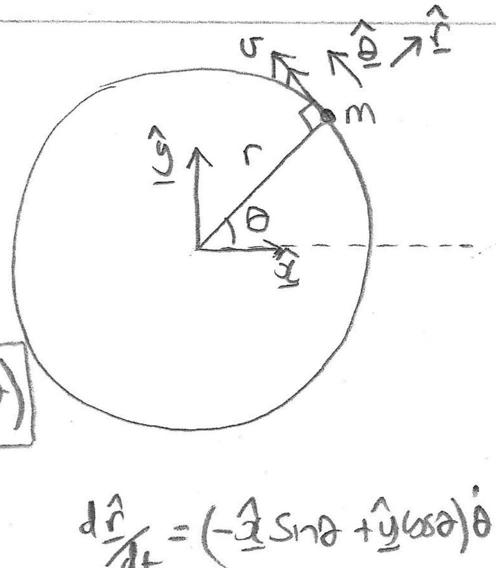
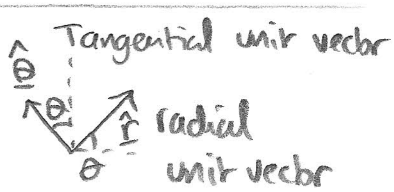
PRE-U REVISION NOTES, MECHANICS AF|3|18

ROTATIONAL MECHANICS : CIRCULAR MOTION, MOMENTS, TORQUE
MOMENTS OF INERTIA, CENTRE OF MASS
ANGULAR MOMENTUM





CIRCULAR MOTION

ie position vector

polars $\boxed{\underline{r} = r\hat{\underline{r}}}$ r = constant

Define $\boxed{\omega \equiv \dot{\theta}}$

Angular velocity (radians/s)

$\boxed{\dot{\theta} = \frac{d\theta}{dt}}$

Cartesians: $\boxed{\underline{r} = r(\hat{\underline{x}}\cos\theta + \hat{\underline{y}}\sin\theta)}$

Now $\hat{\underline{r}} = \hat{\underline{x}}\cos\theta + \hat{\underline{y}}\sin\theta$

$\hat{\underline{\theta}} = -\hat{\underline{x}}\sin\theta + \hat{\underline{y}}\cos\theta$

$\frac{d\hat{\underline{r}}}{dt} = (-\hat{\underline{x}}\sin\theta + \hat{\underline{y}}\cos\theta)\dot{\theta}$

$\boxed{\frac{d\hat{\underline{r}}}{dt} = \dot{\theta}\hat{\underline{\theta}}}$

$\frac{d\hat{\underline{\theta}}}{dt} = (-\hat{\underline{x}}\cos\theta - \hat{\underline{y}}\sin\theta)\dot{\theta}$

$\boxed{\frac{d\hat{\underline{\theta}}}{dt} = -\dot{\theta}\hat{\underline{r}}}$

Velocity $\underline{v} = \frac{d}{dt}(r\hat{\underline{r}}) = r\frac{d\hat{\underline{r}}}{dt} + \hat{\underline{r}}\dot{r}$

$\therefore \boxed{\underline{v} = r\dot{\theta}\hat{\underline{\theta}} + \dot{r}\hat{\underline{r}}}$

if r = constant $\dot{r} = 0$

$\therefore \boxed{\underline{v} = r\dot{\theta}\hat{\underline{\theta}}}$ or $\boxed{v = r\omega}$

Acceleration $\underline{a} = \frac{d\underline{v}}{dt} = \frac{d}{dt}(r\dot{\theta}\hat{\underline{\theta}} + \dot{r}\hat{\underline{r}})$

$= r\dot{\theta}(-\dot{\theta}\hat{\underline{r}}) + \hat{\underline{\theta}}\left(r\ddot{\theta} + \dot{\theta}\dot{r}\right) + \ddot{r}\hat{\underline{r}} + \dot{r}\dot{\theta}\hat{\underline{\theta}}$

$\therefore \boxed{\underline{a} = \hat{\underline{r}}\left(-r\dot{\theta}^2 + \ddot{r}\right) + \hat{\underline{\theta}}\left(2\dot{r}\dot{\theta} + r\ddot{\theta}\right)}$

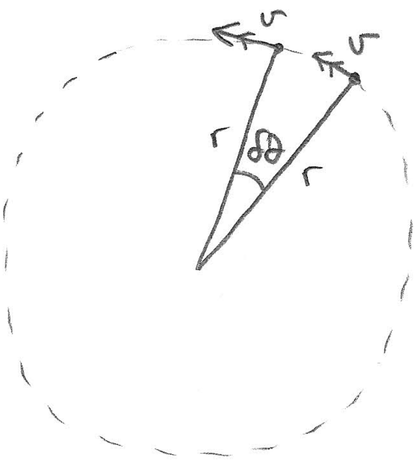
if $\dot{r} = 0$ $\boxed{\underline{a} = \hat{\underline{r}}\left(-r\dot{\theta}^2\right) + \hat{\underline{\theta}}\left(r\ddot{\theta}\right)}$

ie radially inward ("centripetal") acceleration $\boxed{r\omega^2}$

tangential acceleration $r\ddot{\theta} = \boxed{r\dot{\omega}}$

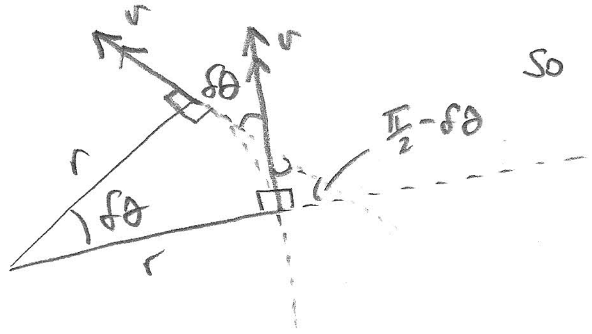
Note since $\boxed{v = r\omega}$, centripetal acceleration $= \boxed{\frac{v^2}{r}}$

