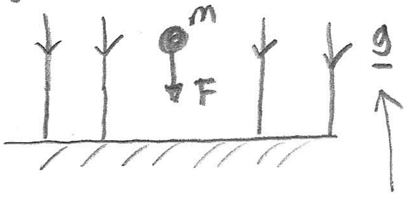
PRE-U REVISION NOTES. MECHANICS AF/3/18

GRAVITATIONAL FIELDS, ORBITS & KEPLER'S LAWS

The Newtonian model of $\boxed{\text{gravity}}$ is that of a $\boxed{\text{force}}$ which permeates all space, and whose magnitude and direction is computable from the spatial distribution of $\boxed{\text{mass}}$ – the source of gravity.

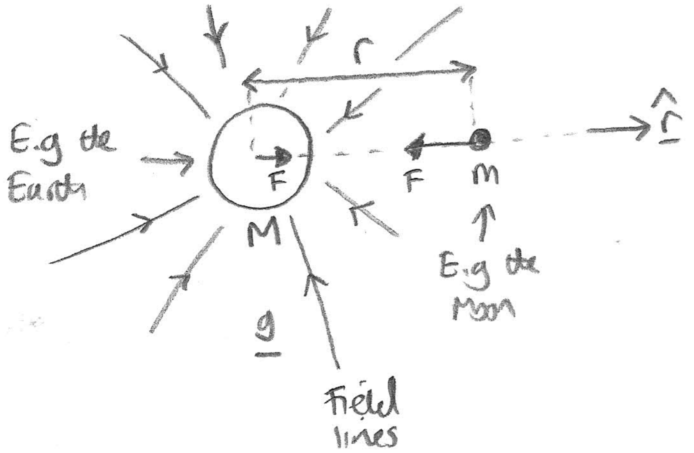
Gravity is therefore a $\boxed{\text{field of vectors}}$ – at any point in space we can draw an arrow pointing the direction of the gravitational force, and with a length proportional to the strength



Force on mass M is $\boxed{\underline{F} = M\underline{g}}$

On Earth, near the surface
$|\underline{g}| \simeq 9.81\ \text{N/kg}$
ie a uniform field of constant strength.

Beyond the surface of the Earth, the gravitational field obeys an $\boxed{\text{inverse square law,}}$ and acts radially inward.



Force on mass m due to gravity of mass M is

$$\boxed{\begin{aligned} \underline{F} &= m\underline{g} \\ \underline{g} &= -\frac{GM}{r^2}\hat{\underline{r}} \end{aligned}}$$

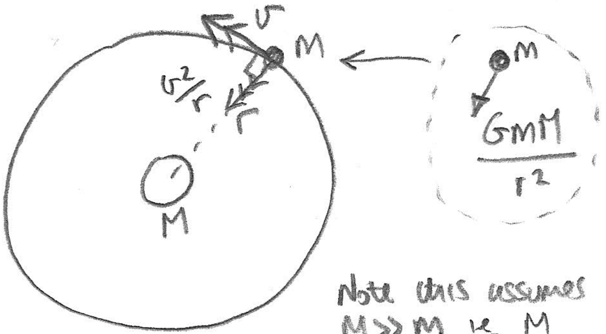
(Note equal and opposite force on M from m)

$\boxed{G = 6.67 \times 10^{-11}\ \text{Nm}^2\ \text{kg}^{-2}}$ Universal Gravitation constant

The centripetal force of gravity is what maintains the orbits of planets, satellites etc. If the orbit is $\boxed{\text{circular}}$ (Venus, Earth & Neptune are the most circular in the Solar System) we can relate orbital period to radius.

$\boxed{v = \frac{2\pi r}{P}}$

P Period
v speed



Note this assumes $M \gg m$ ie M is fixed

Newton II: $\boxed{\frac{mv^2}{r} = \frac{GMm}{r^2}}$

$$\therefore \frac{4\pi^2 r^2}{P^2} = \frac{GM}{r}$$

$$\therefore \boxed{P^2 = \frac{4\pi^2}{GM} r^3}$$ Kepler's Third Law

let M be the Sun $\boxed{M_\odot = 1.99 \times 10^{30}\ \text{kg}} \approx \boxed{332,837\, M_\oplus}$ *

and $r = 1\,\text{AU}$ (Astronomical unit) $= \boxed{1.49598 \times 10^{11}\ \text{m}}$

