PRE-U REVISION NOTES. MECHANICS AF|3|18

SPECIAL RELATIVITY

$\boxed{\text{Historical timeline}}$ (ish!)

- Light — the best understood physical phenomena ↙ Reflection, Refraction, Diffraction, Interference

- Huygens, Fresnel, Young … $\boxed{\text{Wave model of light}}$
- Electricity & Magnetism → ELECTROMAGNETISM
- Foucault & Fizeau use clockwork strobe effect to measure the speed of light. If light is a wave, in what medium does it propagate? The "Luminiferous Aether"
- Maxwell predicts Electromagnetic waves propagate at speed $\boxed{c = \frac{1}{\sqrt{\varepsilon_0 \mu_0}}}$ $\boxed{\text{independent of coordinate system}}$ $\boxed{c = 2.998 \times 10^8 \text{ m/s}}$
- Faraday, Helmholtz, Hertz, Lorentz confirm Maxwell's predictions experimentally
- Michelson & Morley show light can $\boxed{\text{propagate in a vacuum}}$ ie no aether is required
- Experiments with $\boxed{\text{pions}}$ show emitted γ rays travel at c regardless of the speed of the pion which emits them
  - * Alväger 1964
  - * Filippas & Fox 1963
- Muon mystery! Far too many are detected, given known 2.2μs half life [Muons are created in upper atmosphere as a result of cosmic radiation]

Magnetic field from a current carrying wire

$\boxed{B = \frac{\mu_0 I}{2\pi r}}$

$\boxed{\mu_0 = 4\pi \times 10^{-7} \text{ Hm}^{-1}}$

$\boxed{F = \frac{qQ}{4\pi\varepsilon_0 r^2}}$

Force between two charges

$\boxed{\varepsilon_0 = 8.85 \times 10^{-12} \text{ Fm}^{-1}}$

$\pi^0 \rightarrow 2\gamma$

1879–1955 ↓

$\boxed{\text{Einstein}}$ resolves the 'paradox' of the frame invariance of the speed of light (no matter how fast you move a torch, all observers will measure the beam travelling at c) by allowing $\boxed{\text{time and space to change}}$ as speeds → c. This theory is called $\boxed{\text{SPECIAL RELATIVITY}}$

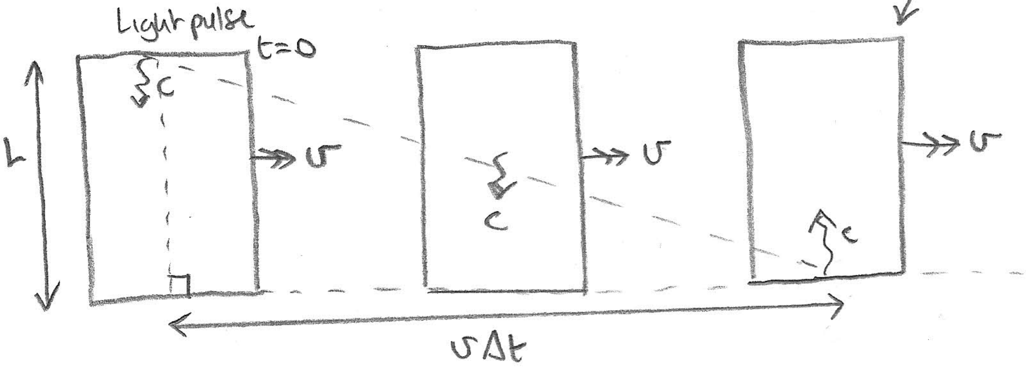
Three effects:

1. Time dilation ("Moving clocks run slow")
2. Length contraction
3. Loss of Simultaneity

→ incorporated into $\boxed{\text{LORENTZ TRANSFORMS}}$

TIME DILATION VIA PROF. FEYNMAN'S GLASS ELEVATOR

← Moves at speed v



Time taken for light pulse to travel top to bottom of elevator is $\boxed{\Delta t' = \frac{L}{c}}$

(measured in elevator frame of reference)

If elevator is transparent and 'refraction free' (!) a stationary observer sees the light pulse take the diagonal path, which takes time Δt.