Simple derivation of centripetal acceleration for <u>uniform</u> circular motion ie $\omega = \dot{\theta} = $ constant, as well as $r =$ constant.

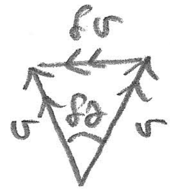


If particle rotates angle $\delta\theta$ radians in δt

$\boxed{v\,\delta t = r\,\delta\theta}$ is the arc length



So



If $\delta\theta \ll 1$

$\boxed{\delta v \approx v\,\delta\theta}$

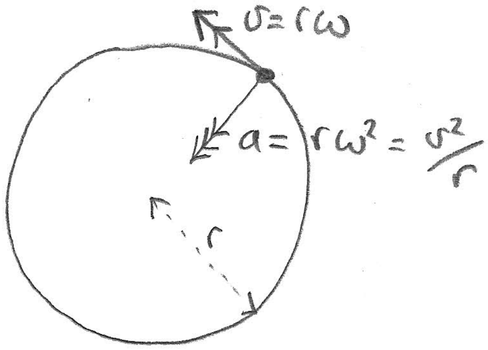
$$\therefore \text{ acceleration } \quad a = \frac{\delta v}{\delta t} = \frac{v\,\delta\theta}{r\delta\theta/v} \Rightarrow \boxed{a = \frac{v^2}{r}}$$

Also, for uniform circular motion $\boxed{v = \frac{2\pi r}{T}}$ where T is the time period for 1 rotation

$\boxed{\omega = \frac{2\pi}{T}}$ so $\boxed{v = r\omega}$

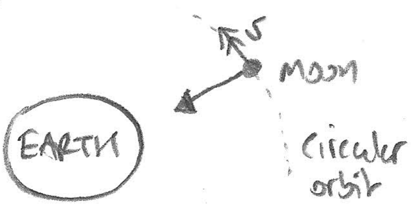
Note if rotations / s $= f$ (Hz) ← rotation frequency

$\boxed{f = \frac{1}{T}}$ $\therefore$ $\boxed{\omega = 2\pi f}$



uniform circular motion $\dot{r} = 0$
$\dot{\omega} = 0$

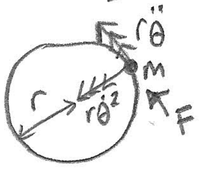
So for uniform circular motion there must be a net force acting radially inwards - for example GRAVITY



Now consider a particle of mass m moving in a circle of radius r but driven by tangential force $F\hat{\underline{\theta}}$

Newton II: $m r\ddot{\theta} = F$
$/\!/ \hat{\underline{\theta}}$

$$m r^2 \ddot{\theta} = Fr$$



Fr is the $\boxed{\text{turning moment}}$ or $\boxed{\text{Torque}}$ τ magnitude ("Force × ⊥ distance")

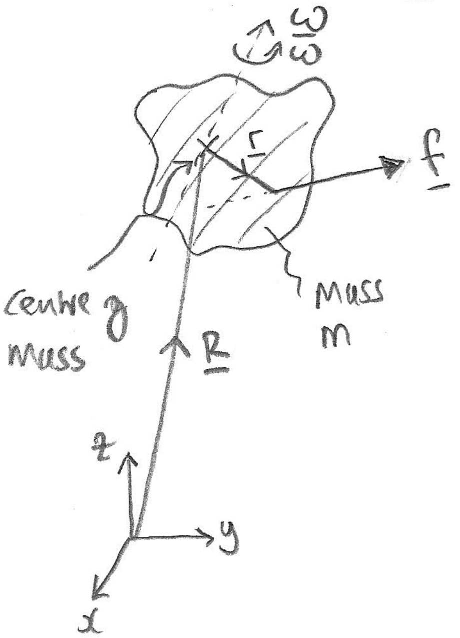
So $\tau = m r^2 \dot{\omega}$

$\boxed{\tau = I\dot{\omega}}$ $\quad I = mr^2$ Moment of inertia

Note $KE = \frac{1}{2}mv^2$

$KE = \frac{1}{2}mr^2\omega^2 = \boxed{\frac{1}{2}I\omega^2}$

In general for a $\boxed{\text{rigid body}}$ acted upon by force $\underline{f}$



Newton II : $\boxed{m\ddot{\underline{R}} = \underline{f}}$

ie acceleration of centre of mass CoM

$\boxed{\underline{\tau} = \underline{r} \times \underline{f} \quad \text{Torque}}$

$\underline{r}$ is displacement of force from centre of mass

$\underline{\tau} = \underline{\underline{I}}\, \dot{\underline{\omega}}$

↑

Moment of inertia matrix ("tensor")