$\therefore\ P = \dfrac{2\pi}{\sqrt{GM}}\, r^{3/2}$ $\therefore\ P_{\oplus\odot} = \dfrac{2\pi}{\sqrt{6.67 \times 10^{-11} \times 1.99 \times 10^{30}}} \times \left(1.49598 \times 10^{11}\right)^{3/2}\ \text{s}$

Note only 3.s.f used for G

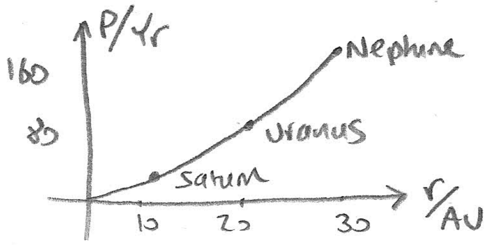
* EARTH MASS
$\boxed{M_\oplus = 5.972 \times 10^{24}\ \text{kg}}$

$\oplus$ Earth
$\odot$ Sun

$P_{\oplus\odot} = 3.15558 \times 10^{7}\,\text{s} = \boxed{1.0006\ \text{years}}$

$365 \times 24 \times 3600\,\text{s} = 1\,\text{Yr}$
$(\approx \pi \times 10^{7}\,\text{s})$

So $\boxed{\dfrac{P}{\text{Yr}} \approx \left(\dfrac{r}{\text{AU}}\right)^{3/2}}$

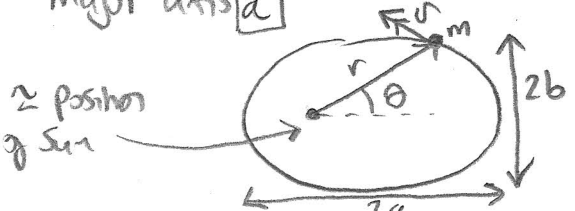


SOLAR SYSTEM

Planet	Period P/Yr	r/AU [radius from Sun] **	$M/M_\oplus$	Rotation Period /days	orbital eccentricity ε
Mercury	0.241	0.387	0.055	58.646	0.21
Venus	0.615	0.723	0.815	243.018	0.01
Earth	1.000	1.000	1.000	1.000	0.02
Mars	1.881	1.523	0.107	1.026	0.09
Jupiter	11.861	5.202	317.85	0.413	0.05
Saturn	29.628	9.576	95.159	0.444	0.06
Uranus	84.747	19.293	14.5	0.718	0.05
Neptune	166.344	30.246	17.204	0.671	0.01
[pluto]	248.348	39.509	0.003	6.387	0.25

Note pluto is officially a 'dwarf planet', and also orbits in a different plane to the other planets in the Solar System. So sometimes it is closer than Neptune.

** orbits are actually $\boxed{\text{ELLIPSES}}$ ← Kepler I
with the centre of mass of the planet, Sun system at one focus. r/AU is the semi-major axis $\boxed{a}$
↑ Table above

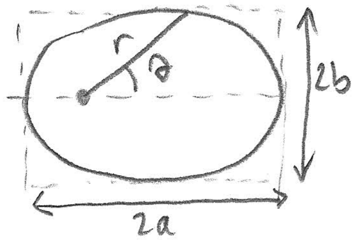


$\boxed{\text{Eccentricity}\quad \varepsilon = \sqrt{1 - \frac{b^2}{a^2}}}$

KEPLER'S LAWS OF ORBITAL MOTION

← Discovered from Tycho Brahe's (1546–1601) astronomical data, prior to Newton!

↑ Johannes Kepler (1571–1630)

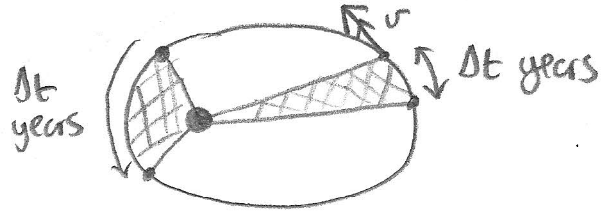


I The orbit of every planet is an ELLIPSE with the centre of mass of the planet, Sun at the focus [Since $M_\odot \gg$ all other masses of planets, the BARYCENTRE is $\simeq$ centre of the Sun ie Sun barely moves]

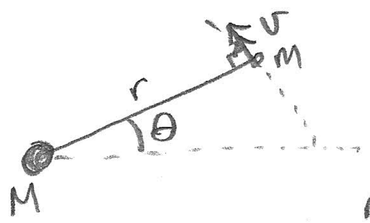
Polar equation of Ellipse $\boxed{r = \dfrac{a(1-\varepsilon^2)}{1-\varepsilon\cos\theta}}$

$\varepsilon = \sqrt{1 - b^2/a^2}$ Eccentricity

II A line joining a planet and the Sun sweeps out equal areas during equal intervals of time



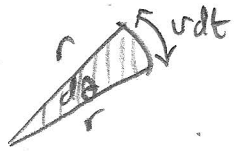
Since the force of gravity is centripetal ⇒ there is no torque $\underline{\tau} = \underline{r} \times \underline{F}$ on a planet, ∴ its angular momentum is constant.



