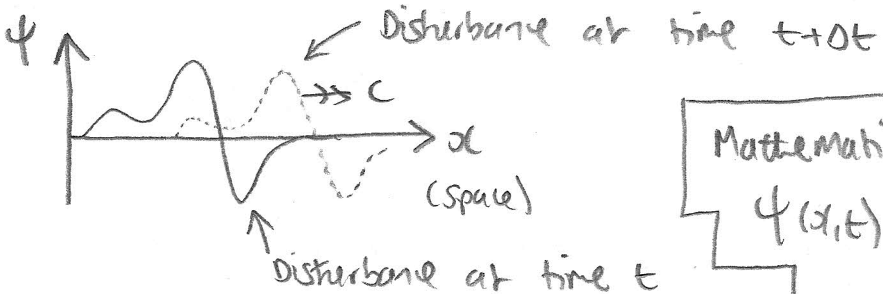
TYPES OF WAVES. MATHEMATICAL ANATOMY OF WAVES. STANDING WAVES
REFLECTION, REFRACTION, DIFFRACTION. INTERFERENCE. DOPPLER EFFECT.

A wave is essentially a <u>disturbance</u> that propagates at speed c through space. The disturbance, which has numeric value ψ could be: gas pressure above ambient, movement of a string from equilibrium under tension, fluctuations in electric and magnetic fields....



Mathematically could write:
$$\psi(x,t) = f(x - ct) \times \text{attenuation factor}$$
ie translation of f(x)

Types of waves

Longitudinal

Disturbance // propagation direction

* Sound waves in gases, liquids, solids
* Earthquake P-waves

Transverse

Disturbance ⊥ to propagation direction

* Electromagnetic waves (radio, microwave, IR, visible, UV, X-ray, gamma)
* Earthquake S-waves
* GRAVITY waves

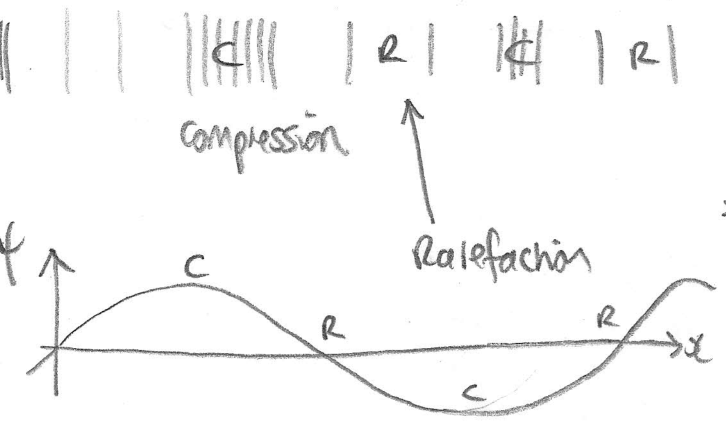
Surface (or interfaces)

* Rayleigh & love waves (Earthquakes)
* Water waves → Surf, Kelvin wedge, Ripples
* Meteorological effects
  ↳ lenticular clouds

↳ Kelvin-Helmholtz instability

Shock waves

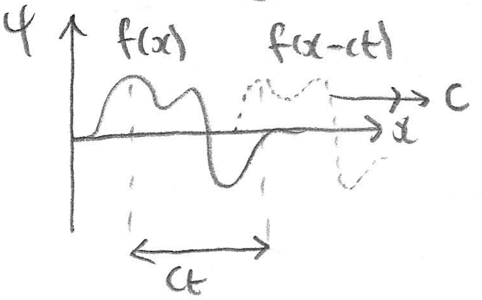
* E.g. from explosions ["Detonation"]
* Aircraft breaking the sound barrier



MOST INFORMATION ABOUT THE UNIVERSE IS TRANSMITTED IN WAVE FORM! BEYOND EARTH OUR ONLY SOURCE IS EM WAVES (+ GRAVITY WAVES)

Mathematical anatomy of waves

A key feature of a wave is that it is a spatial translation of a disturbance $f(x)$ as time progresses. There may also be a decay of amplitude with time ("attenuation"), but this process shall be modelled separately.



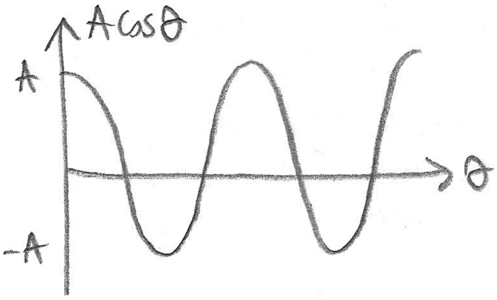
So $\boxed{\psi(x,t) = f(x-ct)}$

One can show that any $\psi(x,t)$ of this form will satisfy the $\boxed{\text{wave equation}}$

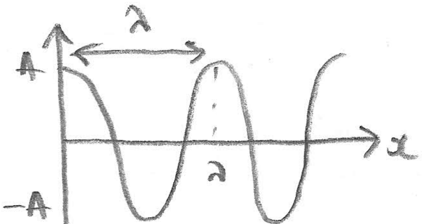
$$\boxed{\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \psi}{\partial t^2}}$$

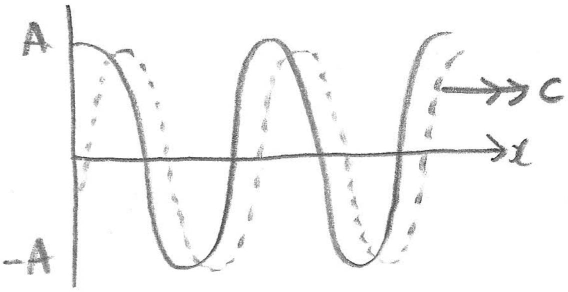
The ideas of $\boxed{\text{Fourier Series}}$ show that <u>any</u> disturbance $\psi(x,t)$ can be assembled from a sum of ("superposition") of $\boxed{\text{sinusoids}}$ of different amplitude, phase shift and frequency. So study

$\boxed{\psi(x,t) = A\cos(kx - \omega t - \phi_0)}$ and you have the basis for all waves.



let $\theta = 2\pi \times \dfrac{x}{\lambda}$ $\Rightarrow$



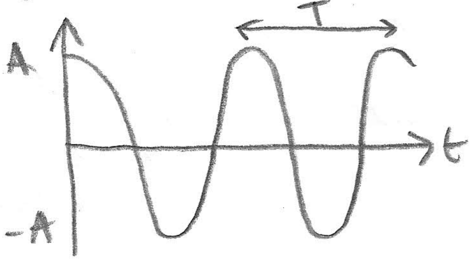


Now incorporate a translation by ct.

— $\psi(x,0) = A\cos\left(\dfrac{2\pi x}{\lambda}\right)$

--- $\psi(x,t) = A\cos\left(\dfrac{2\pi}{\lambda}(x-ct)\right)$

Trace above is a 'snapshot' of a wave. If at position x you could draw the wave trace as it passes.



$\boxed{\begin{array}{l} T = \text{period} \\ f = 1/T \end{array}}$ (s) (Hz)

↑ frequency

Clearly $c = \lambda/T$

$\therefore \boxed{c = f\lambda}$

let us define a few other quantities: PHASE $\boxed{\phi = kx - \omega t - \phi_0}$

WAVENUMBER $\boxed{k = \frac{2\pi}{\lambda}}$ ANGULAR FREQUENCY $\boxed{\omega = 2\pi f}$

INITIAL PHASE $\boxed{\phi_0}$ (IN RADIANS 2π radians $= 360°$)

So:

$$c = f\lambda \qquad \frac{c}{\lambda} = f$$

$$\frac{2\pi c}{\lambda} = 2\pi f \qquad kc = \omega$$

$$\boxed{\omega = ck}$$

$c = \omega/k$ is the 'phase velocity of waves'. Many waves have a dispersion relation (eg water waves)

ie $\omega^2 = gk \quad \therefore \omega = \sqrt{g}\, k^{\frac{1}{2}}$

$$\frac{\omega}{k} = \sqrt{g/k}$$

So $\boxed{c = \sqrt{\frac{g\lambda}{2\pi}}}$ Deep water waves

ie $\lambda \ll$ depth.

