Number facts

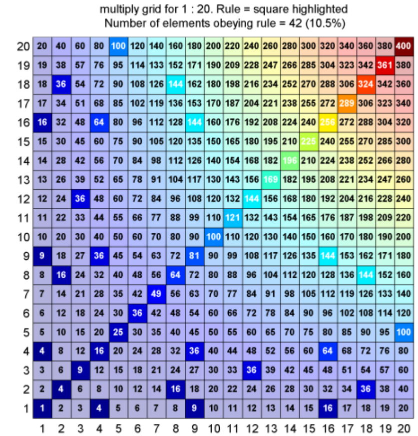
(a) A *prime* number is divisible only by itself and by one.
 e.g. 2, 3, 5, 7, 11, 13 . . .

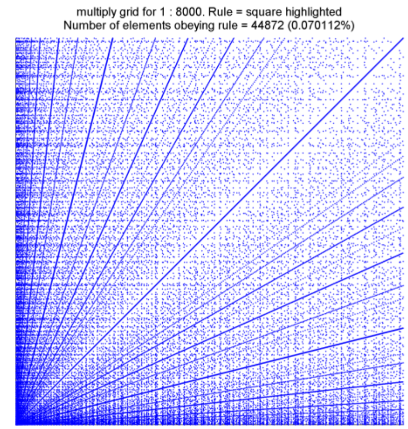
(b) The *multiples* of 12 are 12, 24, 36, 48 . . .

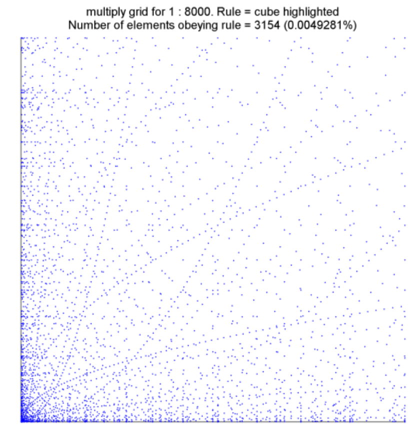
(c) The *factors* of 12 are 1, 2, 3, 4, 6, 12.

(d) Rational and irrational numbers.
 The *exact* value of a *rational* number can be written down as the ratio of two whole numbers.
 e.g. $3,\ 2\frac{1}{2},\ 5{\cdot}72,\ -3\frac{1}{4}$.
 The exact value of an *irrational* number *cannot* be written down.
 e.g. $\pi,\ \sqrt{2},\ \sqrt{3},\ \sqrt{5}$.

$$\pi \approx 3.14$$
$$e \approx 2.72$$
$$\sqrt{2} \approx 1.41$$
$$\sqrt{3} \approx 1.73$$
$$\sqrt{5} \approx 2.24$$
$$\sqrt{7} \approx 2.65$$
$$\ln 2 \approx 0.693$$
$$\log_{10} 2 \approx 0.301$$



