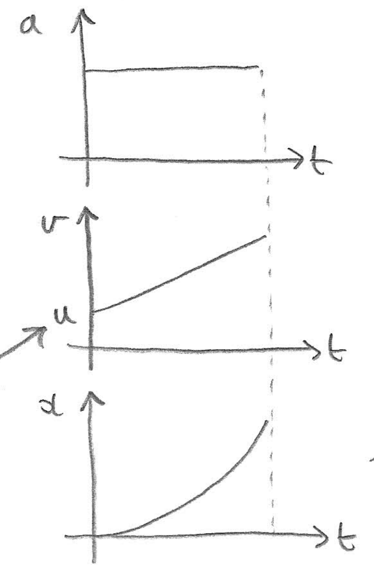
PRE-U REVISION NOTES. MECHANICS. AF. 3/18

LINEAR MECHANICS: KINEMATICS, FORCE, MOMENTUM, ENERGY

ASSUME $\boxed{\text{CLASSICAL}}$ ie $v \ll c$ ← speed of light

$\boxed{\text{KINEMATICS}}$

t Time / s
x Displacement / m
v Velocity / ms^{-1}
a Acceleration / ms^{-2}



$v(t=0) = u$

$a = \frac{dv}{dt}$ $\quad v = \int a\,dt$

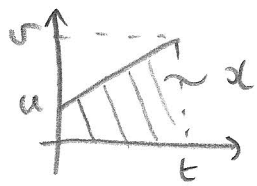
$v = \frac{dx}{dt}$ $\quad x = \int v\,dt$

* Special case *

{Constant acceleration motion: a = constant}   "XUVAT"

$\boxed{a = \frac{v-u}{t}}$ ← ie gradient of linear (t, v) graph $\Rightarrow$ $\boxed{v = u + at}$

Also $\boxed{t = \frac{v-u}{a}}$



$\boxed{x = \frac{1}{2}(u+v)t}$ ie area under (t,v) graph

$\therefore x = \frac{1}{2}(u + u + at)t$

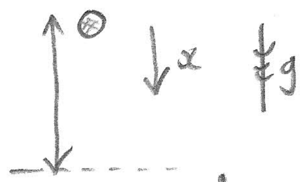
$\boxed{x = ut + \frac{1}{2}at^2}$

$\therefore x = \frac{1}{2}(u+v)(v-u)/a$

$\therefore 2ax = uv + v^2 - u^2 - uv$

$\therefore \boxed{v^2 = u^2 + 2ax}$

So displacement of a ball dropped in time t is $\boxed{x = \frac{1}{2}gt^2}$

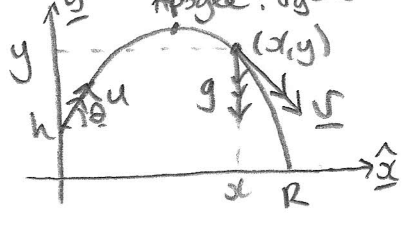


Ball dropped from rest.

Constant acceleration in x and y directions

$\boxed{\text{PROJECTILE MOTION}}$

No horizontal acceleration ie ignore air resistance



$\boxed{\begin{aligned} x &= ut\cos\theta \\ v_x &= u\cos\theta \\ a_x &= 0 \end{aligned}}$ *

$\boxed{\begin{aligned} y &= h + ut\sin\theta - \tfrac{1}{2}gt^2 \\ v_y &= u\sin\theta - gt \\ a_y &= -g \\ v_y^2 &= u^2\sin^2\theta - 2gy \end{aligned}}$ **