Hence: $\boxed{\psi(x,t) = A\cos\left(\frac{2\pi}{\lambda}(x-ct) - \phi_0\right)}$

Add in initial phase

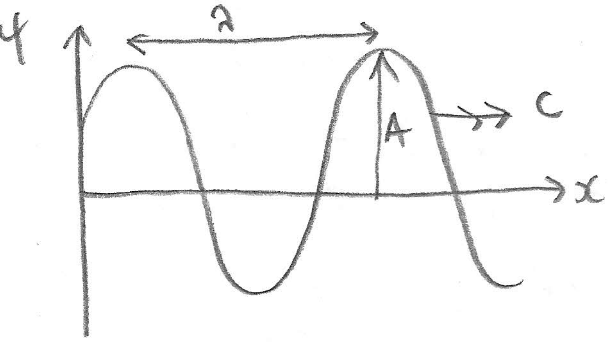
Phase is 'where we are in the cos curve

ie $\boxed{\phi = \frac{2\pi}{\lambda}(x-ct) - \phi_0}$ PHASE

$$\phi = kx - \frac{2\pi c}{\lambda}t - \phi_0$$

$$\frac{c}{\lambda} = f \quad \text{so} \quad \frac{2\pi c}{\lambda}t = \omega t$$

$$\therefore \boxed{\psi(x,t) = A\cos(kx - \omega t - \phi_0)}$$



[Note we often find it convenient to use complex numbers!

$e^{i\theta} = \cos\theta + i\sin\theta$ de Moivre theorem

So let $\psi(x,t) = Ae^{i(kx-\omega t-\phi_0)}$ and take the real part]

$\boxed{\text{Summary of wave parameters}}$

$$\psi(x,t) = A\cos\left(\frac{2\pi}{\lambda}(x-ct) - \phi_0\right)$$

$$\psi(x,t) = A\cos(kx - \omega t - \phi_0)$$

$$\psi(x,t) = A\cos\phi$$

$$c = f\lambda \qquad \omega = ck$$

ϕ Phase / radians

λ wavelength

A Amplitude

ψ Disturbance

$k = 2\pi/\lambda$ wavenumber

$\omega = 2\pi f$ Angular frequency

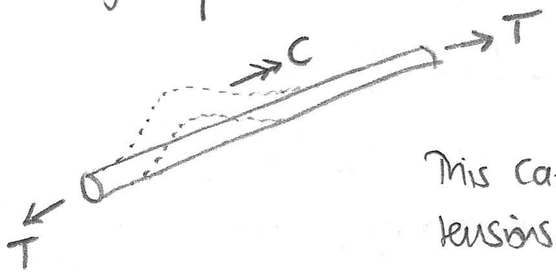
f Frequency

T Period

WAVES ON STRINGS AND WAVE ENERGY

[See Eclecticon note "Mathematical anatomy of waves" for derivations]

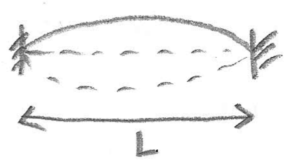
For a string under tension T newtons, and mass per unit length μ

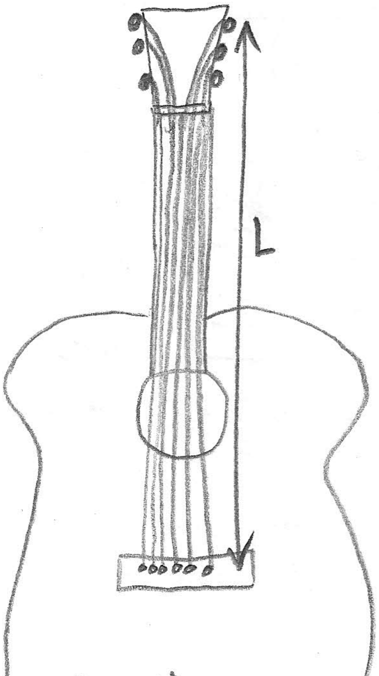


$$\boxed{c = \sqrt{\frac{T}{\mu}}}$$

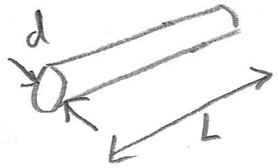
Speed of waves on string

This can be used to work out the required tensions of guitar strings. These are clamped at both ends, so the lowest "fundamental" frequency is such that $\boxed{L = \lambda/2}$





Acoustic guitar



For a string of diameter d and length L of density ρ, mass is:

$$M = \pi\left(\frac{d}{2}\right)^2 L\rho \quad \text{so} \quad \mu = \frac{M}{L}$$

$$\mu = \frac{\pi d^2 \rho}{4}$$

Now $c = f\lambda$ and $\lambda = 2L$

$$\boxed{f = 110 \times 2^{n/12}} \text{ (Hz)}$$

for 'equal tempered scale'

Note	A	Bb	B	C	C#	D	Eb	E	F	F#	G	Ab	A+
n	0	1	2	3	4	5	6	7	8	9	10	11	12
f /Hz	110		123.5			146.8			174.6				220

Next octave

So $c^2\mu = T$ $\therefore T = f^2(2L)^2\mu$

$$T = 4f^2L^2\pi d^2\rho/4$$

Idea is want T to be similar, so not to stress the guitar too much

$$\therefore \boxed{T = \pi\rho(fLd)^2}$$

For my electric guitar:

Note	n	f/Hz	d (inches)	L/m	ρ (kgm^{-3})	T/N
E	19	329.63	0.010	0.75	7690	95.3
B	14	246.94	0.013	0.73	7950	88.5
G	10	196	0.017	0.71	8220	93.5
D	5	146.83	0.026	0.69	6930	97.5
A	0	110	0.036	0.67	6610	94.3
E	-5	[illegible]	0.046	[illegible]	[illegible]	[illegible]

SPEED OF SOUND IN IDEAL GASES

$$\boxed{c = \sqrt{\frac{\gamma p}{\rho}}}$$

$\gamma = \frac{C_p}{C_v}$ ratio of specific heat capacities [see Thermodynamics course notes!]

p = pressure

ρ = density

Using $\boxed{pV = nRT}$

n = # moles of gas

R = molar gas constant $R = 8.314\ \mathrm{J\,mol^{-1}K^{-1}}$

T = temperature /K

$\rho = \frac{nM}{V}$ M = molar mass of the gas

So $\rho = \frac{nM}{nRT} p$ $\therefore$ $\boxed{c = \sqrt{\frac{\gamma RT}{M}}}$

For air: $\gamma \simeq 1.4$

$M \simeq 0.02896$ kg/mol (dry air), so at 298K

$c = \sqrt{\frac{1.4 \times 8.314 \times 298}{0.02896}}$

$= \boxed{346\ \mathrm{m/s}}$

SPEED OF SOUND IN ELASTIC SOLIDS

P waves $\boxed{c = \sqrt{\frac{Y(1-v)}{\rho(1+v)(1-2v)}}}$

S waves $\boxed{c = \sqrt{G/\rho}}$

Y young's modulus

v poisson ratio

G shear modulus

ρ density

Example: Steel

Y 200 GPa

G 80 GPa

v 0.3

ρ 7800 kg/m3

$\Rightarrow$

P velocity	$\approx$	5880 m/s
S "	$\approx$	3202 m/s

$\leftarrow \sqrt{\frac{200 \times 10^9 \times 0.7}{7800 \times 1.3 \times 0.4}}$