$\underline{\omega} = \omega_x \hat{\underline{x}} + \omega_y \hat{\underline{y}} + \omega_z \hat{\underline{z}}$

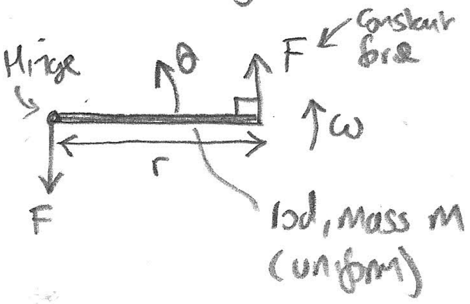
Angular velocity vector (ie rotation axis through CoM)

If no net torque $\underline{\underline{I}}\,\dot{\underline{\omega}} = 0$ so $\underline{\underline{I}}\,\underline{\omega} = \text{constant}$

This is called the $\boxed{\text{angular momentum}}$ $\boxed{\underline{L} = \underline{\underline{I}}\,\underline{\omega}}$

$\underline{\underline{I}} \to I$

For simplicity let's choose objects where rotation axis is fixed



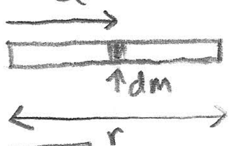
$\boxed{\underbrace{Fr}_{\text{Torque}} = I\dot{\omega}}$

So if $\omega = 0$ when $t = 0$

$\boxed{\omega(t) = \frac{Fr}{I}t}$

$\omega = \frac{d\theta}{dt}$ $\therefore$ $\boxed{\theta(t) = \frac{1}{2}\frac{Fr}{I}t^2}$

Now $I = \int dm\, x^2$



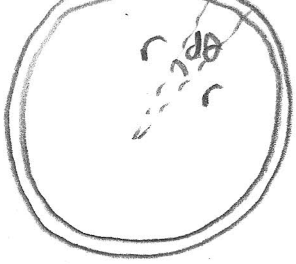
If rod is uniform $\boxed{dm = \frac{dx}{r}m}$

$\therefore$ $I = \int_0^r \frac{m}{r}x^2\,dx$

$I = \frac{m}{r}\left[\frac{1}{3}x^3\right]_0^r$

$\boxed{I = \frac{1}{3}mr^2}$ ← Moment of inertia of rod length r mass m about one end.

$\boxed{\text{Moment of inertia of a hoop of mass m}}$



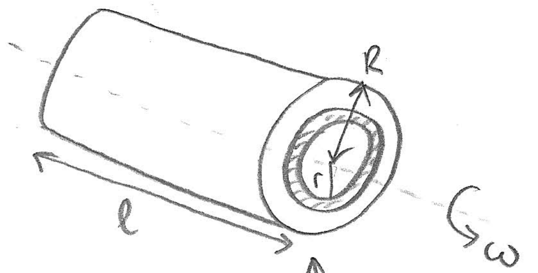
$I = \int dm\, r^2$

$dm = \frac{r\,d\theta}{2\pi r}m$

$\therefore$ $I = \int_0^{2\pi} \frac{mr^2\,d\theta}{2\pi} = \boxed{mr^2}$

Moment of inertia of a solid cylinder

or a disc



Mass M

$\therefore$ density $\boxed{\rho = \frac{M}{\pi R^2 l}}$

$$I = \int dm \times r^2$$

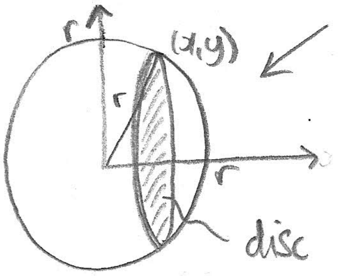
dm = mass of tube of radius r

$$dm = \rho l \times 2\pi r\,dr = \frac{M}{\pi R^2 \cancel{l}} \cancel{l} \times 2\pi r\,dr$$

$$dm = \frac{2M r\,dr}{R^2}$$

$$\therefore I = \int_0^R \frac{2M}{R^2} r^3\,dr = \frac{2M}{R^2}\left[\frac{1}{4}r^4\right]_0^R = \boxed{\frac{1}{2}MR^2}$$

Moment of inertia of a solid sphere



Slice sphere into discs

Mass $dm = \rho\pi y^2 dx$

$$I = \int \frac{1}{2} dm\, y^2$$ ie sum of disc moment of inertias

$$\therefore I = 2\int_{x=0}^{r} \frac{1}{2}\rho\pi y^4\,dx$$

$$\boxed{\rho = \frac{M}{\frac{4}{3}\pi r^3}}$$

Pythagoras: $x^2 + y^2 = r^2$ $\quad \therefore y^4 = (r^2 - x^2)^2$

$$y^4 = r^4 - 2r^2x^2 + x^4$$

$$\therefore I = \frac{3M}{4r^3}\int_0^r (r^4 - 2r^2x^2 + x^4)\,dx = \frac{3M}{4r^3}\left[r^4x - \frac{2}{3}r^2x^3 + \frac{1}{5}x^5\right]_0^r$$