* $\boxed{t = \frac{x}{u\cos\theta}}$

Parabolic trajectory

$\boxed{\underline{v} = v_x\hat{\underline{x}} + v_y\hat{\underline{y}}}$

$\therefore v^2 = v_x^2 + v_y^2 = u^2(\cos^2\theta + \sin^2\theta) - 2gy$

$\Rightarrow \boxed{v^2 = u^2 - 2gy}$

$\hat{x}, \hat{y}$ unit vectors

$|\hat{x}| = |\hat{y}| = 1; \ \hat{x}\cdot\hat{y} = 0$

$\frac{1}{\cos^2\theta} = 1 + \tan^2\theta$

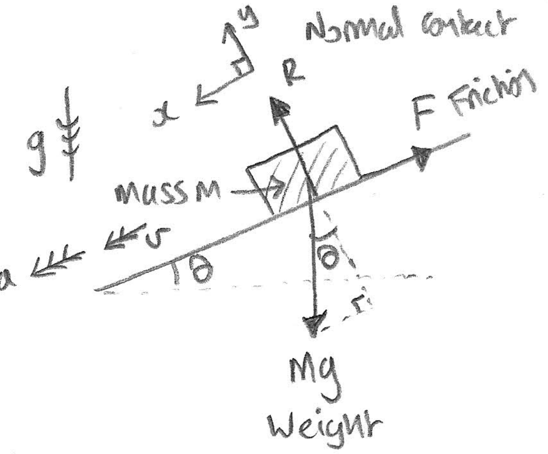
** $y = h + \frac{ux\sin\theta}{u\cos\theta} - \frac{1}{2}g\left(\frac{x}{u\cos\theta}\right)^2$

$\boxed{y = h + x\tan\theta - \frac{gx^2}{2u^2}(1+\tan^2\theta)}$

NEWTON'S SECOND LAW : $\boxed{\text{MASS} \times \text{ACCELERATION} = \text{VECTOR SUM OF FORCES}}$

ie if vector sum is zero we have equilibrium ie zero acceleration and $\therefore$ constant velocity (which could be zero)

["NO FORCE $\Rightarrow$ constant velocity" is NEWTON'S FIRST LAW]



Block on plane problem

ie resistive forces in x, y directions

Newton II: $\boxed{\begin{aligned} /\!/x: &\quad Ma = Mg\sin\theta - F \\ /\!/y: &\quad 0 = R - Mg\cos\theta \end{aligned}}$

If not sliding (ie in equilibrium $a = 0$)

Friction model : $\boxed{F \leq \mu R}$ $\quad$ μ coefficient of friction

$\therefore$ $\left.\begin{aligned} F &= Mg\sin\theta \\ R &= Mg\cos\theta \end{aligned}\right\}$ $\quad$ $F \leq \mu R$

$Mg\sin\theta \leq \mu Mg\cos\theta$

$\Rightarrow \boxed{\mu \geq \tan\theta}$

Put block on a slope. Tilt slope till it slides at angle θ. $\mu = \tan\theta$

If sliding : $F = \mu_D R$ $\quad$ ($\mu_D = \mu$ possibly!)

"Dynamic coefficient of friction"

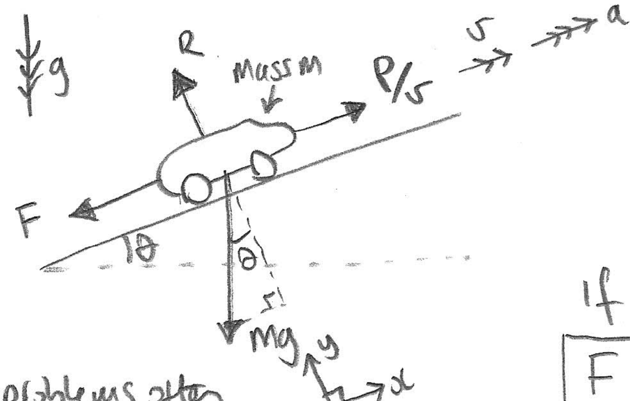
So $\boxed{a = g(\sin\theta - \mu_D\cos\theta)}$ then XUVAT to find v, x etc

Driving force & drag problem

Power = force $\times$ velocity

$P = fv$

$\boxed{\text{So driving force } f \text{ is } f = P/v}$



Newton II: $/\!/x:$ $\boxed{M\dfrac{dv}{dt} = \dfrac{P}{v} - F - Mg\sin\theta}$

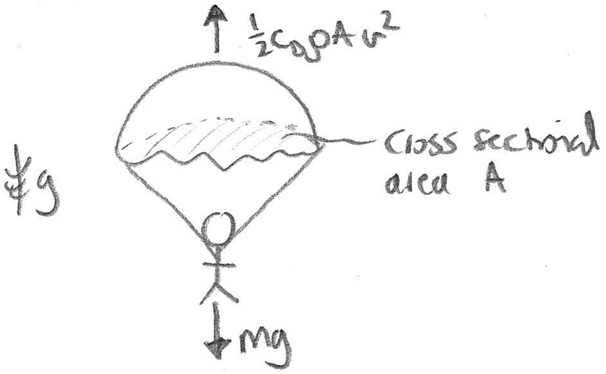
$/\!/y:$ $0 = R - Mg\cos\theta$

In problems often v = constant ie input power = rate of work done against gravity and resistive forces