POWER IN A WAVE

Imagine length dl of a string oscillating. $KE = \frac{1}{2} m v^2$

$v = \frac{\partial \psi}{\partial t}$ $m = \mu\, dl$ $\psi = A\cos(kx - \omega t - \phi_0)$ So if dl is set oscillating in dt and wave speed is c $dl = c\,dt$

So $\underbrace{dE}_{\text{energy input}} = \frac{1}{2}\mu c\,dt\, \overline{\left(\frac{\partial\psi}{\partial t}\right)^2}$ Average of $\left(\frac{\partial\psi}{\partial t}\right)^2$ is $\frac{\omega^2 A^2}{2}$

So $\frac{dE}{dt} = \frac{1}{2}\mu c \omega^2 A^2 \times \frac{1}{2}$ $\leftarrow$ actually we need to consider EPE of string

So this

Hence input power to wave on a string is

$$\boxed{P = \frac{1}{2}\mu c A^2 \omega^2}$$

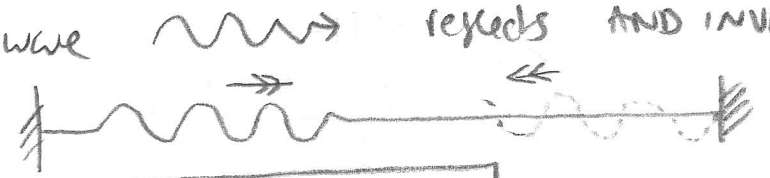
ie $\boxed{\text{power} \propto \text{amplitude}^2 \text{ and frequency}^2}$

This is a general result for waves
[$P = \frac{1}{2} Z A^2 \omega^2$ where Z is the IMPEDANCE
for a string $Z = \mu c$]

STANDING WAVES ON STRINGS AND IN PIPES

Standing waves – a superposition of 'rightwards' → and 'leftwards' ← waves such that the nodes and antinodes of the resulting interference pattern don't move.

For a guitar string a wave reflects AND INVERTS off the clamped end



so
$$\boxed{\underset{\text{'rightwards →'}}{\psi(x,t) = A\sin(kx-\omega t)} \; - \underset{\text{'leftwards ←'}}{A\sin(-kx-\omega t)}}$$

using trig identities $\sin(A+B) = \sin A \cos B + \cos A \sin B$
$\sin(-x) = -\sin x$
$\cos(-x) = \cos x$

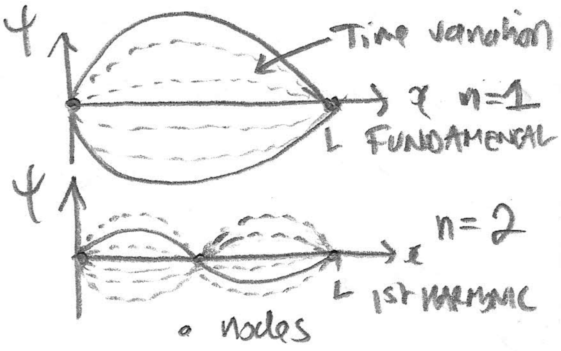
See Eclection notes (Standing waves).

Can show : $\boxed{\psi(x,t) = 2A \sin kx \cos \omega t}$ ie separation of spatial and temporal parts

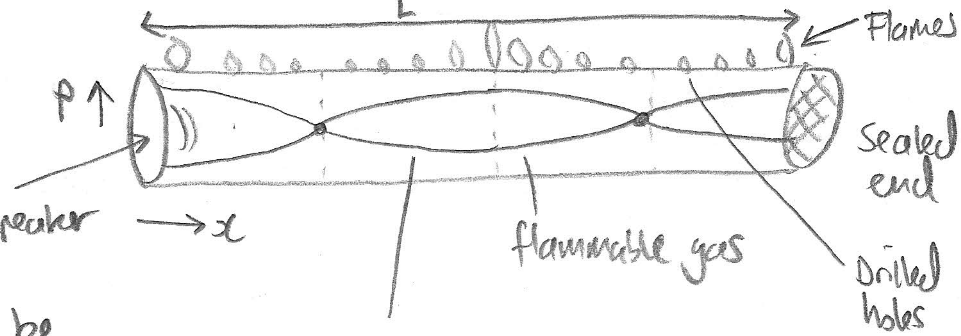
A guitar string must have nodes at both ends so : $\boxed{n \times \frac{1}{2}\lambda = L}$ n = integer so $\boxed{\lambda = \frac{2L}{n}}$

$\therefore \; k = \frac{2\pi}{\lambda} = 2\pi \times \frac{n}{2L} = \frac{n\pi}{L}$

$\therefore$ $\boxed{\psi(x,t) = 2A \sin\left(\frac{n\pi}{L}\right) \cos\left(\frac{2\pi}{T} t\right)}$



RUBEN'S TUBE



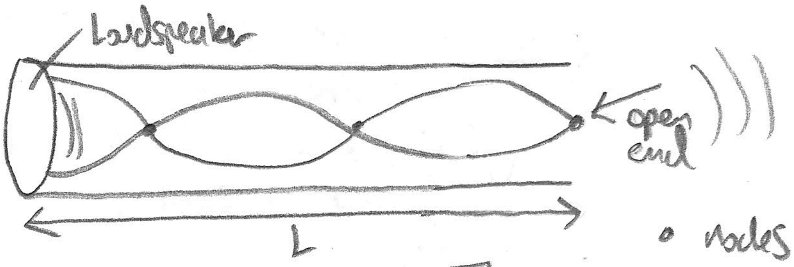
Pressure waves must be $\boxed{\text{antinodes}}$ at loudspeaker and sealed end.

So $\boxed{\frac{\lambda}{2} \times n = L}$ $\therefore$ pressure $\boxed{p(x,t) = P_{max} \cos\left(\frac{2\pi x}{\lambda_n}\right) \cos(2\pi f_n t)}$

$\boxed{\lambda_n = \frac{2L}{n}}$ $f_n = \frac{c}{\lambda_n} = \boxed{\frac{nc}{2L}}$ c is wave speed in gas [use $c = \sqrt{\gamma RT/M}$]

DRIVEN PIPE WITH OPEN END eg KUNDT'S tube

* Pressure antinode at loudspeaker end
* pressure node at open end



END CORRECTION
TUBE VIBRATING CAUSES $L \to L + 0.6r$ where r is tube radius

So $\boxed{L = \text{odd} \# \text{ of } \lambda/4}$

ie $L = (2n-1)\lambda/4$ $\therefore$ $\boxed{\lambda_n = \frac{4L}{2n-1}}$ $\boxed{f_n = \frac{c}{\lambda_n} = \frac{c(2n-1)}{4L}}$

So for an organ pipe $f = 440\text{Hz}$, $n = 1$, $c = 345$ m/s

$f_1 = \frac{c(2\times1-1)}{4L}$ $\therefore$ $L = \frac{345 \text{ m/s}}{4 \times 440 \text{ Hz}} = \boxed{0.196\text{ m}}$

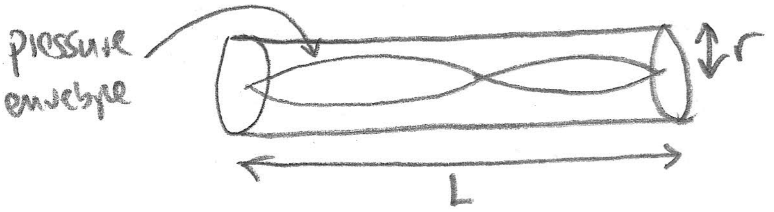
For a very low note (8Hz, below human hearing range)

$L = \frac{345 \text{ m/s}}{4 \times 8} = \boxed{10.78\text{ m}}$

[NOTE: pressure nodes correspond to DISPLACEMENT ANTINODES and vice versa]