Hence $\boxed{c = \frac{\sqrt{L^2 + v^2 \Delta t^2}}{\Delta t}}$ since light always travels at c

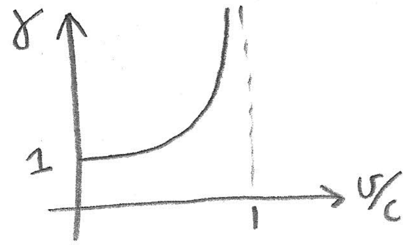
Hence using $L = c\Delta t'$ $\Rightarrow$ $c^2 \Delta t^2 = c^2 \Delta t'^2 + v^2 \Delta t^2$

$$\Rightarrow \Delta t'^2 = \frac{c^2 - v^2}{c^2} \Delta t^2$$

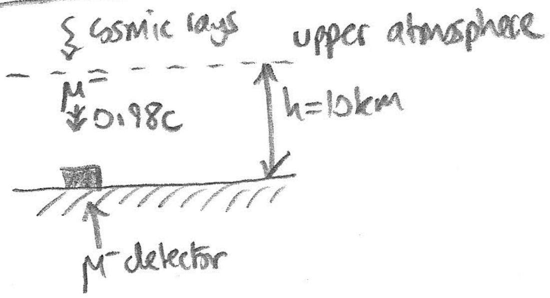
$$\therefore \Delta t' = \sqrt{1 - \frac{v^2}{c^2}}\, \Delta t$$

Define $\boxed{\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}}}$ $\therefore$ $\boxed{\Delta t' = \Delta t / \gamma}$

Since $\gamma \geq 1$ this means "moving clocks run slow" ie TIME DILATION



MUON MYSTERY SOLVED



Muons decay: $\mu^- \to e^- + \bar{\nu}_e + \nu_\mu$ with half life 2.2μs

To travel 10km should take a muon $\Delta t = \frac{h}{0.98c}$

$\Delta t \approx 34\mu s$

34μs ≅ 15.5 half lives. $\therefore$ Expect $\frac{1}{2^{15.5}} \approx \frac{1}{45{,}000}$ the rate of muon detections at ground level as at 10km. BUT WE SEE MORE LIKE $\frac{1}{8}$ ....

The explanation? Time dilation $\boxed{\Delta t = \Delta t' \gamma}$ and $\Delta t'$ corresponds to $2.2\,\mu s$ ie the $\frac{1}{2}$ life of the muon in its rest frame.

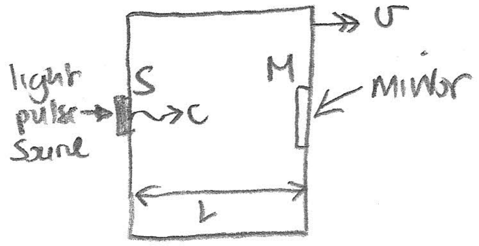
$\gamma = (1-0.98)^{-\frac{1}{2}} \approx 5.03$ $\therefore$ $\Delta t \approx 5.03 \times 2.2\,\mu s = 11.1\,\mu s$

Hence muon half life, $\boxed{\text{from detector's perspective}}$ is $\boxed{11.1\,\mu s}$

$\therefore$ expect fraction $\boxed{2^{-34/11.1} \simeq \frac{1}{8.4}}$ to be detected, which is about right experimentally.

LENGTH CONTRACTION

(using the glass elevator again)



Light pulse reflects off mirror and returns to source in time $\boxed{\Delta t' = 2L/c}$

A stationary observer sees the pulse travel a different distance, assuming from their perspective the elevator has width l.

$\boxed{c\Delta t_{SM} = l + v\Delta t_{SM}}$ (M moves away from original S position at speed v)

$\boxed{c\Delta t_{MS} = l - v\Delta t_{MS}}$ (S moves towards reflected light pulse)

Δt_{SM} Source to mirror time

Δt_{MS} Mirror to Source time

$\therefore$ $\boxed{\Delta t_{SM} = \frac{l}{c-v}}$ $\therefore$ $\boxed{\Delta t_{MS} = \frac{l}{c+v}}$

$\therefore$ Total there and back time is $\Delta t = \Delta t_{SM} + \Delta t_{MS} = l\left(\frac{1}{c-v} + \frac{1}{c+v}\right)$

$\therefore$ $\Delta t = l\left(\frac{c+v+c-v}{c^2-v^2}\right) = \frac{2lc}{c^2}\frac{1}{1-v^2/c^2}$ $\therefore$ $\boxed{\Delta t = \frac{2l}{c}\gamma^2}$ $\gamma = \left(1-\frac{v^2}{c^2}\right)^{-\frac{1}{2}}$

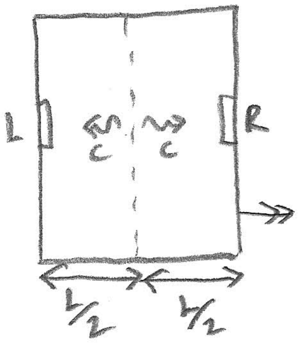
Now using $\boxed{\Delta t' = 2L/c}$ and time dilation result $\boxed{\Delta t' = \Delta t/\gamma}$

$\therefore$ $\frac{\Delta t}{\gamma} = \frac{2L}{c}$ $\therefore$ $\boxed{\Delta t = \frac{2L}{c}\gamma}$