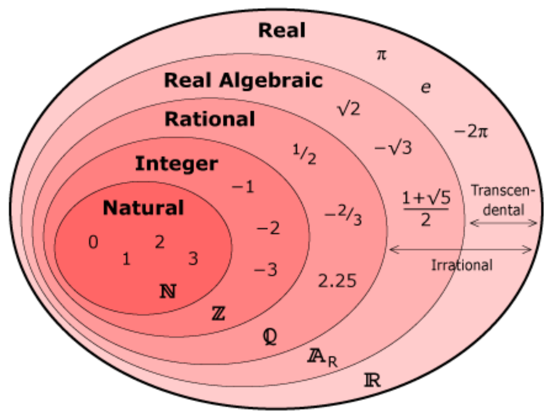




Integers > 0

even + even = even
even + odd = odd
odd + odd = even

even x even = even
even x odd = even
odd x odd = odd

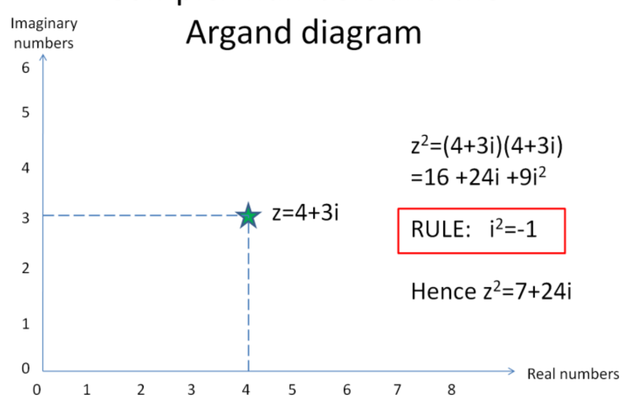


$$(-1)^{\text{even}} = 1$$

$$(-1)^{\text{odd}} = -1$$

0 + 0 = 0
0 x 0 = 0
0 / 0 = ?

Complex numbers and the Argand diagram



X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	X
20	20	40	60	80	100	120	140	160	180	200	220	240	260	280	300	320	340	360	380	400	20
19	19	38	57	76	95	114	133	152	171	190	209	228	247	266	285	304	323	342	361	380	19
18	18	36	54	72	90	108	126	144	162	180	198	216	234	252	270	288	306	324	342	360	18
17	17	34	51	68	85	102	119	136	153	170	187	204	221	238	255	272	289	306	323	340	17
16	16	32	48	64	80	96	112	128	144	160	176	192	208	224	240	256	272	288	304	320	16
15	15	30	45	60	75	90	105	120	135	150	165	180	195	210	225	240	255	270	285	300	15
14	14	28	42	56	70	84	98	112	126	140	154	168	182	196	210	224	238	252	266	280	14
13	13	26	39	52	65	78	91	104	117	130	143	156	169	182	195	208	221	234	247	260	13
12	12	24	36	48	60	72	84	96	108	120	132	144	156	168	180	192	204	216	228	240	12
11	11	22	33	44	55	66	77	88	99	110	121	132	143	154	165	176	187	198	209	220	11
10	10	20	30	40	50	60	70	80	90	100	110	120	130	140	150	160	170	180	190	200	10
9	9	18	27	36	45	54	63	72	81	90	99	108	117	126	135	144	153	162	171	180	9
8	8	16	24	32	40	48	56	64	72	80	88	96	104	112	120	128	136	144	152	160	8
7	7	14	21	28	35	42	49	56	63	70	77	84	91	98	105	112	119	126	133	140	7
6	6	12	18	24	30	36	42	48	54	60	66	72	78	84	90	96	102	108	114	120	6
5	5	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90	95	100	5
4	4	8	12	16	20	24	28	32	36	40	44	48	52	56	60	64	68	72	76	80	4
3	3	6	9	12	15	18	21	24	27	30	33	36	39	42	45	48	51	54	57	60	3
2	2	4	6	8	10	12	14	16	18	20	22	24	26	28	30	32	34	36	38	40	2
1	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	1
X	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	X

How to divide by 11 without using a calculator

in **decimal** form

We can determine the division of an integer by eleven (and any remainder) just by using the digits of the number we wish to divide.


```
THE AMAZING DIVIDE BY ELEVEN METHOD

Define integer (>=11) to be divided by eleven
: N = 12345

x = 12345/11
x = 1,
x = 1,1,
x = 1,1,2,
x = 1,1,2,2,
x = 1,1,2,2,3,

x = 12345/11 = 1122, remainder 3

CHECK: 12345/11 = 1122.2727 = 1122 + 3/11
```

Recipe (or 'algorithm')

Write the digits of `N` as a list of integers
e.g. `N = 12345`, therefore `D = 1,2,3,4,5`

Create a new list `x`, where the first element in the list is `x(1) = D(1)`
e.g. `x(1) = 1`

The next digit in list `x` is: `D(2) - x(1)`
e.g. `x(2) = 2 - 1 = 1`

If this subtraction `x(n) = D(n) - x(n-1)` is negative,
subtract one from `x(n-1)` and add `11` to `x(n)`

Keep going till you get to the end of the list `D`
e.g. `x = 1,1,2,2,3`

`N/11` = all the digits of list `x`, with the final digit being the *remainder*
e.g. `12345/11 = 1122 + 3/11`

Why it works:

$$N = abc = 100a + 10b + c$$

$$\text{if} \quad b > a,\ c > (b-a)$$

$$x = a, b-a, c-(b-a)$$

Note this is a proof of the *special case* when we don't have any *negative* digits in the list `x`