If air resistance is the main resistive force F

$\boxed{\begin{aligned} F &= kv \\ F &= kv^2 \end{aligned}}$ $\quad$ (low speed, "viscous" drag) (high speed, sub-sonic (!) drag)

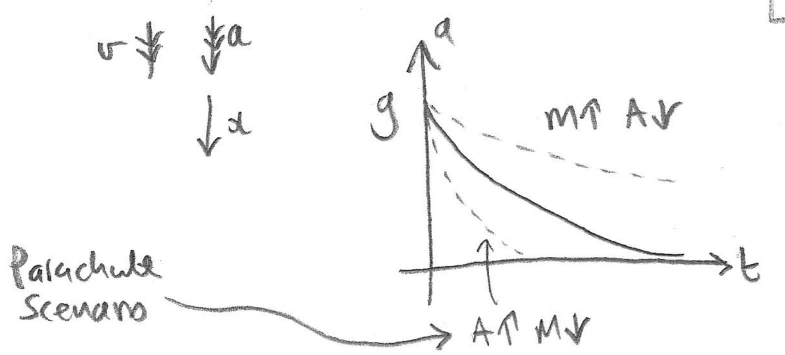
Parachute drag & terminal velocity problem



Newton II: $ma = mg - \frac{1}{2}C_D\rho A v^2$

C_D Drag coefficient
ρ Air density

so $\boxed{a = g - \frac{1}{2}C_D\rho A v^2/m}$



when $a = 0$ we have terminal speed

$$\frac{1}{2}v_T^2 C_D \rho A = mg$$

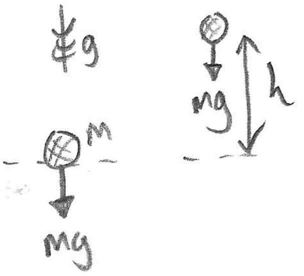
$$\boxed{v_T = \sqrt{\frac{2mg}{C_D\rho A}}}$$

$v_T \downarrow$ for parachute ie $A \uparrow$

so $v_T \propto \sqrt{m}$ and $\propto \frac{1}{\sqrt{A}}$

WORK DONE = FORCE × DISPLACEMENT (in direction of force) d

ie $\boxed{W = Fd}$ This explains why GRAVITATIONAL POTENTIAL ENERGY (GPE) is $\boxed{mgh}$



In general Force F may vary with displacement (x)

So $\boxed{W = \int f\,dx}$ ie the area under the (x,f) graph

Note by Newton II: $f = m\frac{dv}{dt} = mv\frac{dv}{dx}$ $\left[v\frac{dv}{dx} = \frac{dx}{dt}\frac{dv}{dx} = \frac{dv}{dt} \right]$

So $W = \int mv\frac{dv}{dx}\,dx = m\int_u^v v\,dv = \boxed{\frac{1}{2}mv^2 - \frac{1}{2}mu^2}$

ie Associate $\boxed{\frac{1}{2}mv^2}$ with $\boxed{\text{kinetic energy}}$, ie where the work done "goes" if a system has no losses or potential energy.

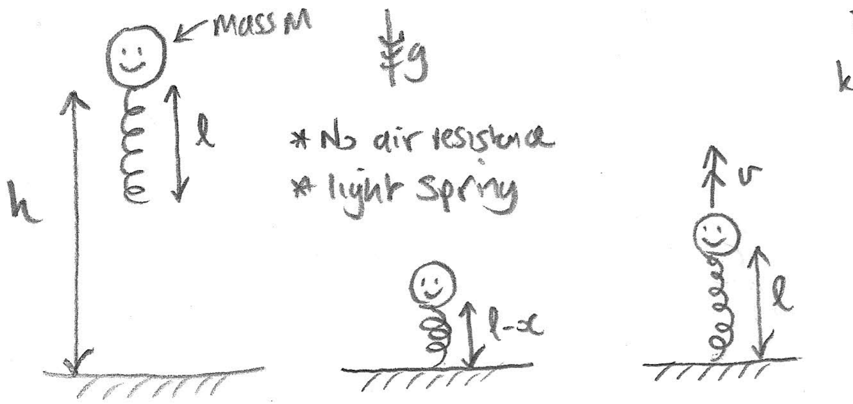
LAW OF CONSERVATION OF ENERGY	KE + PE (Potential energy) + losses = Constant