PIPE WITH BOTH ENDS OPEN TO THE ATMOSPHERE

e.g pan pipes. In this case pressure nodes at either end



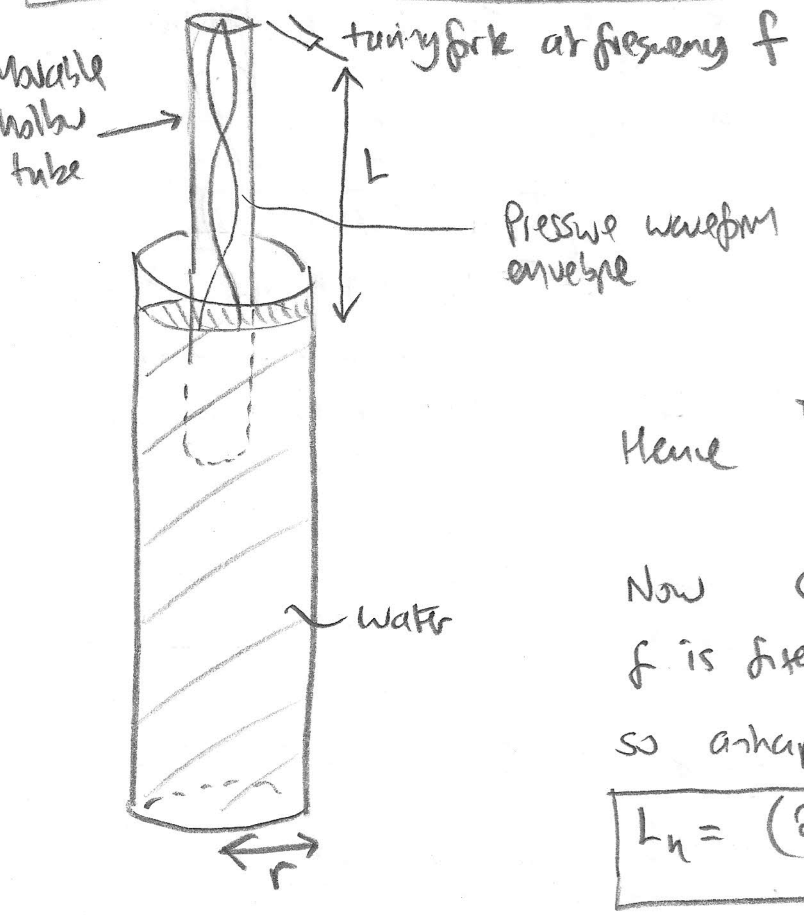
so $n\frac{\lambda}{2} = L$

or $\lambda_n = \frac{2L}{n}$

$f_n = \frac{c}{\lambda_n} = \boxed{\frac{nc}{2L}}$

[For $\boxed{\text{end correction}}$ L = Actual length + 2 x 0.66r
since both ends vibrate the air]

WATER TUBE RESONANCE EXPERIMENT



For air column in movable tube:

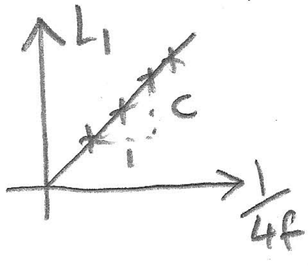
* Must be pressure node at open end
* Must be pressure antinode at water surface (HUGE change in density ie ∞ impedance)

Hence $\boxed{L = (2n-1) \times \lambda/4}$

Now $c = f\lambda$ so $\lambda = c/f$
f is fixed here (driven by tuning fork)
so anticipate maximum loudness when

$$\boxed{L_n = (2n-1) \times \frac{c}{4f}}$$

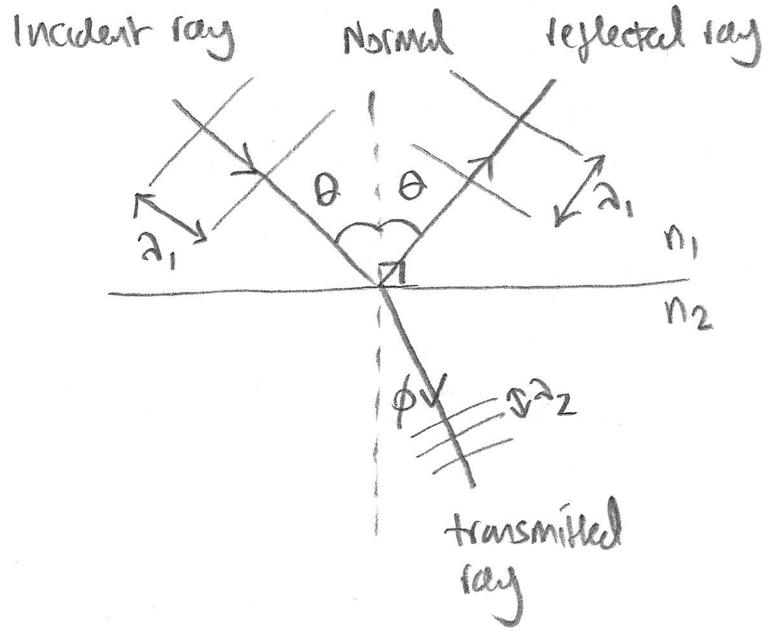
We can use this to find the speed of sound in air by using different tuning forks, and finding L_1
plot L_1 vs $\frac{1}{4f}$ and gradient should be c.



[Note L_1 will be actual L + 0.66r]

REFLECTION & REFRACTION

n is the refractive index $\begin{cases} \approx 1 \text{ for air} \\ \approx 1.5 \text{ for glass} \end{cases}$



Consider plane waves incident on a boundary.

* Region 1: Speed of light is $c/n_1 = v_1$
* Region 2: Speed of light is $c/n_2 = v_2$

c is speed of light in a vacuum.

[If working with other waves simply use wave speeds v_1 and v_2]

Law of reflection: Angle of incidence = angle of reflection

Snell's law of refraction: $\boxed{\dfrac{\sin\theta}{v_1} = \dfrac{\sin\phi}{v_2}}$ or if using light $\boxed{n_1 \sin\theta = n_2 \sin\phi}$

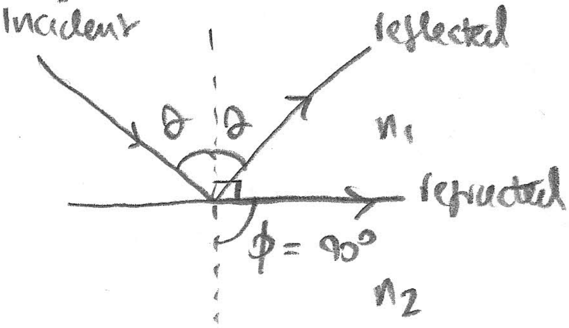
Now frequency of waves is the <u>same</u> either side of the boundary. So since $v_1 = f\lambda_1$, $v_2 = f\lambda_2$ $\therefore$ $\boxed{\lambda_2 = \lambda_1 \dfrac{v_2}{v_1}}$

For light: $\boxed{\lambda_2 = \lambda_1 \times \dfrac{n_1}{n_2}}$ So if from air $(n_1 \approx 1)$ to glass $(n_2 \approx 1.5) \Rightarrow \lambda_2 \approx \lambda_1 \times \frac{2}{3}$ <u>ie</u> wavelength reduces.

Note both the law of reflection and refraction satisfy Fermat's principle <u>ie</u> the light path is the one which $\boxed{\text{minimizes the time taken}}$.

Total internal reflection if $n_2 > n_1 \Rightarrow \phi > \theta$. when $\phi = 90°$

This means no further refraction can occur.