Hence $\frac{2L}{c}\gamma = \frac{2l\gamma^2}{c}$ $\therefore$ $\boxed{l = \frac{L}{\gamma}}$ ie lengths contract, since $\gamma \geq 1$

LOSS OF SIMULTANEITY

ie clocks 'go out of sync'



Light source placed in centre of transparent elevator

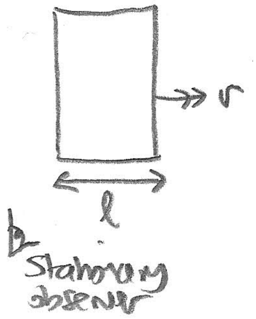
(R) means "right detector receives the pulse"

(L) means " left " " " " "

$\boxed{\text{Clearly they happen at the same time in the elevator}}$

What does the stationary observer see?

(ie outside the elevator, which has a relative velocity of v)



$$c\Delta t_R = \tfrac{1}{2}l + v\Delta t_R \Rightarrow \Delta t_R = \frac{l/2}{c-v}$$

$$c\Delta t_L = \tfrac{1}{2}l - v\Delta t_L \Rightarrow \Delta t_L = \frac{l/2}{c+v}$$

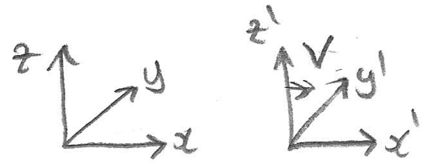
Define $\Delta t = \Delta t_R - \Delta t_L = \frac{1}{2}l\left(\frac{1}{c-v} - \frac{1}{c+v}\right)$

$$\Delta t = \frac{1}{2}l\left(\frac{c+v-c+v}{c^2-v^2}\right) = \frac{lv}{c^2}\frac{1}{1-v^2/c^2} = \frac{lv\gamma^2}{c^2}$$

Now using length contraction $\boxed{l = \frac{L}{\gamma}}$ so $\boxed{\Delta t = \frac{Lv\gamma}{c^2}}$

ie (R) and (L) events are NOT SIMULTANEOUS in the stationary observer's frame!

THE LORENTZ TRANSFORMS



S frame S' frame

$$\boxed{\begin{aligned} x &= \gamma(x' + Vt') \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t' + Vx'/c^2\right) \\ x' &= \gamma(x - Vt) \\ t' &= \gamma\left(t - Vx/c^2\right) \end{aligned}}$$

$$\underbrace{\gamma t'}_{\text{Time dilation}} + \underbrace{\gamma \frac{Vx'}{c^2}}_{\text{Loss of simultaneity}}$$

$$\boxed{\gamma = \left(1 - V^2/c^2\right)^{-\frac{1}{2}}}$$

Use differential forms to transform velocities

$$\boxed{\begin{aligned} dx &= \gamma(dx' + Vdt') \\ dt &= \gamma(dt' + Vdx'/c^2) \end{aligned}}$$

$$\therefore \boxed{v_x = \frac{dx}{dt}} = \frac{\gamma(dx' + Vdt')}{\gamma(dt' + Vdx'/c^2)} = \boxed{\frac{v_x' + V}{1 + Vv_x'/c^2}}$$

$$\boxed{v_x' = \frac{dx'}{dt'}}$$

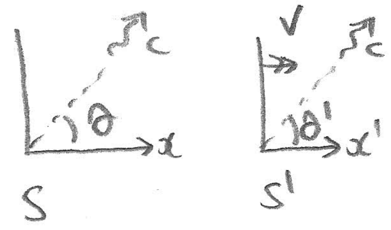
$$v_y = \frac{dy}{dt} = \frac{dy'}{\gamma(dt' + V dx'/c^2)} \Rightarrow \boxed{v_y = \frac{v_y'}{\gamma(1 + V v_x'/c^2)}}$$

Similarly $\boxed{v_z = \frac{dz}{dt} = \frac{v_z'}{\gamma(1 + V v_x'/c^2)}}$

For light

$v_x = c\cos\theta$

$v_x' = c\cos\theta'$



Relativistic aberration

$$\boxed{\cos\theta = \frac{\cos\theta' + V/c}{1 + \frac{V}{c}\cos\theta'} \qquad \cos\theta' = \frac{\cos\theta - V/c}{1 - \frac{V}{c}\cos\theta}}$$

Summary of other important relativistic results

$u^2 = |\underline{u}|^2$

Momentum $\boxed{\underline{p} = \gamma m \underline{u}}$ $\boxed{\gamma = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}}}$

Newton II $\underline{f} = \frac{d}{dt}(\gamma m \underline{u}) = m\gamma \underline{a} + m\underline{u}\frac{d\gamma}{dt}$