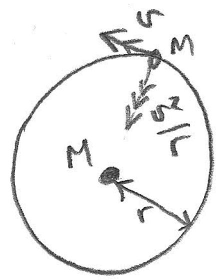
If the orbit was circular $\boxed{v = r\dfrac{d\theta}{dt}}$ ← "rω"

Angular momentum $\boxed{L = mrv}$

$L = mr^2 \, d\theta/dt$ ∴ $r^2 \dfrac{d\theta}{dt} = \dfrac{L}{m}$



In dt, area swept is $\boxed{\tfrac{1}{2} r^2 \, d\theta = dA}$



$$\therefore \frac{dA}{dt} = \frac{1}{2} r^2 \frac{d\theta}{dt}$$

$$\therefore \boxed{\frac{dA}{dt} = \frac{1}{2}\frac{L}{m}} \quad \text{ie constant}$$

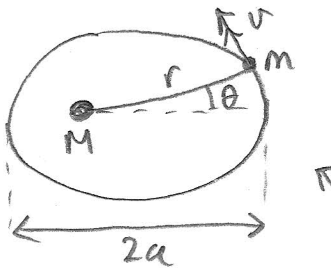
Now by Newton II: $\dfrac{mv^2}{r} = \dfrac{GMm}{r^2}$ so $v = \sqrt{\dfrac{GM}{r}}$ so $\dfrac{L}{m} = rv$

$$\Rightarrow \frac{L}{m} = r\sqrt{\frac{GM}{r}} = \sqrt{\frac{r^2 GM}{r}} = \sqrt{GMr}$$

so for circular orbits: $\boxed{\dfrac{dA}{dt} = \dfrac{1}{2}\sqrt{GMr}}$

[In general, for a two-body system $\boxed{\dfrac{dA}{dt} = \dfrac{1}{2}\sqrt{G(m+M)(1-\varepsilon^2)a}}$]

III $$\boxed{p^2 = \frac{4\pi^2}{G(M+m)} a^3}$$

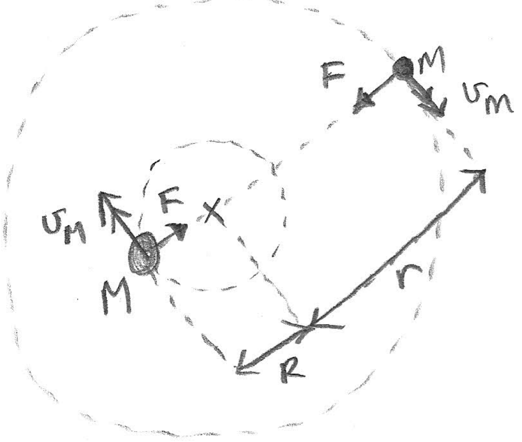


$M \gg m$ in this case

"orbital period squared $\propto$ semi-major axis cubed"

Two body Kepler problem (circular orbits)

x centre of mass "Barycentre"



Define $\boxed{a = R + r}$

Newton II: $m r \omega^2 = \frac{GMm}{a^2}$ ← F ①

$M R \omega^2 = \frac{GMm}{a^2}$ ②

$\therefore R\omega^2 + r\omega^2 = G(M+m)/a^2$ ① + ②

$\therefore \omega^2 = G(M+m)/a^3$

$\frac{4\pi^2}{p^2} = G(M+m)/a^3$

$\therefore \boxed{p^2 = \frac{4\pi^2}{G(M+m)} a^3}$ Kepler III

Since no net torque on system, angular momentum on each planet is constant
$\therefore$ orbital periods are constant and equal
$\therefore \boxed{\omega = \frac{2\pi}{p}}$

Total energy for system is: $\boxed{E = \frac{1}{2}M(R\omega)^2 + \frac{1}{2}m(r\omega)^2 - \frac{GMm}{a}}$

See next page!