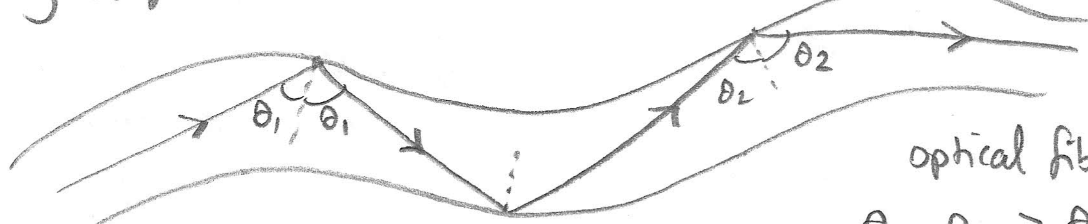
So if $n_2 > n_1$

CRITICAL angle is θ_c

$$n_1 \sin\theta_c = n_2 \sin 90^\circ$$

$$\therefore \boxed{\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)}$$

for $\theta > \theta_c$, no refraction, only reflection.
Very useful to minimize transmission losses!

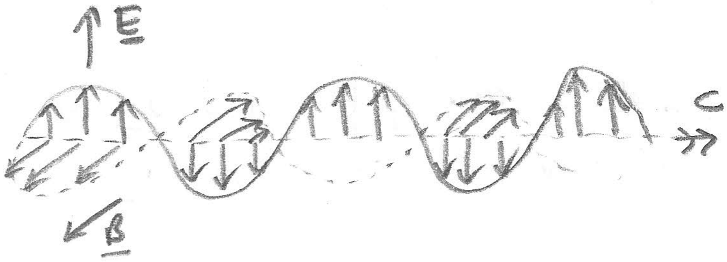


optical fibre

$\theta_1, \theta_2 > \theta_c$

For glass $\rightarrow$ air interface
$n_1 \approx 1.5$ $\quad n_2 \simeq 1$

$$\theta_c \simeq \sin^{-1}\left(\frac{1}{1.5}\right) \approx \boxed{41.8^\circ}$$



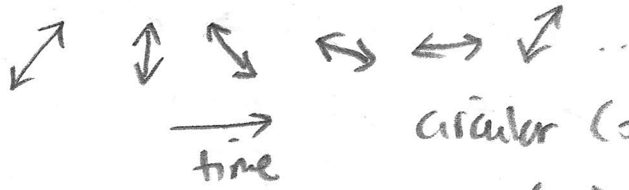
POLARIZATION OF EM WAVES

Electromagnetic waves consist of electric ($\underline{E}$) and Magnetic ($\underline{B}$) vector fields oscillating at 90° to each other, and both 90° to the propagation direction

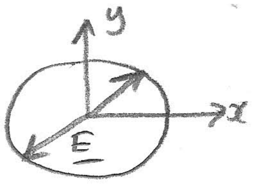
Polarization is the direction $\underline{E}$ oscillates

unpolarized

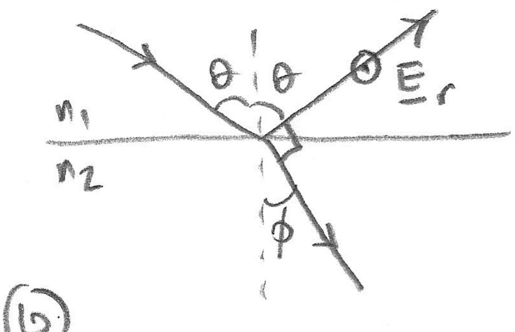
linear polarization



circular (or elliptical) polarization



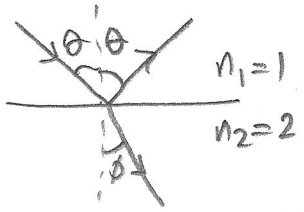
unpolarized

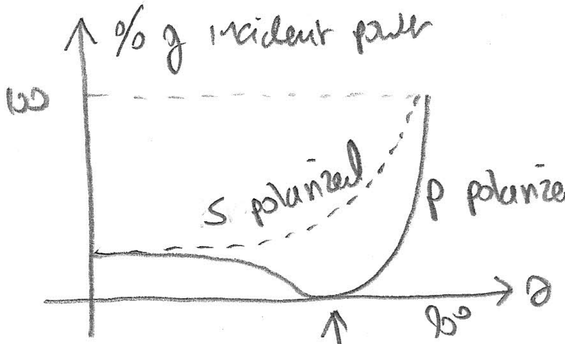


Now polarization direction must be $\perp$ to the propagation direction. If reflected ray is 90° to refracted ray $\underline{E}_r$ must be $\perp$ to the plane $\odot$ or $\otimes$ ie only "S" polarized light is reflected, no "P" polarization $\parallel$ $\underline{E}$ // plane is allowed!

(b)

If we plot the reflected power vs θ





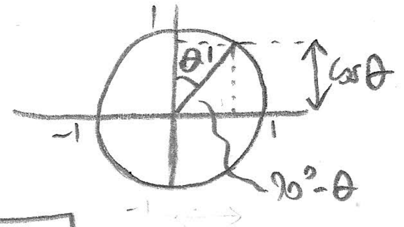
$\theta_B \leftarrow$ $\boxed{\text{Brewster angle}}$

S $\perp$ plane $\odot$ $\otimes$ $\underline{E}$

P $//$ plane $\underline{E}$

Now if $180^\circ = \theta_B + 90^\circ + \phi \Rightarrow 90^\circ = \theta_B + \phi \quad \therefore \phi = 90^\circ - \theta_B$

Snell's law: $n_1 \sin\theta = n_2 \sin(90^\circ - \theta)$



Now $\sin(90^\circ - \theta) = \cos\theta$

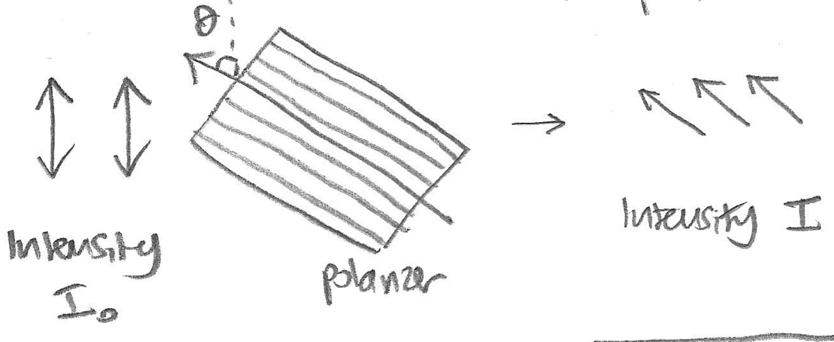
so $n_1 \sin\theta_B = n_2 \cos\theta_B$

$\therefore \quad \tan\theta_B = \dfrac{n_2}{n_1} \quad \therefore \boxed{\theta_B = \tan^{-1}\left(\dfrac{n_2}{n_1}\right)}$

So if $n_2 = 2$, $n_1 = 1 \Rightarrow \theta_B = \tan^{-1}(2) \cong \boxed{63.4^\circ}$

MALUS' LAW

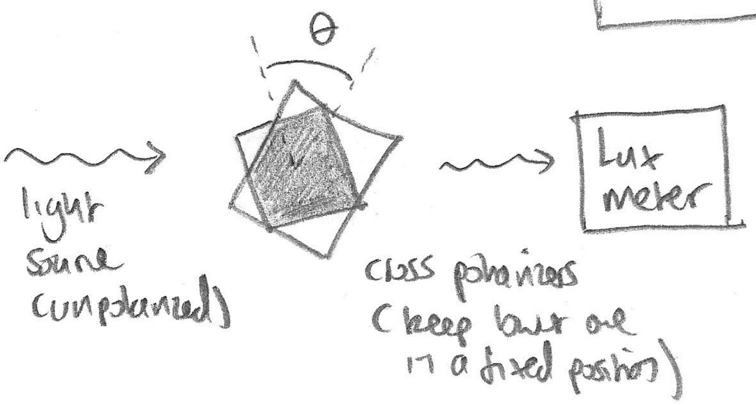
If linear polarized light is incident upon a linear polarizer tilted at angle θ