(if no external work is done)

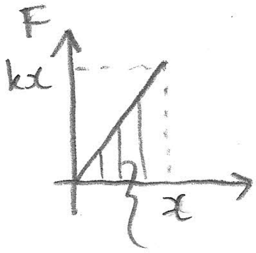
MASS - SPRING EXAMPLE

Hookean springs: $\boxed{F = kx}$

restoring force — extension

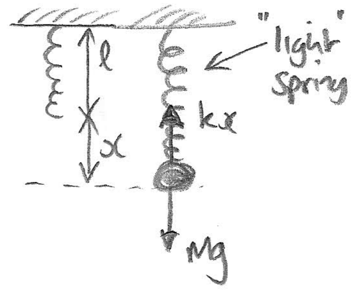


* No air resistance
* light spring



Work done stretching spring is $W = \int F\,dx$

$\boxed{W = \frac{1}{2}kx^2}$ ← Elastic Potential Energy



In equilibrium $Mg = kx$

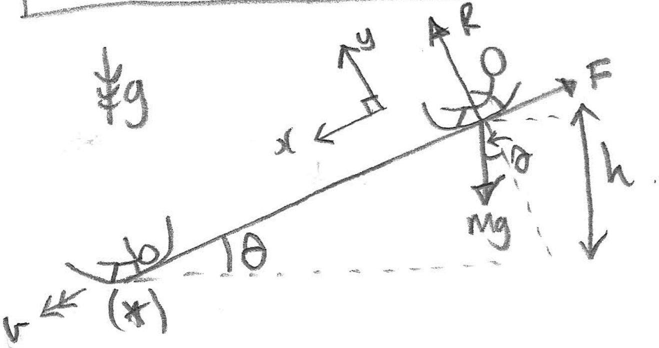
GPE = mgh	GPE = $mg(l-x)$	GPE = mgl
EPE = 0	EPE = $\frac{1}{2}kx^2$	EPE = 0
KE = 0	KE = 0	KE = $\frac{1}{2}mv^2$

By conservation of energy: $mgh = mgl + \frac{1}{2}mv^2$

$$\Rightarrow \boxed{v = \sqrt{2g(h-l)}}$$

Also $mgh = mgl - Mgx + \frac{1}{2}kx^2 \longrightarrow$ Solve quadratic to find x.

HILL SLIDING EXAMPLE



If start from rest at height h what constant resistive force is required to result in speed v at (*), the bottom of the hill? [Slope of hill has length $l = \frac{h}{\sin\theta}$]

Conservation of energy

Top of hill: $\boxed{E = mgh}$

Bottom of hill: $\boxed{\underbrace{E - Fl}_{\text{Work done against friction force } F} = \frac{1}{2}mv^2}$

So $mgh - \frac{Fh}{\sin\theta} = \frac{1}{2}mv^2 \quad \therefore \quad mgh - \frac{1}{2}mv^2 = \frac{Fh}{\sin\theta}$

$$\therefore \boxed{F = mg\sin\theta\left(1 - \frac{v^2}{2gh}\right)}$$

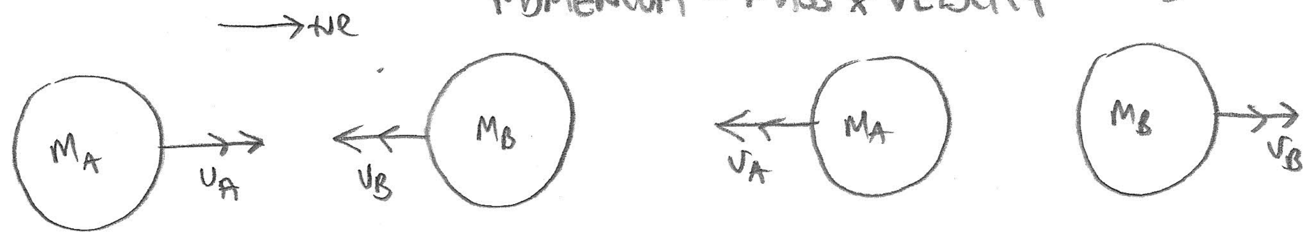
Note $\frac{v^2}{2gh} = \frac{\frac{1}{2}mv^2}{mgh}$

ie KE/GPE for vertical drop.