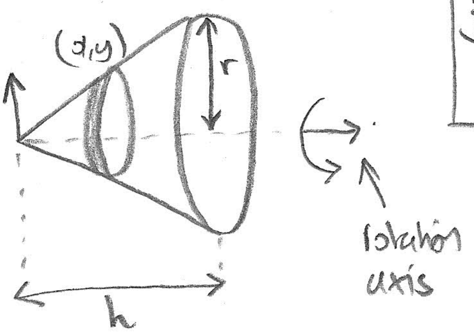
$$I = \frac{3M}{4r^3}\left(1 - \frac{2}{3} + \frac{1}{5}\right)r^5$$

$$I = \frac{3}{4}\left(\frac{1}{3} + \frac{1}{5}\right)Mr^2$$

$$I = \frac{3}{4} \times \frac{8}{15} Mr^2$$

$$\boxed{I = \frac{2}{5}Mr^2}$$

Moment of inertia of a solid cone



$$\boxed{\rho = \frac{M}{\frac{1}{3}\pi r^2 h}}$$

$I = \int \frac{1}{2} dm \times y^2$ i.e. sum of disc moment of inertias

$$\boxed{\frac{y}{x} = \frac{r}{h}}$$ for cone boundary

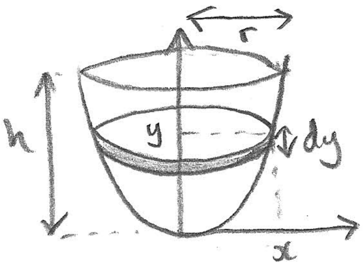
$dm = \pi y^2 dx\, \rho$

$\therefore dm = \pi \rho \frac{r^2 x^2}{h^2} dx$

So $I = \frac{1}{2}\pi \frac{M}{\frac{1}{3}\pi r^2 h} \frac{r^2}{h^2} \int_0^h x^2 \frac{r^2 x^2}{h^2} dx$

$I = \frac{3}{2} m r^2 \frac{1}{h^5} \left[\frac{1}{5} x^5\right]_0^h \Rightarrow \boxed{I = \frac{3}{10} m r^2}$

Moment of inertia of a parabolic cap (of mass M)



Volume $V = \int_0^h \pi x^2 dy = \frac{\pi r^2}{h} \int_0^h y\, dy$

$= \frac{\pi r^2}{h} \left[\frac{1}{2} y^2\right]_0^h$

$= \frac{\pi r^2}{2h} h^2 \Rightarrow \boxed{V = \frac{1}{2}\pi r^2 h}$

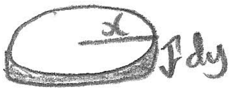
let $\boxed{y = h\frac{x^2}{r^2}}$

ie $x^2 = \frac{r^2}{h} y$

$\therefore$ Density $\boxed{\rho = \frac{M}{\frac{1}{2}\pi r^2 h}}$

So moment of inertia about y rotation axis is $I = \int \frac{1}{2} dm\, x^2$

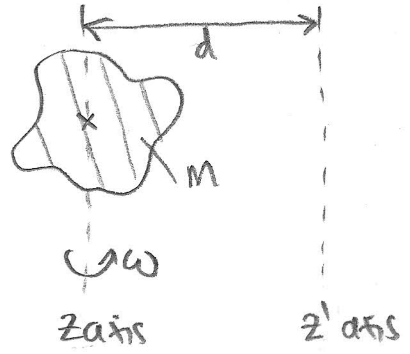
$dm = \rho \pi x^2 dy = \frac{2M}{\pi r^2 h} \pi \left(\frac{r^2}{h} y\right) dy = \frac{2M}{h^2} y\, dy$. Note $x^2 = \frac{r^2 y}{h}$



$\therefore I = \frac{2M}{h^2} \frac{r^2}{h} \frac{1}{2} \int_0^h y^2 dy = \frac{M r^2}{h^3} \left[\frac{1}{3} y^3\right]_0^h$

$\therefore \boxed{I = \frac{1}{3} m r^2}$

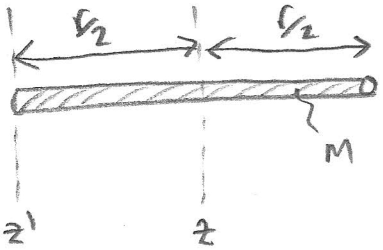
// axis theorem



If I_z is the moment of inertia of a rigid body of mass about an axis of rotation z through the object centre of mass

$$\boxed{I_{z'} = I_z + md^2}$$

So for rigid rod :



From above

$I_{z'} = \frac{1}{3}mr^2$

$\therefore \; I_z = \frac{1}{3}mr^2 - m\left(\frac{r}{2}\right)^2$

$I_z = mr^2\left(\frac{1}{3} - \frac{1}{4}\right)$

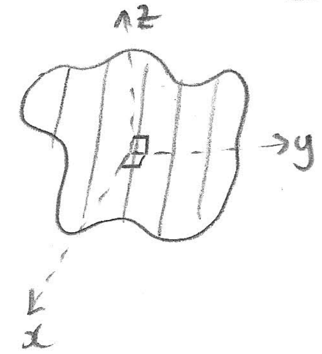
$$\boxed{I_z = \frac{1}{12}mr^2}$$

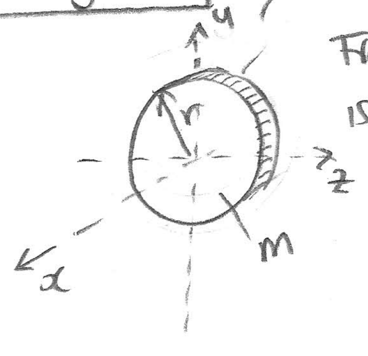
For 2D rigid objects ie "Laminae" there is also the