[Acceleration $\underline{a} = d\underline{u}/dt$] $\boxed{\frac{d\gamma}{dt} = \gamma^3 \frac{\underline{a}\cdot\underline{u}}{c^2}}$

$\therefore \boxed{\underline{f} = m\gamma\underline{a} + m\gamma^3\left(\frac{\underline{a}\cdot\underline{u}}{c^2}\right)\underline{u}}$ [$\gamma \to 1$, $\underline{f} \to m\underline{a}$]

1D Example: → "$E = mc^2$"

f → m → u

$\boxed{f = \frac{dp}{dt}}$ "Force = rate of change of momentum"

$\boxed{p = \gamma m u}$ $\therefore$ if m constant $f = m\frac{d}{dt}(\gamma u) = m\gamma\frac{du}{dt} + mu\frac{d\gamma}{dt}$

$\boxed{\gamma = \left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}}}$ $\therefore \frac{d\gamma}{du} = -\frac{1}{2}\left(1 - \frac{u^2}{c^2}\right)^{-3/2}\left(-\frac{2u}{c^2}\right) = \frac{u}{c^2}\gamma^3$

$\frac{d\gamma}{dt} = \frac{d\gamma}{du} \times \frac{du}{dt}$ $\therefore$ $\boxed{\frac{d\gamma}{dt} = \frac{u\gamma^3}{c^2}\frac{du}{dt}}$

$\therefore f = m\frac{du}{dt}\left(\gamma + \frac{u^2}{c^2}\gamma^3\right)$

$f = m\gamma^3\frac{du}{dt}\left(\gamma^{-2} + \frac{u^2}{c^2}\right)$

$\gamma^{-2} = 1 - u^2/c^2$ $\therefore$ $\boxed{f = m\gamma^3\frac{du}{dt}}$

Work done $W = \int f\,dx$

$W = \int f u\,dt$ since $u = \frac{dx}{dt}$

$\therefore W = \int m\gamma^3\frac{du}{dt}u\,dt$

$W = m\int\gamma^3 u\,du$

Now $\frac{d\gamma}{du} = u\gamma^3/c^2$

$\therefore c^2\,d\gamma = u\gamma^3\,du$

Hence Work done $W = mc^2 \int_{\gamma_1}^{\gamma_2} d\gamma$

$$\boxed{W = mc^2(\gamma_2 - \gamma_1)}$$

Hence total energy of mass m is $\boxed{E = \gamma mc^2}$

when $\gamma = 1, u = 0$ $\boxed{E = mc^2}$

So interpret

$$\boxed{E = \underbrace{mc^2}_{\text{Rest mass energy}} + \underbrace{(\gamma-1)mc^2}_{\text{kinetic energy}}}$$

Note if $\frac{u}{c} \ll 1$ $\left(1 - \frac{u^2}{c^2}\right)^{-\frac{1}{2}} \approx 1 + \frac{u^2}{2c^2}$

$\therefore \gamma - 1 \approx \frac{1}{2}\frac{u^2}{c^2}$ $\therefore (\gamma-1)mc^2 \approx \boxed{\frac{1}{2}mu^2}$ Classical KE.

ENERGY MOMENTUM INVARIANT

$$\boxed{\begin{aligned} E &= \gamma mc^2 \\ \underline{p} &= \gamma m\underline{u} \end{aligned}}$$

$$\begin{aligned} \therefore E^2 - |\underline{p}|^2c^2 &= \gamma^2 m^2 c^4 - \gamma^2 m^2 u^2 c^2 \\ &= m^2c^4\left(\gamma^2 - \gamma^2 u^2/c^2\right) \\ &= m^2c^4\gamma^2\left(1 - u^2/c^2\right) \\ &= m^2c^4\gamma^2\gamma^{-2} \\ &= \boxed{m^2c^4} \end{aligned}$$

Hence $\boxed{E^2 - |\underline{p}|^2c^2 = m^2c^4}$

ie independent of the coordinate frame.

Very useful result in particle physics! (eg Compton Scattering...)

Note for $\boxed{\text{photons}}$ $m = 0$ $\therefore E^2 - p^2c^2 = 0$

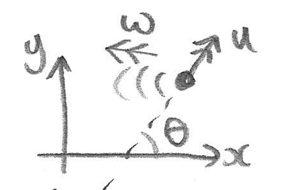
$E = hf$

planck law

$\therefore$ photons have $\boxed{\text{momentum}}$ $\boxed{p = E/c}$

ie 'radiation pressure' from photon collisions with a Solar Sail, using $\boxed{p = hf/c}$

RELATIVISTIC DOPPLER SHIFT



w is wave speed

$$\boxed{f = \frac{f'}{\gamma\left(1 + \frac{u\cos\theta}{w}\right)}}$$

f' source frequency

f observed frequency