CONSERVATION OF MOMENTUM AND RESTITUTION

[1D example]

MOMENTUM = MASS × VELOCITY



BEFORE AFTER

Conserving momentum: $\boxed{M_A u_A - M_B u_B = M_B v_B - M_A v_A}$ (1)

Restitution: $\boxed{\varepsilon = \dfrac{\text{speed of separation}}{\text{speed of approach}}}$

↑ coefficient of restitution

$\boxed{\begin{matrix} \varepsilon = 0 & \text{INELASTIC} \\ \varepsilon = 1 & \text{ELASTIC} \end{matrix}}$

$\boxed{\varepsilon = \dfrac{v_A + v_B}{u_A + u_B}}$ (2)

Usually know M_A M_B u_A u_B ε
↳ so solve (1) and (2) to find v_A, v_B.

Can show: $\varepsilon = 1 \Rightarrow$ KE is conserved.
$\varepsilon = 0 \Rightarrow$ max KE is lost due to heat, sound, friction etc.

e.g.

BEFORE AFTER

Conserve momentum: $3m(3u) - 2m(2u) = 3m v_A + 2m v_B$

$$5mu = m(3v_A + 2v_B)$$

$$5u = 3v_A + 2v_B \quad ①$$

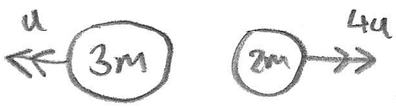
Restitution: $\varepsilon = \dfrac{v_B - v_A}{5u}$ $\therefore$ $v_B - v_A = 5u\varepsilon$ ②

So $v_B = 5u\varepsilon + v_A$ $\therefore$ $5u = 3v_A + 10u\varepsilon + 2v_A$

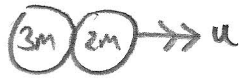
$$5u - 10u\varepsilon = 5v_A$$

$$\boxed{v_A = u(1-2\varepsilon)}$$

$\therefore$ $\boxed{v_B = u(1+3\varepsilon)}$



ELASTIC



INELASTIC

Note $\boxed{\text{Impulse} = \text{Momentum change}}$ [Momentum = p]

So for Elastic example: Impulse $\boxed{\Delta p_A = -12mu}$ for mass A

$p_0 = 9mu$ $P_1 = -3mu$

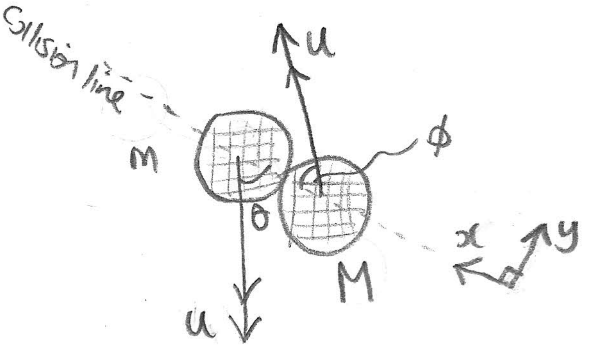
BEFORE AFTER

$P_0 = -4mu$ $P_1 = 8mu$

Impulse $\boxed{\Delta p_B}$ $\boxed{= 12mu}$ for mass B

ie net impulse $\Delta p_A + \Delta p_B = 0$ i.e. the impulses are 'equal and opposite'

$\boxed{\text{2D Collisions}}$ are essentially 1D (!) ie consider conservation of momentum along the $\boxed{\text{Collision line}}$ joining object centres. No change to momentum $\perp$ to this line.



BEFORE

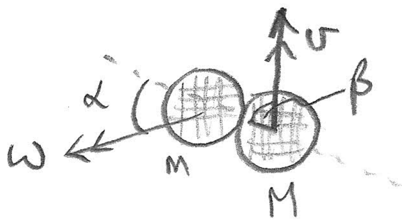
'Collision' is $\parallel x$:

$$\boxed{\begin{aligned} &-mu\cos\theta + Mu\cos\phi \\ &= mw\cos\alpha + Mv\cos\beta \end{aligned}}$$ $\boxed{\text{Conserve momentum}}$

$$\boxed{\varepsilon = \frac{w\cos\alpha - v\cos\beta}{u\cos\theta + u\cos\phi}}$$ $\boxed{\text{RESTITUTION}}$

Also "No collision $\parallel y$:"

So $\boxed{\begin{aligned} v\sin\beta &= u\sin\phi \\ w\sin\alpha &= u\sin\theta \end{aligned}}$ ie y component velocities don't change.