E projection is $E_0 \cos\theta$

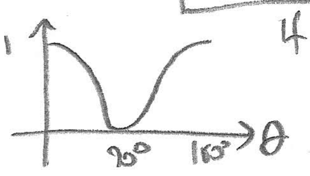
... and since power $\propto$ amplitude^2

$\therefore \boxed{I = I_0 \cos^2\theta}$



Expect $\boxed{\dfrac{I - I_{min}}{I_{max} - I_{min}} = \cos^2\theta}$

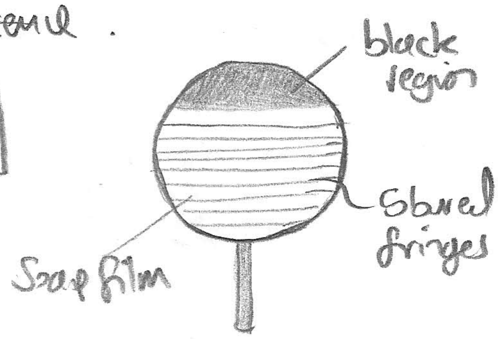
if $\theta = 0$ when $I = I_{max}$

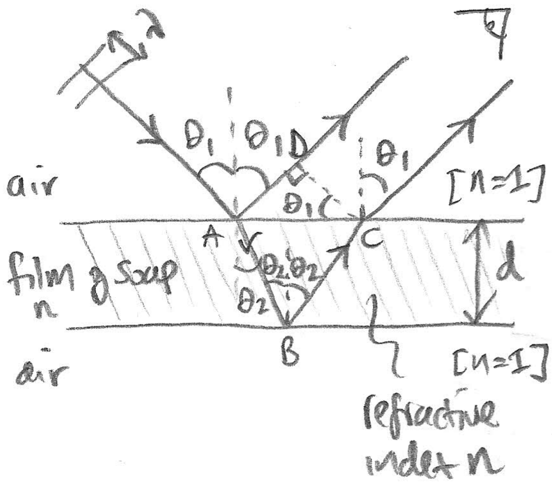


Use of phase #1: Thin film interference

For two wave propagation paths, if the difference in spatial parts of phase $\boxed{kx}$ is $2\pi \times$ integer this means constructive interference. ie a 'maxima'. If the phase difference is $\pi \times$ odd integer then destructive interference.

We can see this effect in a $\boxed{\text{Soap film}}$ held within a hoop, and illuminated with white light. As the film drops down under gravity, the upper part thins. Initially we will see vivid rainbow coloured fringes, but as time passes these → black.





Get constructive interference when $\boxed{\Delta\phi_{ABC} - \Delta\phi_{AD} = 2\pi m}$ ← integer

$$\Delta\phi_{ABC} = \frac{2\pi}{\lambda_{film}} AB + \frac{2\pi}{\lambda_{film}} BC$$

$\lambda_{film} = \frac{\lambda}{n}$ $\quad AB = BC$

$AB \cos\theta_2 = d$ $\quad \therefore AB = d/\cos\theta_2$

$$\therefore \quad \boxed{\Delta\phi_{ABC} = \frac{4\pi n}{\lambda} \frac{d}{\cos\theta_2}}$$

$$\Delta\phi_{AD} = \frac{2\pi}{\lambda} AD - \pi$$

{ π phase change (π) on reflection from air, soap boundary ie inversion }

$AD = AC \sin\theta_1$

$AC = 2 AB \sin\theta_2$

$AC = \frac{2d}{\cos\theta_2} \sin\theta_2$ $\quad \rightarrow$ so $\boxed{\Delta\phi_{AD} = \frac{2\pi}{\lambda} \times \frac{2d \sin\theta_1 \sin\theta_2}{\cos\theta_2} - \pi}$

$\therefore$ constructive interference when:

$$\frac{4\pi n}{\lambda}\frac{d}{\cos\theta_2} + \pi - \frac{4\pi d}{\lambda}\frac{\sin\theta_1 \sin\theta_2}{\cos\theta_2} = 2\pi m$$

Snell's law $\boxed{1 \times \sin\theta_1 = n\sin\theta_2}$

So $$\frac{4\pi d n}{\lambda\cos\theta_2}\left(1 - \sin^2\theta_2\right) = 2\pi m - \pi$$

$$\frac{2dn}{\lambda\cos\theta_2}\cos^2\theta_2 = m - \frac{1}{2}$$

$$\cos\theta_2 = \frac{\lambda}{2nd}\left(m - \frac{1}{2}\right)$$

Now $$\sin\theta_2 = \sqrt{1 - \cos^2\theta_2}$$

So $$\sin\theta = n\sqrt{1 - \frac{\lambda^2\left(m-\frac{1}{2}\right)^2}{4n^2d^2}}$$

$$\therefore \boxed{\theta = \sin^{-1}\left\{n\sqrt{1 - \frac{\lambda^2\left(m-\frac{1}{2}\right)^2}{4n^2d^2}}\right\}}$$

Now for real θ, $$1 - \frac{\lambda^2\left(m-\frac{1}{2}\right)^2}{4n^2d^2} > 0$$

$$\therefore 4n^2d^2 > \lambda^2\left(m-\frac{1}{2}\right)^2$$

$$\therefore \boxed{d > \frac{\lambda\left(m-\frac{1}{2}\right)}{2n}}$$

$\therefore$ Thinnest soap film for interference fringes is when $m = 1$

ie $d_{min} = \lambda/4n$ .... But not quite all, since $\{\ldots\}$ must be ≤ 1 for real θ.

So $\quad n\sqrt{1-\frac{\lambda^2(m-\frac{1}{2})^2}{4n^2d^2}} \leq 1 \quad$ Since $|\sin\theta| \leq 1$

$$\therefore \quad 1-\frac{\lambda^2(m-\frac{1}{2})^2}{4n^2d^2} \leq \frac{1}{n^2}$$

$$1-\frac{1}{n^2} \leq \frac{\lambda^2(m-\frac{1}{2})^2}{4n^2d^2} \qquad\qquad 1-\frac{1}{n^2} = \frac{n^2-1}{n^2}$$

$$d^2 \leq \frac{\lambda^2(m-\frac{1}{2})^2}{4n^2}\,\frac{n^2}{n^2-1}$$

$$\boxed{d \leq \frac{\lambda(m-\frac{1}{2})}{4(n^2-1)^{\frac{1}{2}}}}$$

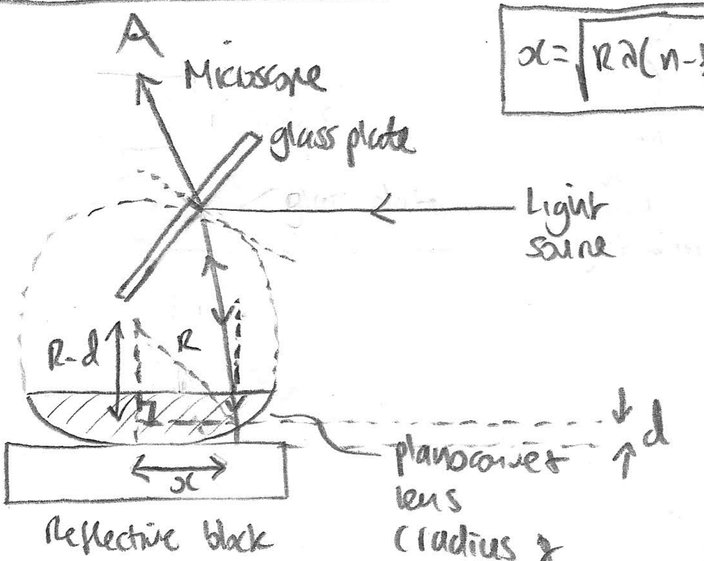
So when $\quad m=1 \quad d \leq \frac{\lambda}{4(n^2-1)^{\frac{1}{2}}}$

let $n \simeq 1.3$ (water) : so $\boxed{\frac{\lambda}{4n} < d \leq \frac{\lambda}{4(n^2-1)^{\frac{1}{2}}}}$

$$\therefore \quad \boxed{0.769\,\frac{\lambda}{4} < d < 1.204\,\frac{\lambda}{4}}$$

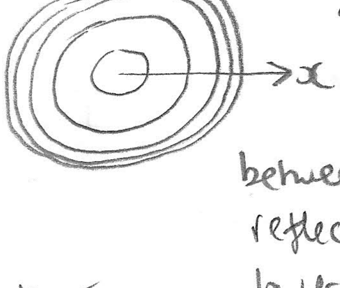
for constructive interference.