Define $$M = 10a + (b-a) + \frac{c-b+a}{11}$$

$$11M = 110a + 11b - 11a + c - b + a$$

$$11M = 100a + 10b + c$$

$$\therefore M = \frac{N}{11}$$

This is a **proof** for a three digit integer `N`, but the argument can be extended to larger numbers of digits i.e. higher powers of ten.

```
Define integer (>=11) to be divided by eleven
: N = 54321

x = 54321/11
x = 5,
x = 5,-1,
x = 4,10,
x = 4,10,-7,
x = 4,9,4,
x = 4,9,4,-2,
x = 4,9,3,9,
x = 4,9,3,9,-8,
x = 4,9,3,8,3,

x = 54321/11 = 4938, remainder 3

CHECK: 54321/11 = 4938.2727 = 4938 + 3/11
```

`x(2)` is *negative*, so we *subtract* `1` from `x(1)` and *add* `11` to `x(2)`

Why the digits of numbers that are a multiple of 9 add up to a number in the 9 times table

Let $abcd$ be a four-digit decimal number that is a multiple of 9. i.e. integer n is $abcd$ divided by 9.

$$9n = abcd$$

$n \in \mathbb{Z}^+$ This means n is a member of the *set* of ***positive integers***

In decimal notation a,b,c,d can only be the integers 0 ... 9 → $a,b,c,d \in \{0,1,2,3,4,5,6,7,8,9\}$

$$\therefore 9n = a \times 10^3 + b \times 10^2 + c \times 10^1 + d \times 10^0$$

The idea is extensible to larger multiples of nine, where of course we must include more digits i.e.

$$9n = abcdefgh.....$$

Now if the **sum of the digits of a number which is a multiple of nine** *also* **add up to a number which is a multiple of nine**

$$a + b + c + d = 9m$$

$m \in \mathbb{Z}^+$ This means m is a positive integer

$$\therefore d = 9m - a - b - c$$

$$9n = a \times 10^3 + b \times 10^2 + c \times 10^1 + d$$

$$9n = a \times 10^3 + b \times 10^2 + c \times 10^1 + 9m - a - b - c$$

$$9(n - m) = 999a + 99b + 9c$$

$$n - m = 111a + 11b + c$$

$$\therefore \boxed{m = n - 111a - 11b - c}$$

So if n and m are carefully chosen integers, the equation above will always be satisfied since the right hand side must be a positive integer.

Example: $117 \times 9 = 1{,}053$

i.e.
$a = 1,\ b = 0,\ c = 5,\ d = 3$
$a + b + c + d = 9$

$$n = 117$$
$$m = n - 111a - 11b - c$$
$$m = 117 - 111(1) - 11(0) - 5$$
$$m = 1$$

```
Specify an positive integer multiple of nine
 n = 123456789

9 x 123456789 = 1111111101

Digit sum of 1111111101 is:

1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 0 + 1 = 9


Specify an positive integer multiple of nine
 n = 7648576348

9 x 7648576348 = 68837187132

Digit sum of 68837187132 is:

6 + 8 + 8 + 3 + 7 + 1 + 8 + 7 + 1 + 3 + 2 = 54

Digit sum of 54 is: 5 + 4 = 9
```

Proof That $\sqrt{2}$ Is Irrational

This document proves that $\sqrt{2}$ is irrational (i.e. one which can't be expressed as a fraction of one integer over another). The technique used is one of *proof by contradiction*. It is a technique widely used by mathematicians, but most A Level students will not have seen it.

Proof

We are trying to prove that $\sqrt{2}$ cannot be expressed as a fraction. If we are trying to prove that something *cannot* be true, it is often useful to assume that it *is* true and attempt to prove a contradiction. So let us assume that

$$\sqrt{2} = \frac{a}{b}$$

where $\frac{a}{b}$ is a fraction in its lowest form.

Let us play around with this formula and see what we can come up with.

$$\begin{aligned} \sqrt{2} &= \frac{a}{b} \\ 2 &= \frac{a^2}{b^2} && \text{squaring both sides} \\ 2b^2 &= a^2 && \text{multiplying by } b^2 \end{aligned}$$

So a^2 is an even number $\Rightarrow a$ is an even number. We can therefore express a as $2c$ where c is also an integer.