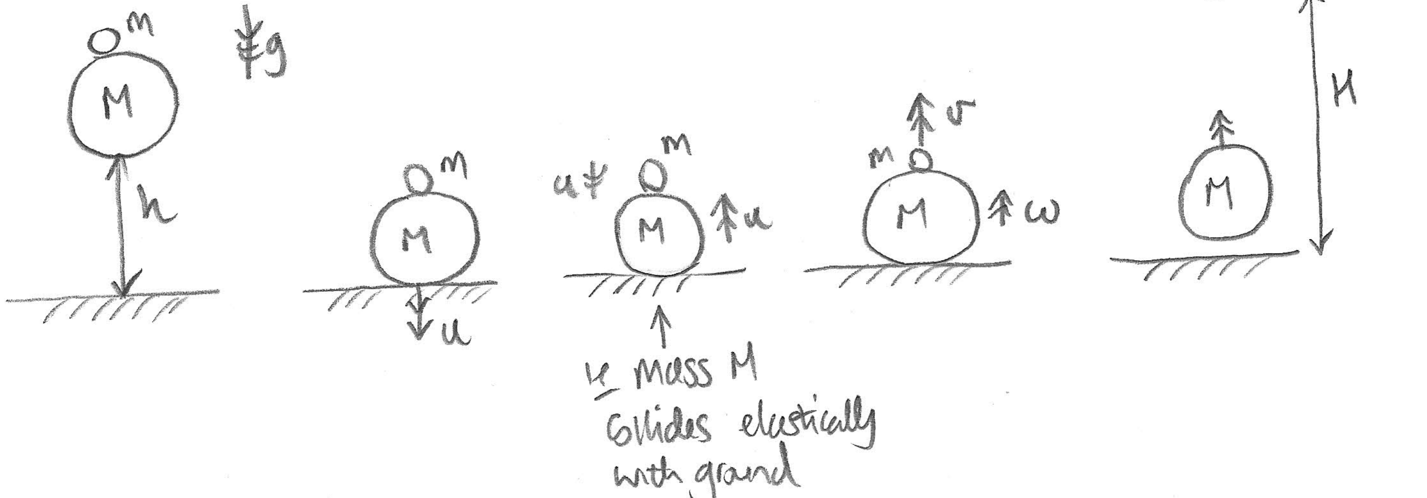


After

Solve these four equations to find w, v, α, β given BEFORE inputs.

TWO BALL BOUNCE PROBLEM

Assume all collisions are elastic



Conservation of energy: $Mgh = \frac{1}{2}Mu^2 \quad \therefore \boxed{u^2 = 2gh}$ So $\boxed{\frac{H}{h} = \left(\frac{v}{u}\right)^2}$

$\frac{1}{2}mv^2 = mgH \quad \therefore \boxed{v^2 = 2gH}$

Conservation of momentum: $Mu - mu = Mw + mv$

Restitution (elastic) : $1 = \frac{v-w}{2u} \quad \therefore v - w = 2u$

$w = v - 2u$

$\therefore (M-m)u = M(v-2u) + mv$

$\therefore (3M-m)u = (m+M)v$

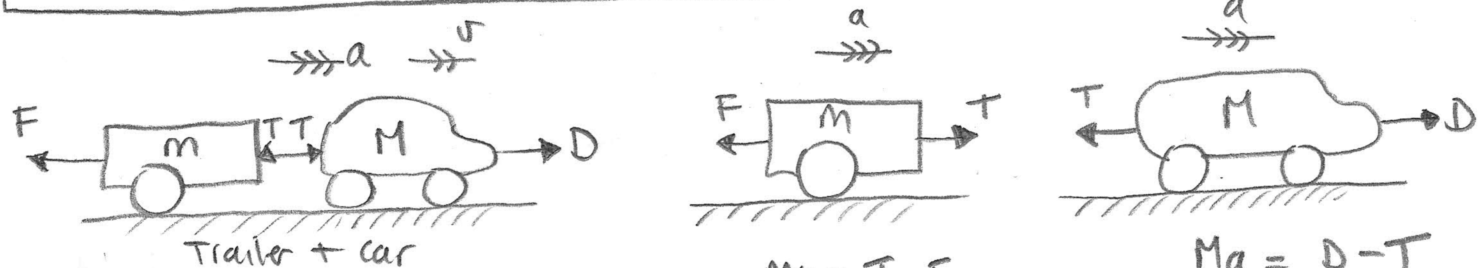
$\therefore \frac{v}{u} = \frac{3M-m}{m+M}$