USE OF PHASE #2: NEWTON'S RINGS



$$\boxed{x=\sqrt{R\lambda(n-\tfrac{1}{2})}}$$

observe coloured fringes

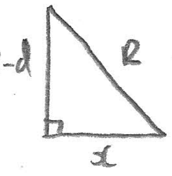


due to constructive interference between light reflecting off the lower surface of the lens and the block

integer

$$\boxed{2\pi n = \Delta\phi = \frac{2\pi}{\lambda}\times 2d + \pi}$$

Block reflection imparts this shift



$R^2=(R-d)^2+x^2$

$R^2=R^2-2Rd+d^2+x^2$

$2Rd = d^2+x^2$

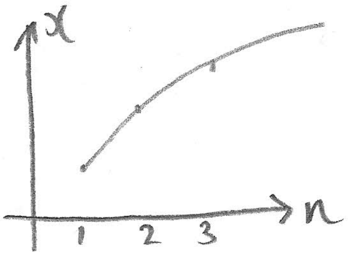
$d = \frac{d^2}{2R}+\frac{x^2}{2R} \simeq \frac{x^2}{2R}$ if $d \ll R$

14 $\left[d\left(1-\frac{d}{2R}\right)=\frac{x^2}{2R} \;\therefore\; d \simeq \frac{x^2}{2R}\right]$

So. $2\pi n = \frac{2\pi}{\lambda} \times \frac{2x^2}{2R} + \pi$

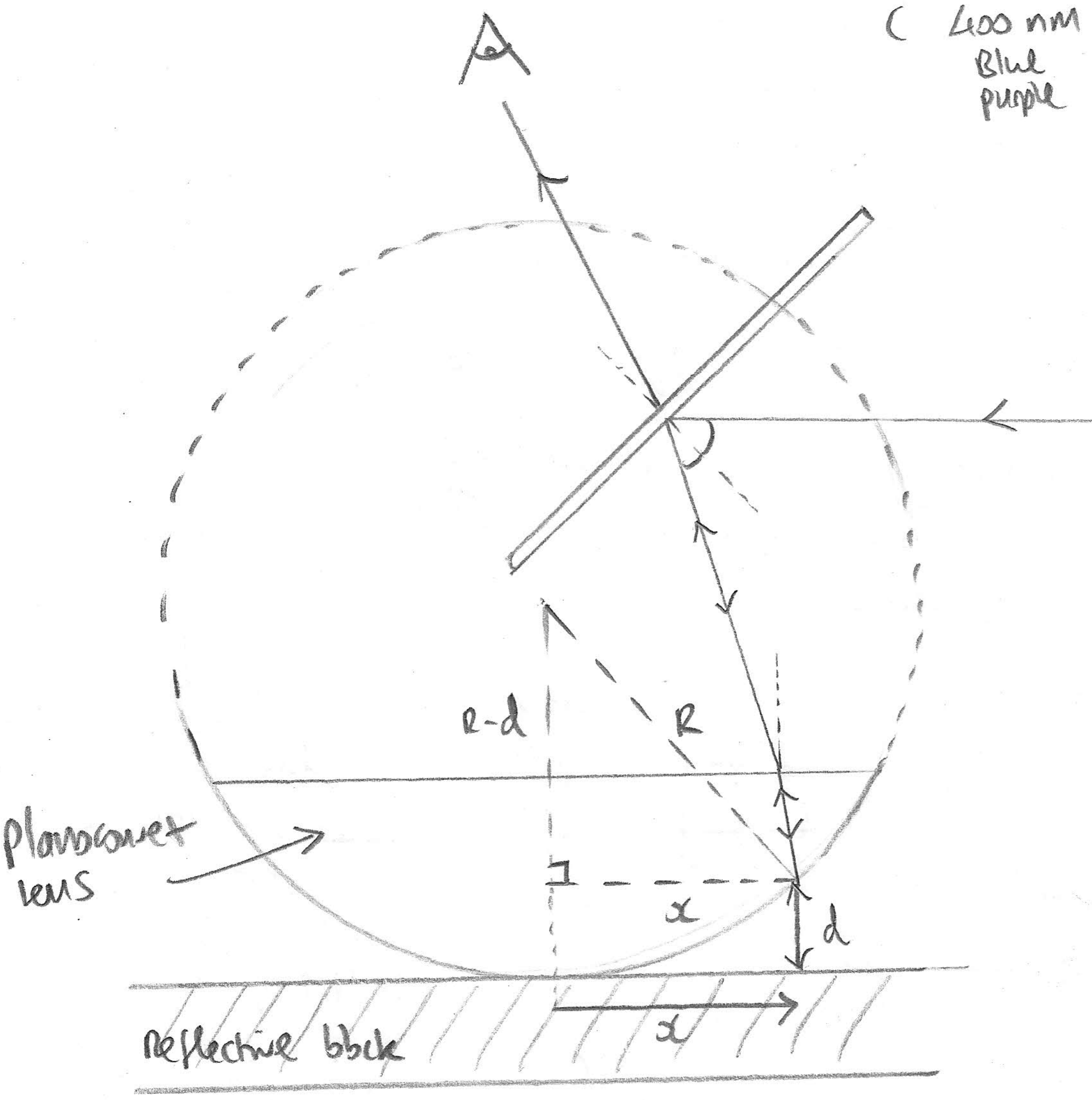
$n = \frac{x^2}{\lambda R} + \frac{1}{2}$

$\therefore \boxed{\sqrt{(n-\tfrac{1}{2})\lambda R} = x}$

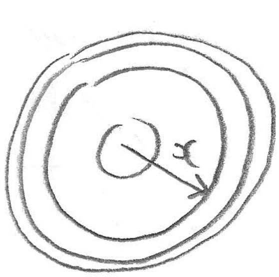


This explains why fringe separation reduces and if white light is used, each fringe has a rainbow colour

(400 nm < λ < 800nm)
Blue purple — Red



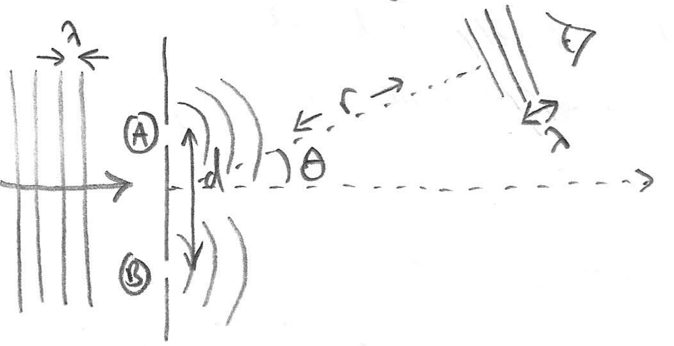
Geometry of Newton's rings



$x_n = \sqrt{(n-\tfrac{1}{2})\lambda R}$

DIFFRACTION FROM DOUBLE SLITS, SINGLE SLIT AND GRATINGS

Consider plane waves striking an obstacle with two thin slits separated by distance d. Slit width $\ll \lambda, d$

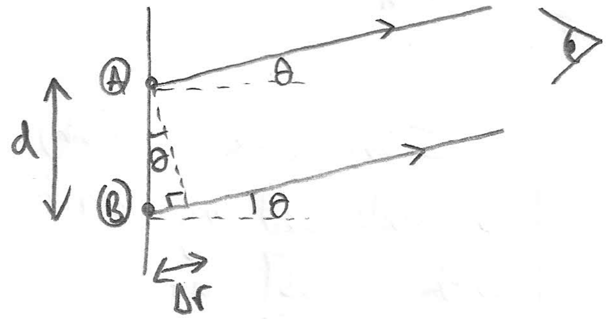


Thin slits A, B separated by distance d

observe superposition of diffracted waves at distance $\boxed{r \gg d^2/\lambda}$

This is the 'Far field' and at this point can regard waves at angle θ to be planar

We observe an interference pattern vs θ ie a series of bright and dark fringes.



