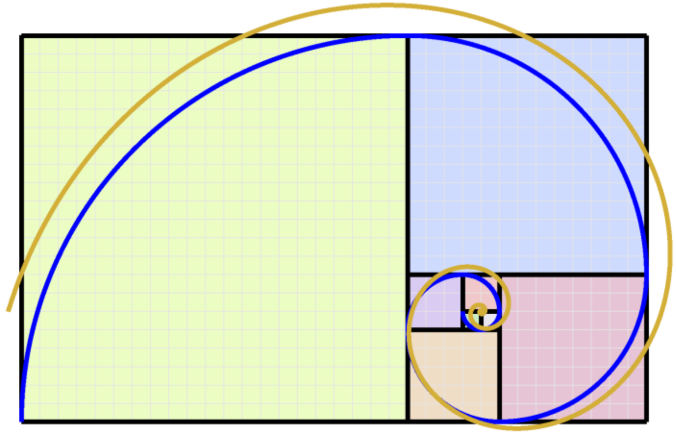
$$\phi^2 - \phi - 1 = 0$$

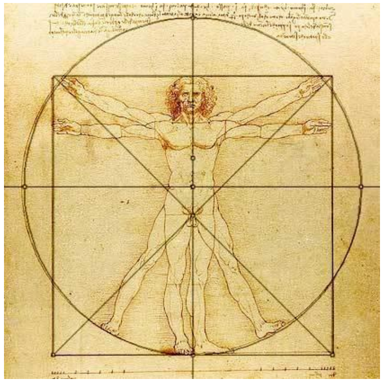
$$\left(\phi - \tfrac{1}{2}\right)^2 - \tfrac{1}{4} - 1 = 0$$

$$\phi = \tfrac{1}{2} \pm \tfrac{\sqrt{5}}{2}$$

If we assert that $\phi > 1$ we can take the positive root

$$\phi = \tfrac{1}{2}\left(1 + \sqrt{5}\right)$$





Note the Golden Spiral is often mistakenly drawn as a Fibonacci spiral. The latter has a 'nested square structure,' with each square bounding the quarter turn arc.

I have fitted a golden spiral approximately, by working out a scaling constant by plotting the ratio of the Fibonacci spiral radius to a Golden spiral radius, (with a unit scaling constant)

Vitruvian Man (c 1490)
Leonardo da Vinci (1452-1519)

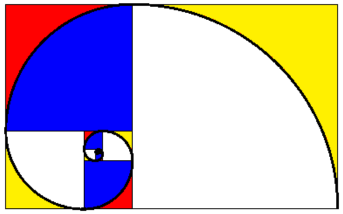
A **Fibonacci spiral** (blue) are arcs drawn every quarter turn, whose radii equal terms of the Fibonacci series

The **Golden spiral**, starting from the same coordinates is overlaid.

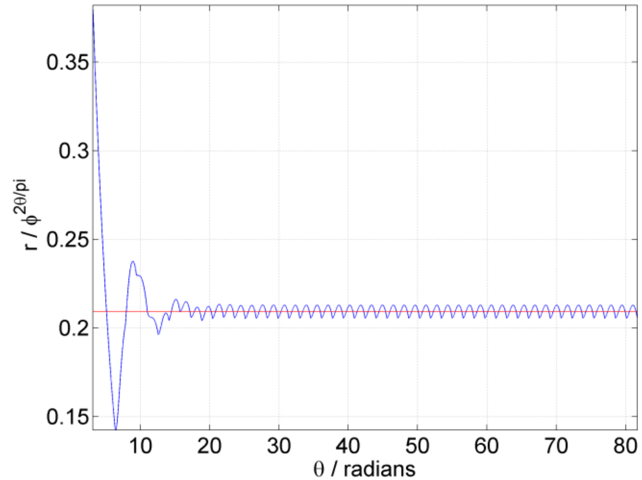
$$r = a\phi^{\frac{2\theta}{\pi}} \;\; ; \;\; a = 0.2092001398$$

$$x = r\cos\theta \;\; ; \;\; y = r\cos\theta$$

$$\theta = -16\pi \;\; \dots \;\; 8 \times \tfrac{\pi}{2} + \pi$$

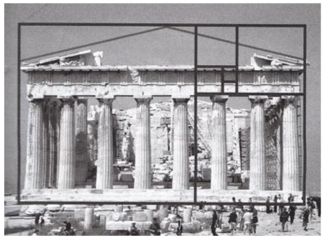


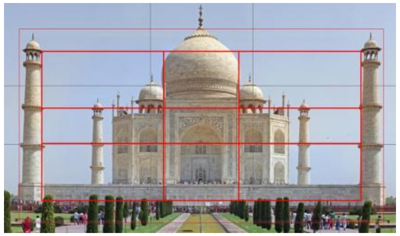
$r / \phi^{2\theta/pi}$ tends towards 0.2092001398