$\therefore \frac{H}{h} = \left(\frac{3M-m}{m+M}\right)^2$

$\therefore \frac{H}{h} = \left(\frac{3 - m/M}{1 + m/M}\right)^2$ Let $M \gg m$ $\therefore \boxed{\frac{H}{h} \to 9}$

CONNECTED OBJECTS & NEWTON III

$\boxed{\text{Contact forces}}$: "For every action there is an equal and opposite reaction"



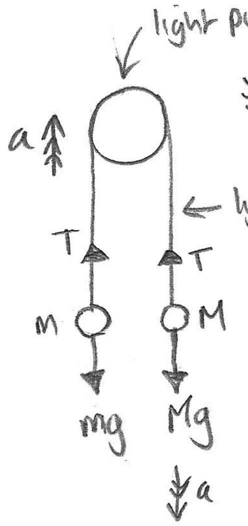
$Ma = D - T$

$ma = T - F$

$(m+M)a = D - F$ ← Newton II

Decompose system to find internal forces

Pulley Systems



[M>m]

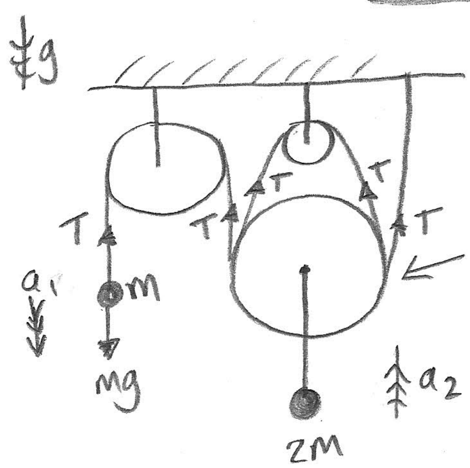
Newton II: $Ma = Mg - T$ ① $\quad ma = T - mg$ ②

① + ②: $(M+m)a = (M-m)g$

$$a = \frac{M-m}{M+m}g$$

$$\boxed{a = \frac{1-\frac{m}{M}}{1+\frac{m}{M}}g}$$

Atwood Machine #1 (Morin 3.27)



Newton II: $Ma_1 = Mg - T$ ①

$2Ma_2 = 4T - 2Mg$ ②

"Conservation of string"

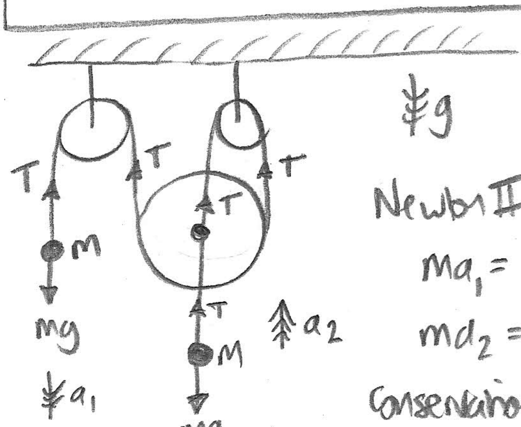
$a_1 = 4a_2$ ③

So $4Ma_1 = 4Mg - 4T$ ④

$2M\left(\frac{a_1}{4}\right) = 4T - 2Mg$ ⑤

④ + ⑤: $\frac{9}{2}Ma_1 = 2Mg \quad \therefore \boxed{a_1 = \frac{4}{9}g} \quad \therefore \boxed{a_2 = \frac{1}{9}g}$

Atwood Machine #2 (Morin 3.28)



Newton II:

$Ma_1 = Mg - T$ ①

$Ma_2 = 3T - Mg$ ②

Conservation of string

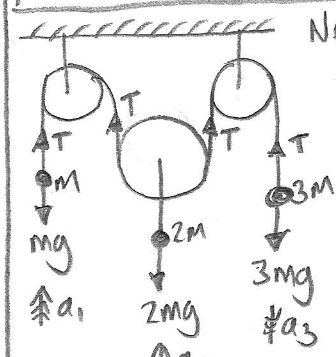
$a_1 = 3a_2$ ③

$\therefore 3M(3a_2) = 3Mg - 3T$ ④

$10Ma_2 = 2Mg$ ② + ④

$\therefore \boxed{a_2 = \frac{g}{5}} \quad \therefore \boxed{a_1 = \frac{3g}{5}}$

Atwood Machine #3 (Morin 3.29)



Newton II: $Ma_1 = T - Mg$ ①

$2Ma_2 = 2T - 2Mg$ ②

$3Ma_3 = 3Mg - T$ ③

String conservation:

$a_3 = a_1 + 2a_2$ ④

$\therefore 3M(a_1 + 2a_2) = 3Mg - T$ ⑤

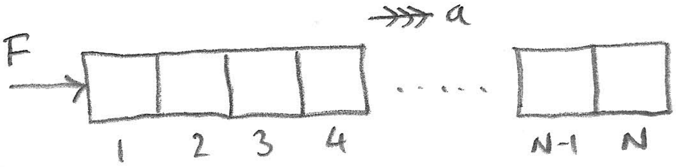
$\therefore 3Ma_1 + 6Ma_2 = 3Mg - T$

$\therefore 3(T - Mg) + 6Ma_2 = 3Mg - T \quad \therefore 6Ma_2 = 6Mg - 4T$

②: $6Ma_2 = 6T - 6Mg \quad \therefore 6T - 6Mg = 6Mg - 4T$

$\therefore 10T = 12Mg \quad \therefore \boxed{T = \frac{6}{5}Mg} \Rightarrow \boxed{a_1 = \frac{g}{5}, a_2 = \frac{g}{5}, a_3 = \frac{3g}{5}}$

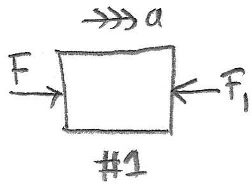
Force on N cubes problem (or train pushing N carriages)



N cubes, all with mass m
Assume cubes are rigid and move as a single unit

Newton II (all N cubes): $\boxed{F = Nma}$ $\therefore$ $\boxed{a = F/Nm}$ and $\boxed{ma = \frac{F}{N}}$

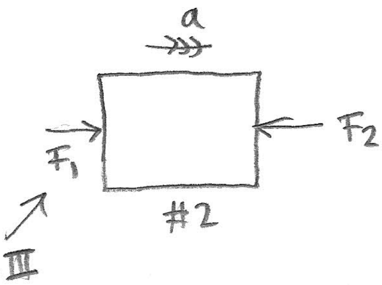
Cube #1:



F_1 ← Contact force of other cubes on cube #1

Newton II: $ma = F - F_1$ $\therefore$ $\frac{mF}{Nm} = F - F_1$ $\therefore$ $\boxed{F_1 = F\left(1 - \frac{1}{N}\right)}$

Newton II (cube #2):

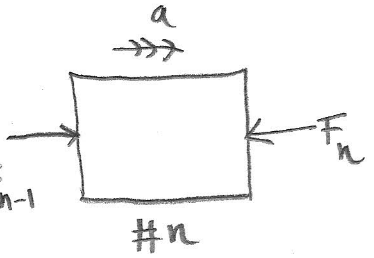


$ma = F_1 - F_2$

$\therefore F_2 = \underbrace{F - \frac{F}{N}}_{F_1} - \underbrace{\frac{F}{N}}_{ma}$

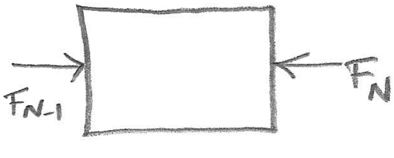
$\therefore \boxed{F_2 = F(1 - 2/N)}$

This generalizes



$\boxed{F_n = F\left(1 - n/N\right)}$

So force on last cube is



$F_N = F\left(1 - \frac{N}{N}\right) = 0$

As expected since no more carriages!