⊥ axis theorem

$$\boxed{I_x = I_y + I_z}$$

i.e neglect thickness ($\ll r$)



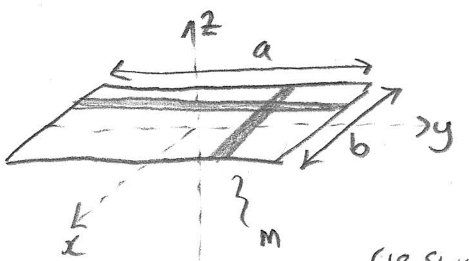


From above I_x for a disc is $\frac{1}{2}mr^2$. Clearly from symmetry $I_y = I_z$

$\therefore \; \frac{1}{2}mr^2 = 2I_y$

$\therefore \; \boxed{I_y = \frac{1}{4}mr^2}$

Moments of inertia of a rectangular plate



$I_y = 2\int_0^{b/2} \frac{a\,dx}{ab} m\,x^2 = \frac{2M}{b}\left[\frac{1}{3}x^3\right]_0^{b/2}$

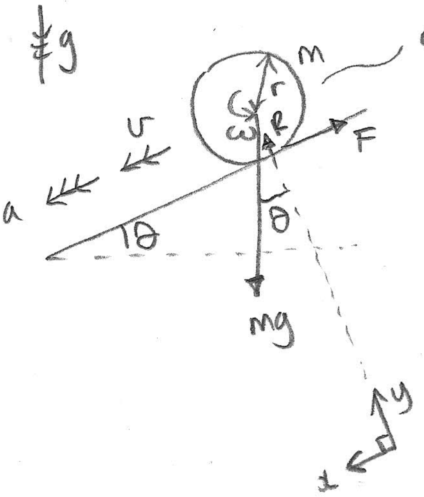
$= \frac{2mb^2}{24} = \boxed{\frac{mb^2}{12}}$

$\therefore \; a \leftrightarrow b \Rightarrow \boxed{I_x = \frac{ma^2}{12}}$

(ie swap a to b for x to y)

$\therefore$ by ⊥ axis theorem $\boxed{I_z = \frac{M}{12}(a^2+b^2)}$

ROLLING A CYLINDER OR SPHERE DOWN A SLOPE WITHOUT SLIPPING



cylinder or sphere of mass m, rolling at angular speed ω

If no slip $\boxed{v = r\omega}$ also $\boxed{F \leq \mu R}$

Newton II: $//x$: $ma = mg\sin\theta - F$

$//y$: $0 = R - mg\cos\theta$

"Torque = moment of inertia × angular acceleration"

$\uparrow$ only turning moment about cylinder/sphere COM is $F \times r$

So $\boxed{Fr = I\dot{\omega}}$

$a = \dot{v} = r\dot{\omega}$

$\therefore \boxed{\dot{\omega} = \frac{a}{r}}$

$$\boxed{\begin{aligned} I_{sphere} &= \tfrac{2}{5} Mr^2 \\ I_{cyl} &= \tfrac{1}{2} Mr^2 \end{aligned}}$$

$\therefore Fr = I a/r$

$\therefore \boxed{F = I a/r^2}$

$\therefore ma = mg\sin\theta - Ia/r^2 \Rightarrow a\left(m + I/r^2\right) = mg\sin\theta$

$$\therefore \boxed{a = \frac{g\sin\theta}{1 + I/Mr^2}}$$

For no slip $\mu \geq \frac{F}{R}$

$$\mu \geq \frac{Ia/r^2}{mg\cos\theta}$$

$$\mu \geq \frac{I}{r^2} \frac{g\sin\theta}{1 + I/Mr^2} \frac{1}{mg\cos\theta}$$

$$\mu \geq \frac{I}{Mr^2} \frac{1}{1 + I/Mr^2} \tan\theta$$

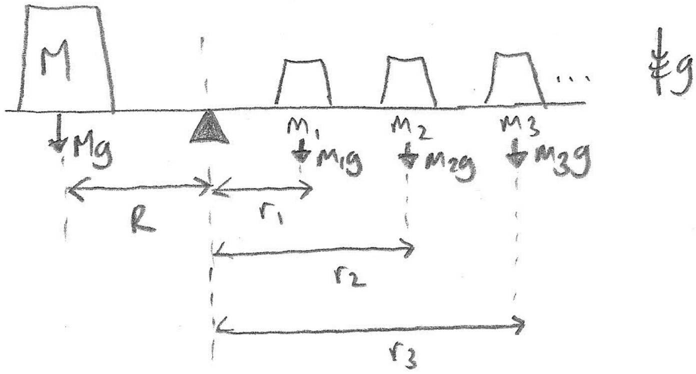
$$\boxed{\mu \geq \frac{\tan\theta}{\frac{Mr^2}{I} + 1}}$$

Particle on slope sliding
$a = g\sin\theta$, $\mu < \tan\theta$

Rolling cylinder, no slip
$a = \frac{2}{3} g\sin\theta$, $\mu \geq \frac{1}{3}\tan\theta$