Using Newton II: $r\omega^2 = \frac{GM}{a^2}$ $\quad R\omega^2 = \frac{Gm}{a^2}$

$\therefore E = \frac{1}{2}MR\,Gm/a^2 + \frac{1}{2}mr\,GM/a^2 - \frac{GMm}{a}$

$E = -\frac{GMm}{2a}\left(-\frac{R}{a} - \frac{r}{a} + 2\right)$

$E = -\frac{GMm}{2a}\left(\underbrace{-\frac{(R+r)}{a}}_{=1} + 2\right)$

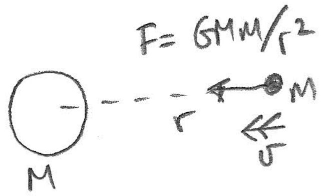
This works for elliptical orbits too

$\therefore \boxed{E = -\frac{GMm}{2a}}$

negative total energy $\Rightarrow$ BOUND orbits

trajectories have $E > 0$

GRAVITATIONAL POTENTIAL ENERGY



Work done to move mass m from $r = a$ to $r = b$ $(b > a)$ is

$$W = \int_a^b F\,dr = GMm \int_a^b \frac{1}{r^2}\,dr$$

$$= GMm \left[-\frac{1}{r} \right]_a^b$$

$$\therefore \boxed{W = GMm\left(\frac{1}{a} - \frac{1}{b}\right)}$$

In limit $b \to \infty$ $\boxed{W_\infty = \frac{GMm}{a}}$

Now if mass m is accelerated by gravity from $r = \infty$, (at rest) to r, it will gain kinetic energy. The total energy at $r = \infty$ is zero, which in a closed system must also be true at r.

Hence write total energy $0 = \frac{1}{2}mv^2 - \frac{GMm}{r}$, since when $v = 0$ at $r = \infty$, work done is $\frac{GMm}{r}$.

In general total energy may not be zero (in a bound system it is $-\frac{GMm}{2a}$) so write

$$\boxed{E = \frac{1}{2}mv^2 - \frac{GMm}{r}}$$

$-\frac{GMm}{r}$ is the $\boxed{\text{gravitational potential energy}}$

Note $\boxed{\underline{g} = -\frac{GMm}{r^2}\hat{\underline{r}}}$ $\therefore$ $\boxed{\underline{g} = -\frac{d\phi}{dr}\hat{\underline{r}}}$ $\boxed{\phi = -\frac{GMm}{r}}$

$\underline{F} = m\underline{g}$

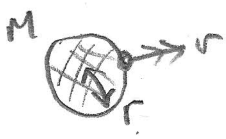
"Field = potential gradient"

A general result for gravity, electric fields etc.

To $\boxed{\text{escape}}$ a planet, $\boxed{E > 0}$ $\therefore$ If velocity v is launch velocity of a rocket (short burn!)

$$\frac{1}{2}v^2 - \frac{GM}{r} > 0 \Rightarrow \boxed{v > \sqrt{\frac{2GM}{r}}}$$

ESCAPE VELOCITY



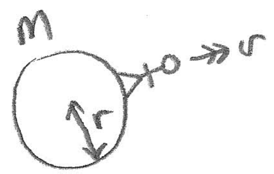
For earth, the escape velocity is $v = \sqrt{\dfrac{2 \times 6.67 \times 10^{-11} \times 5.97 \times 10^{24}}{6.38 \times 10^{6}}} \approx \boxed{11.2\,km/s}$.
($R = 6.38 \times 10^6 m$)

Note when $v = c$, we can work out the 'radius' of a Black Hole
(The Schwarzschild radius) $c^2 = \dfrac{2GM}{r}$ $\therefore$ $\boxed{r = \dfrac{2GM}{c^2}}$

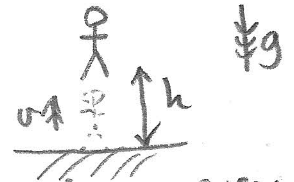
↑ ie minimum radius for a black hole to form.

Astronaut escape problem

An astronaut can jump (fully kitted in space suit) height h on earth. What mass, radius of asteroid can he escape?