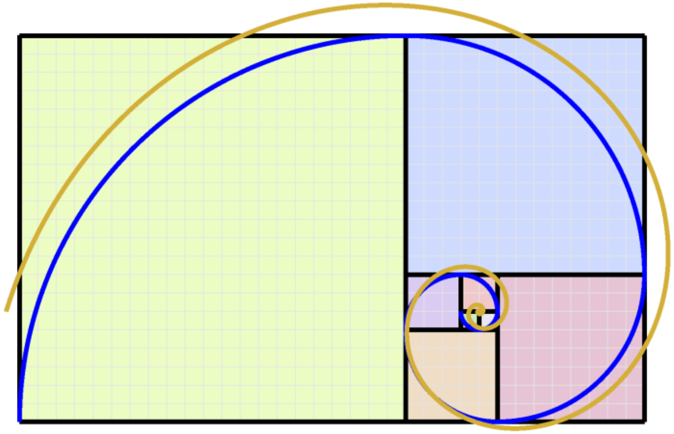


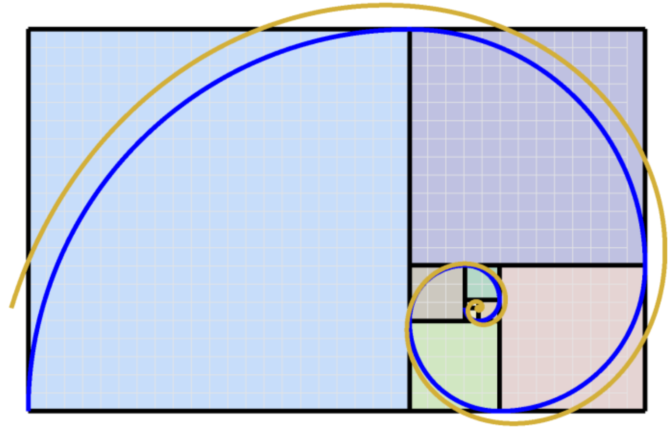
Objects with an *aspect ratio* (i.e. width to height or vice-versa) = ϕ are thought to be very *aesthetically pleasing.*

Many naturally occurring structures (especially in the human body) are in this proportion, or close to it.









A **Fibonacci spiral** (blue) are arcs drawn every quarter turn, whose radii equal terms of the Fibonacci series

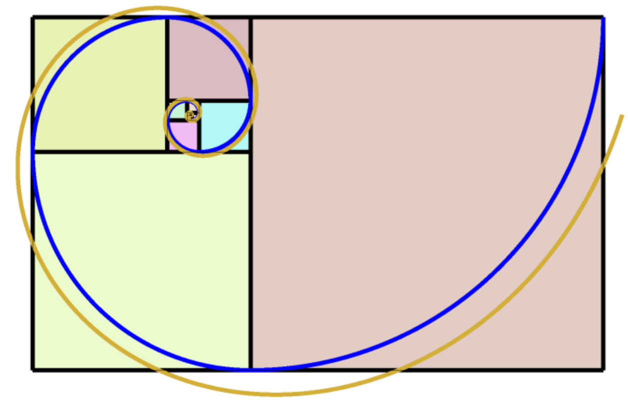
The **Golden spiral**, starting from the same coordinates is overlaid.

$$r = a\phi^{\frac{2\theta}{\pi}} \;\; ; \;\; a = 0.2092001398$$
$$x = r\cos\theta \;\; ; \;\; y = r\cos\theta$$
$$\theta = -16\pi \;\; \dots \;\; 8\times\tfrac{\pi}{2} + \pi$$

An approximate **Fibonacci spiral** (blue) are arcs drawn every quarter turn, whose radii increase in golden ratios. $F_n = \frac{1}{\sqrt{5}}\phi^n$

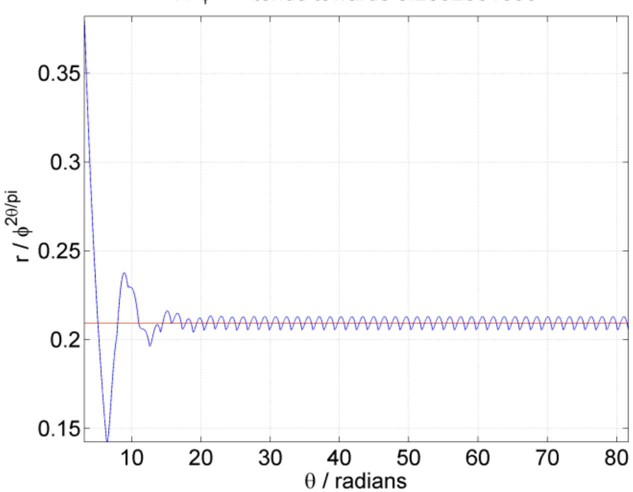
The **Golden spiral**, starting from the same coordinates is overlaid.

$$r = a\phi^{\frac{2\theta}{\pi}} \;\; ; \;\; a = 0.20920013982327$$
$$x = r\cos\theta \;\; ; \;\; y = r\cos\theta$$
$$\theta = -16\pi \;\; \dots \;\; 8\times\tfrac{\pi}{2} + \pi$$



Extending the spiral to 50 quarter turns. You can see how the Golden spiral starts to diverge.

r / $\phi^{2\theta/pi}$ tends towards 0.2092001398



r / $\phi^{2\theta/pi}$ tends towards 0.2092001398