Rolling sphere, no slip
$a = \frac{5}{7} g\sin\theta$, $\mu \geq \frac{2}{7}\tan\theta$

CENTRES OF MASS



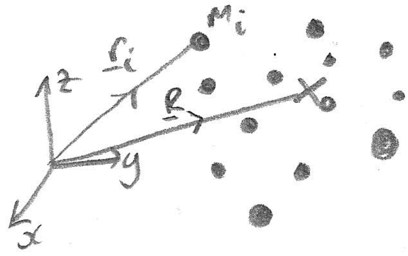
For the see-saw on the right to balance there can be no net turning moment $\underline{\tau} = \underline{r} \times \underline{f}$

All forces are weights so, taking $\circlearrowleft +$ moments, at eq:

$$0 = \sum_i r_i M_i g - RMg \quad \therefore \quad \boxed{R = \frac{\sum_i M_i r_i}{M}}$$

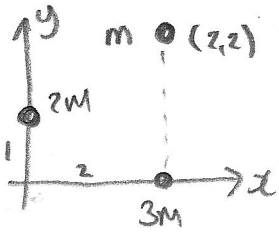
If $M = \sum_i M_i$ we can "represent the total weight on the right as acting at R from pivot"

Hence of a discrete system of masses



$$\boxed{\underline{R} = \frac{\sum_i M_i \underline{r}_i}{\sum_i M_i}}$$

$\uparrow$ centre of mass

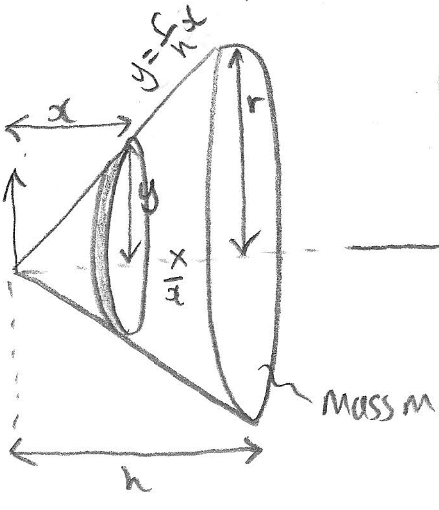


Centre of mass is $\dfrac{2m\begin{pmatrix}0\\1\end{pmatrix} + 3m\begin{pmatrix}2\\0\end{pmatrix} + m\begin{pmatrix}2\\2\end{pmatrix}}{2m + 3m + m} = \underline{R}$

$$\therefore \underline{R} = \frac{1}{6}\begin{pmatrix}8\\4\end{pmatrix} \Rightarrow \boxed{\underline{R} = \begin{pmatrix}4/3\\2/3\end{pmatrix}}$$

Integration needed for solid objects.

CENTRE OF MASS OF A SOLID CONE



$y/x = r/h$ for cone envelope

Volume of cone is $\frac{1}{3}\pi r^2 h$

Density $\rho = \dfrac{M}{\frac{1}{3}\pi r^2 h}$

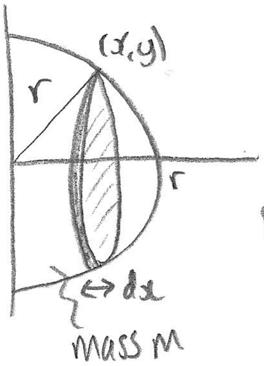
$\longrightarrow$ CoM $\bar{x} = \frac{1}{m}\int dm \times x$

$$dm = \rho \times \pi y^2 dx = \frac{M}{\frac{1}{3}\pi r^2 h}\pi \frac{r^2 x^2}{h^2} dx$$

$$\therefore \bar{x} = \frac{1}{M}\frac{3M}{h^3}\int_0^h x^3 dx \Rightarrow \boxed{\bar{x} = \frac{3}{4}h}$$

ie $\frac{1}{4}h$ from base

CENTRE OF MASS OF A SOLID HEMISPHERE



Density $\boxed{\frac{M}{\frac{2}{3}\pi r^3} = \rho}$

Pythagoras: $r^2 = x^2 + y^2$

$\therefore y^2 = r^2 - x^2$

COM $\bar{x} = \frac{1}{M}\int x \times dm$

$dm = \rho \times \pi y^2 dx$

$dm = \frac{3M}{2\pi r^3}\pi y^2 dx$

$dm = \frac{3}{2}M \frac{y^2}{r^3} dx$

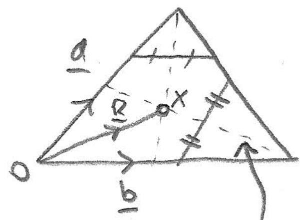
$dm = \frac{\frac{3}{2}M(r^2 - x^2)}{r^3} dx$

So $\bar{x} = \frac{3}{2r^3}\int_0^r (r^2 x - x^3)\,dx \Rightarrow \bar{x} = \frac{3}{2r^3}\left[\frac{1}{2}r^2x^2 - \frac{1}{4}x^4\right]_0^r$