$\boxed{v^2 > \dfrac{2GM}{r}}$ to escape



on earth: $\frac{1}{2}v^2 = gh$ [Constant acc motion, or conservation of energy]
$\therefore$ $\boxed{v^2 = 2gh}$ ($g = 9.81\ m/s^2$)

So $gh > \dfrac{GM}{r}$

let asteroid have density $\rho = \dfrac{M}{\frac{4}{3}\pi r^3}$

$\therefore$ $gh > G\frac{4}{3}\pi r^3 \rho / r$

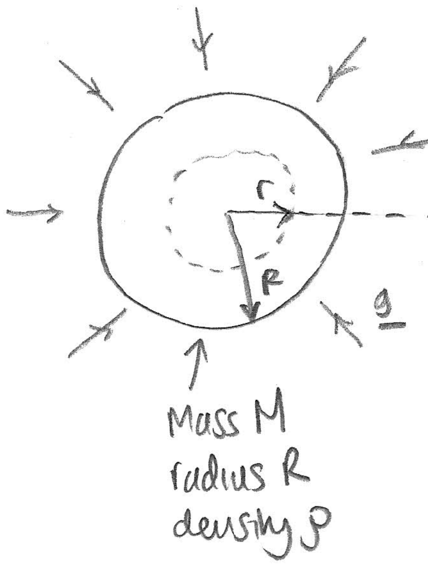
$\therefore$ $r^2 < \dfrac{3gh}{4\pi G\rho}$ $\therefore$ $\boxed{r < \sqrt{\dfrac{3gh}{4\pi G\rho}}}$

let's assume $h = 0.1\ m$, $g = 9.81\ m/s^2$, $\rho = 5000\ kg$
$\Rightarrow$ $\boxed{r < 838\ m}$

So asteroids need to have radius > few km to be 'safe' to walk on untethered!

GRAVITATIONAL FIELD INSIDE (AND OUTSIDE) A SPHERICAL MASS OF UNIFORM DENSITY

Density $\boxed{\rho = \dfrac{M}{\frac{4}{3}\pi R^3}}$



outside the sphere the field strength is:

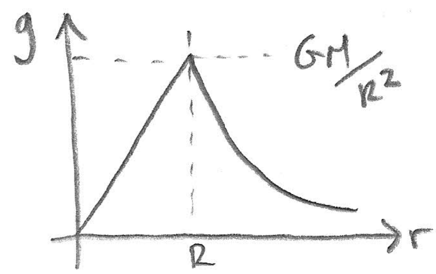
$$\boxed{\underline{g} = -\frac{GM}{r^2}\,\hat{\underline{r}}} \qquad (r \geq R)$$

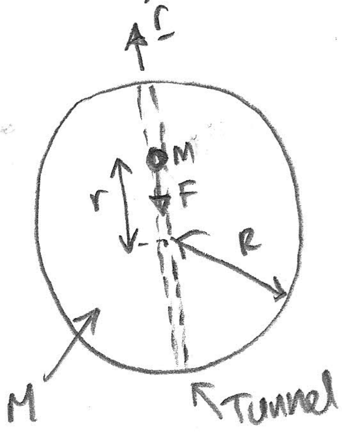
Inside the sphere $\boxed{\underline{g} = -\dfrac{GM(r)}{r^2}\,\hat{\underline{r}}}$ ie same form

ie $\boxed{\text{only mass within radius } r \text{ contributes}}$ This makes sense by symmetry, as there is equal mass beyond r diametrically opposite.

Now $\boxed{M(r) = \frac{4}{3}\pi r^3 \rho} \Rightarrow \boxed{M(r) = M\left(\frac{r}{R}\right)^3}$

$\therefore$ for $0 < r < R$: $\boxed{\underline{g} = -\dfrac{GM}{R^3}\, r\,\hat{\underline{r}}}$





If a tunnel is drilled through a planet of uniform density of mass M and radius R

Newton II: $m\ddot{r} = -F$ $\quad F = mg$
(r is +ve radially) $\quad F = \dfrac{GMm}{R^3}r$

$\therefore \boxed{\ddot{r} = -\dfrac{GM}{R^3}r}$ Simple Harmonic Motion (SHM)

ie mass will oscillate with period P. $\boxed{r(t) = R\cos\left(2\pi t/P\right)}$

(assume starts from rest at surface ie $r=R$) $\quad \ddot{r} = -\dfrac{4\pi^2}{P^2}r$

$\therefore \dfrac{4\pi^2}{P^2} = \dfrac{GM}{R^3} \Rightarrow \boxed{P = 2\pi\sqrt{\dfrac{R^3}{GM}}}$

For $M = M_\oplus$, $R = R_\oplus$ (Earth)
$\Rightarrow \boxed{P \simeq 84 \text{ mins}}$ ie $\boxed{42 \text{ mins for a diameter}}$