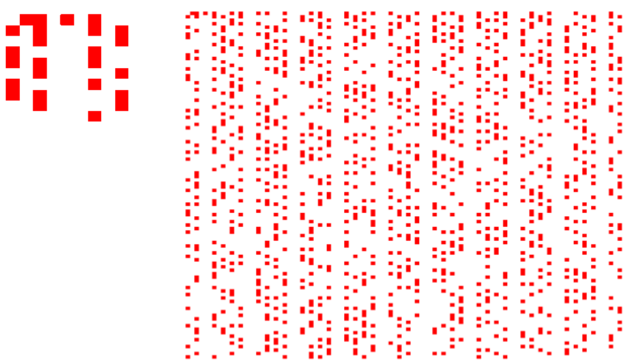
$$\begin{aligned} 2b^2 &= a^2 \\ 2b^2 &= (2c)^2 && \text{substituting } 2c \text{ for } a \\ 2b^2 &= 4c^2 && \text{getting rid of brackets} \\ b^2 &= 2c^2 && \text{cancelling the 2} \end{aligned}$$

We can now see that b^2 is also and even number $\Rightarrow b$ is even.

But we have assumed that $\frac{a}{b}$ is a fraction in its lowest form, which it clearly is not since both a and b are even numbers (and could therefore be cancelled further). So we have a contradiction and have to conclude that our original assumption that $\sqrt{2}$ can be expressed as a fraction is false $\Rightarrow \sqrt{2}$ is irrational. ■

Number miscellany!

Powers of 2	Powers of 3	Powers of 5	Squares	Cubes
$2^0 = 1$	$3^0 = 1$	$5^0 = 1$	$1^2 = 1$	$1^3 = 1$
$2^1 = 2$	$3^1 = 3$	$5^1 = 5$	$2^2 = 4$	$2^3 = 8$
$2^2 = 4$	$3^2 = 9$	$5^2 = 25$	$3^2 = 9$	$3^3 = 27$
$2^3 = 8$	$3^3 = 27$	$5^3 = 125$	$4^2 = 16$	$4^3 = 64$
$2^4 = 16$	$3^4 = 81$	$5^4 = 625$	$5^2 = 25$	$5^3 = 125$
$2^5 = 32$	$3^5 = 243$		$6^2 = 36$	$6^3 = 216$
$2^6 = 64$	$3^6 = 729$		$7^2 = 49$	$7^3 = 343$
$2^7 = 128$			$8^2 = 64$	$8^3 = 512$
$2^8 = 256$			$9^2 = 81$	$9^3 = 729$
$2^9 = 512$			$10^2 = 100$	$10^3 = 1000$
$2^{10} = 1024$			$11^2 = 121$	$11^3 = 1331$
			$12^2 = 144$	$12^3 = 1728$
			$13^2 = 169$	
$2^{11} = 2048$			$14^2 = 196$	
$2^{12} = 4096$			$15^2 = 225$	
$2^{16} =$ 65,536			$16^2 = 256$	
$2^{24} =$ 16,777,216			$17^2 = 289$	
$2^{32} =$ 4,294,967,296			$18^2 = 324$	
			$19^2 = 361$	
			$20^2 = 400$	



PRIMES

2	3	5	7	11	13	17	19	23	29	31	37	41	43	47	53
59	61	67	71	73	79	83	89	97	101	103	107	109	113	127	131
137	139	149	151	157	163	167	173	179	181	191	193	197	199	211	223
227	229	233	239	241	251	257	263	269	271	277	281	283	293	307	311
313	317	331	337	347	349	353	359	367	373	379	383	389	397	401	409
419	421	431	433	439	443	449	457	461	463	467	479	487	491	499	503
509	521	523	541	547	557	563	569	571	577	587	593	599	601	607	613
617	619	631	641	643	647	653	659	661	673	677	683	691	701	709	719
727	733	739	743	751	757	761	769	773	787	797	809	811	821	823	827
829	839	853	857	859	863	877	881	883	887	907	911	919	929	937	941
947	953	967	971	977	983	991	997	1009	1013	1019	1021	1031	1033	1039	1049