$\bar{x} = \frac{3}{2}r\left(\frac{1}{2} - \frac{1}{4}\right)$

$\boxed{\bar{x} = \frac{3}{8}r}$

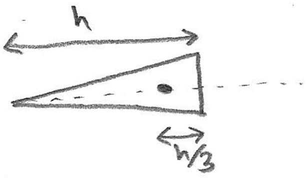
CENTRE OF MASS OF TRIANGULAR LAMINA



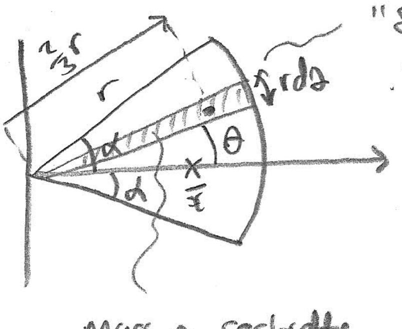
$\underline{R} = \vec{OX} = \boxed{\frac{1}{3}(\underline{a} + \underline{b})}$

Can prove (see Eclecticon Maths | Centre of mass in mechanics) using intersection of median lines

So



CENTRE OF MASS OF CIRCULAR LAMINA (well, sector) ← of angular width 2α



"Sectorette" $d\theta \to 0$ so treat as triangular lamina

$\therefore x$ coordinate of COM of sectorette is $\frac{2}{3}r\cos\theta$

$\therefore \bar{x} = \frac{1}{M}\int dm \times x$

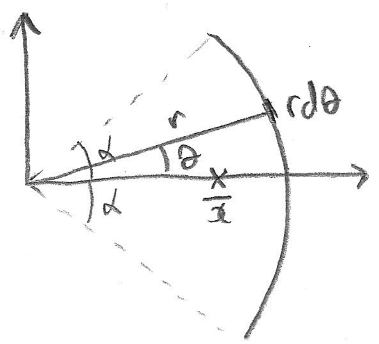
$\bar{x} = \frac{1}{2\alpha}\frac{2}{3}r\int_{-\alpha}^{\alpha}\cos\theta\, d\theta \Rightarrow \bar{x} = \frac{r}{3\alpha}\left[\sin\theta\right]_{-\alpha}^{\alpha}$

Mass of sectorette

$dm = \frac{\frac{1}{2}r^2 d\theta}{\frac{1}{2}r^2(2\alpha)} M$ ← total mass

$\therefore \boxed{\bar{x} = \frac{2r}{3\alpha}\sin\alpha}$

CENTRE OF MASS OF A WIRE ARC



let mass per unit length be $\rho = \dfrac{M}{2r\alpha}$

$$\bar{x} = \frac{1}{M}\int dm \times r\cos\theta$$

$$dm = rd\theta \times \rho \qquad \therefore \bar{x} = \frac{1}{M}\int_{-\alpha}^{\alpha} \frac{M}{2r\alpha} r\cos\theta \, d\theta$$

$$\bar{x} = \frac{1}{2\alpha} r \Big[\sin\theta\Big]_{-\alpha}^{\alpha}$$

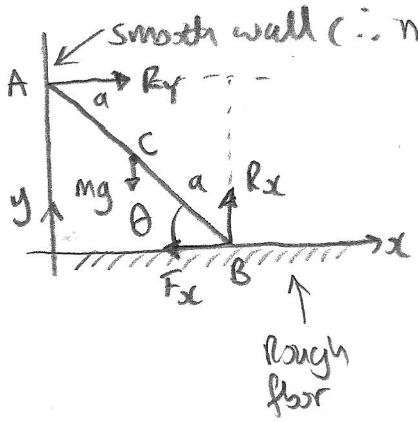
$$\boxed{\bar{x} = \frac{r\sin\alpha}{\alpha}}$$

EQUILIBRIUM PROBLEMS FOR RIGID BODIES

Newton II: $\boxed{\text{No net force}}$ No slip (Momentum conserved)

$\boxed{\text{No net torque}}$ (Angular Momentum conserved)

* key idea * Don't have to take moments about centre of mass. Since torque $= 0$ we can choose any location!



What are the conditions for eq (static) for a ladder of mass m of length $2a$ leant against a smooth vertical wall?

Newton II: $/\!/x$: $\boxed{0 = R_y - F_x}$ ①

$/\!/y$: $\boxed{0 = R_x - mg}$ ②

No slip: $\boxed{F_x \le \mu R_x}$ ③

No torque: Take ↺+ moments about B

$$\boxed{0 = R_y \times 2a\sin\theta - mg \times a\cos\theta} \quad ④$$

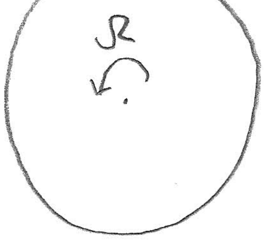
So: ④: $\boxed{R_y = \dfrac{mg\cos\theta}{2\sin\theta}}$ $\therefore$ in ① $\boxed{F_x = R_y}$ in ② $\boxed{R_x = mg}$

So in ③: $\mu \ge \dfrac{F_x}{R_x} \Rightarrow \mu \ge \dfrac{\frac{mg\cos\theta}{2\sin\theta}}{mg} \therefore \boxed{\mu \ge \frac{1}{2}\cot\theta}$

CONSERVATION OF ANGULAR MOMENTUM

A small, sticky mass m is dropped at radius r on a turntable of moment of inertia I, initially rotating at Ω rads^{-1}

BEFORE



AFTER

$I = \frac{1}{2}MR^2$

If turntable has mass M and radius R

Conservation of angular momentum ↺+

$$\boxed{I\Omega = I\omega + \underbrace{mr^2\omega}}$$

$L = mrv$ Angular momentum of mass m

$v = r\omega$

$$\therefore \boxed{\omega = \frac{I}{I+mr^2}\Omega}$$

Energy loss
$$\Delta E = \frac{1}{2}I\Omega^2 - \frac{1}{2}I\omega^2 - \underbrace{\frac{1}{2}mr^2\omega^2}_{\frac{1}{2}mv^2}$$

$$\Delta E = \frac{1}{2}I\Omega^2 - \frac{1}{2}\left(\frac{I}{I+mr^2}\right)^2(I+mr^2)\Omega^2$$

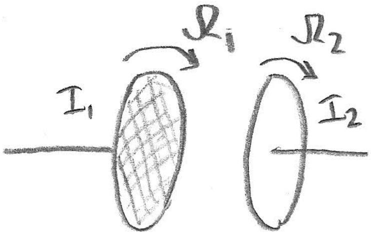
$$\boxed{\Delta E = \frac{1}{2}I\Omega^2\left(1 - \frac{I}{I+mr^2}\right)}$$

So as $\frac{mr^2}{I}$ increases a greater fraction of original energy $\frac{1}{2}I\Omega^2$ is lost.

CLUTCH EXAMPLE

← eg in a car engine rotating shafts come together

BEFORE



AFTER

Conservation of angular momentum

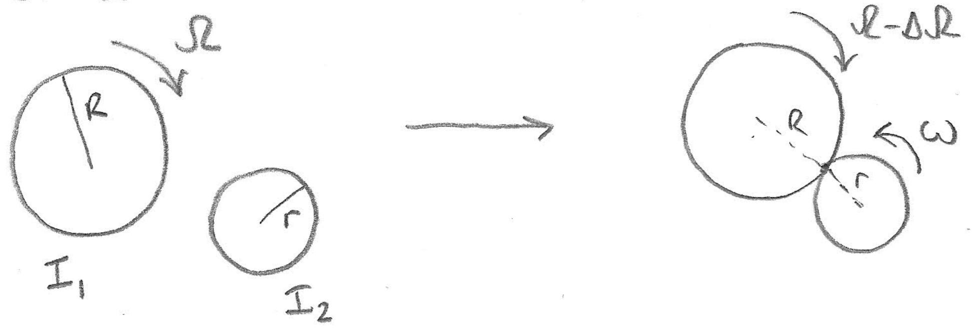
$$I_1\Omega_1 + I_2\Omega_2 = (I_1+I_2)\omega$$

$$\therefore \boxed{\omega = \frac{I_1\Omega_1 + I_2\Omega_2}{I_1+I_2}}$$

$$\boxed{\Delta E = \frac{1}{2}I_1\Omega_1^2 + \frac{1}{2}I_2\Omega_2^2 - \frac{1}{2}(I_1+I_2)\omega^2}$$

GEAR SYSTEMS OFF AXIS (OR ROLLERS)

No external torque so total angular momentum must be conserved?



Two cylindrical rollers / gears

velocity at contact point must be the same for both gears

$$\therefore \boxed{R(\Omega - \Delta\Omega) = r\omega}$$

NOT true!

Total angular momentum conserved? $I_1\Omega = I_1(\Omega - \Delta\Omega) - I_2\omega$

This doesn't make sense, since for this to be true $\Delta\Omega < 0$ and this means roller 2 speeds up! So 'no external torque' can't be used to justify conservation of angular momentum → the act of putting the gears together must supply an input of angular momentum.

Using $\boxed{\text{energy}}$ instead: $$\boxed{\underbrace{\Delta E}_{\text{loss}} = \tfrac{1}{2}I_1\Omega^2 - \tfrac{1}{2}I_2\omega^2 - \tfrac{1}{2}I_1(\Omega - \Delta\Omega)^2}$$

let $\Delta E = \varepsilon \frac{1}{2} I_1 \Omega^2$ $\quad\varepsilon = 0$ is a lossless meshing of gears. *

$$\therefore \varepsilon \tfrac{1}{2}I_1\Omega^2 = \tfrac{1}{2}I_1\Omega^2 - \tfrac{1}{2}I_2\omega^2 - \tfrac{1}{2}I_1\left(\frac{r\omega}{R}\right)^2$$

$$\therefore \omega^2\left(I_1 r^2/R^2 + I_2\right) = \Omega^2(I_1 - \varepsilon I_1)$$

$$\omega^2 = \frac{\Omega^2 I_1(1-\varepsilon)}{I_1 r^2/R^2 + I_2}$$

$$\omega^2 = \frac{\Omega^2(1-\varepsilon)}{r^2/R^2 + I_2/I_1}$$

$$\boxed{\omega = \Omega\left(\frac{1-\varepsilon}{r^2/R^2 + I_2/I_1}\right)^{\frac{1}{2}}}$$

Then $\boxed{\Delta\Omega = \Omega - \frac{r\omega}{R}}$

Angular impulses are $I_1\Delta\Omega$ ↺+ and $I_2\omega$ respectively ie a net amount

* $\varepsilon = 1 \Rightarrow \omega = 0$ which also makes sense - ie all energy lost!