Using the approximation of plane waves (ie // rays) at θ

constructive interference when path difference $\boxed{\Delta r = n\lambda}$

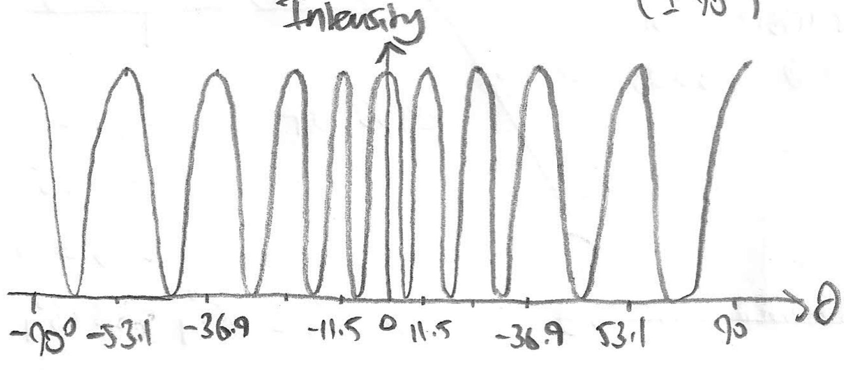
integer

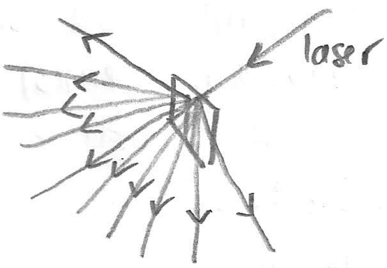
From diagram above: $d\sin\theta = \Delta r$

so expect maxima when $\boxed{\theta = \sin^{-1}\left(\frac{n\lambda}{d}\right)}$

Example: if a red laser illuminates a pair of slits with $d = 5\lambda$ ($\lambda = 650\text{nm}$)

Maxima at: $\boxed{\theta = \sin^{-1}\left(\frac{n}{5}\right)} = 0°, \pm 11.5°, \pm 23.6°, \pm 36.9°, \pm 53.1°$ $(\pm 90°)$



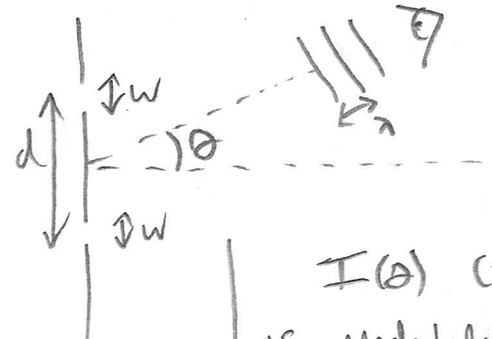


RESOLVING POWER

If $\theta \ll 1$ radian $\sin\theta = \frac{n\lambda}{d}$ becomes $\theta \simeq \frac{\lambda}{d}$ if $n = 1$ (ie 'main grating lobe'). For any optical instrument, the 'resolving power' is likely to be diffraction limited. Hence $\boxed{\Delta\theta \simeq \frac{\lambda}{d}}$ is the minimum angular deviation that two objects could be resolved via an optical system with characteristic dimension d (eg aperture width).

eg for the human eye, $d \simeq 2mm$, so for red light $\lambda = 600nm$ $\therefore$ $\Delta\theta \simeq \frac{600 \times 10^{-9} m}{2 \times 10^{-3} m} \simeq \boxed{3 \times 10^{-4} \text{ radians}}$ $(\simeq 0.02°)$

FINITE SLIT WIDTHS

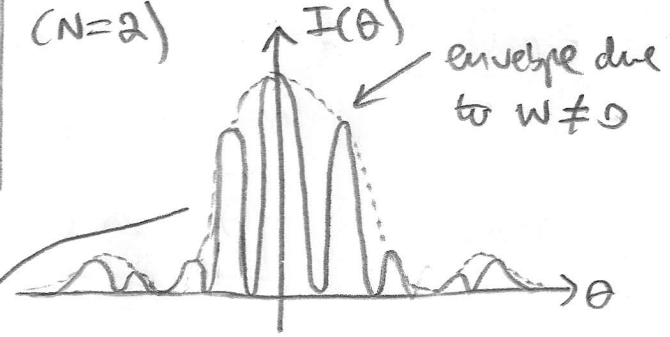


The smallest geometric feature of a diffraction grating (N slits of separation d and width w) gives rise to the LARGEST feature of the diffraction pattern ie $I(\theta)$. This inverse relation is characteristic of FOURIER TRANSFORMS, which is a more general way of describing the diffraction pattern of an aperture.

$I(\theta)$ (the intensity pattern) is modulated by an envelope with zeros at

$$\boxed{\sin\theta = \frac{m\lambda}{w}} \quad (m \neq 0)$$

So if $w = 2\lambda$, $d = 5\lambda$ $(N=2)$



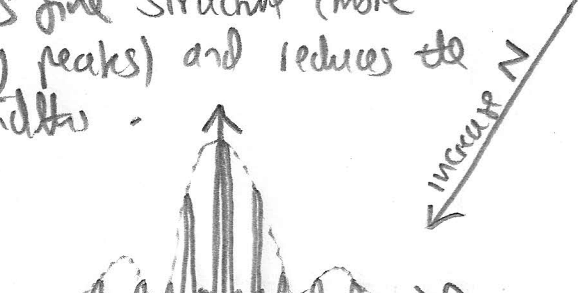
Zeros at $\theta = \pm\sin^{-1}(\frac{1}{2}), \pm\sin^{-1}(1)$
$= \pm 30°, \pm 90°$

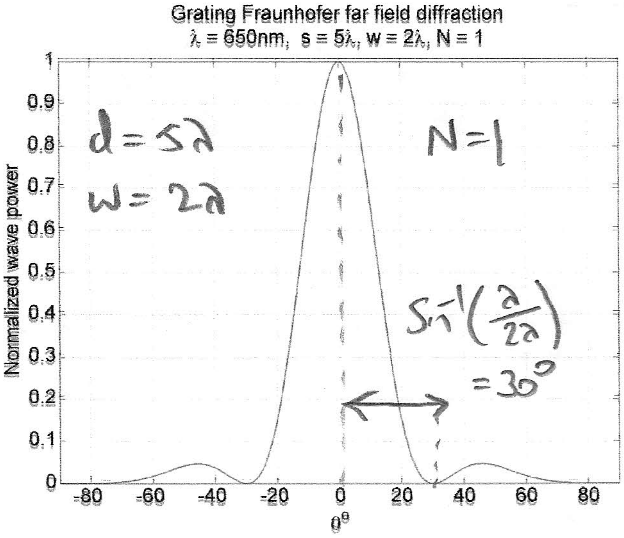
Maxima at $\sin\theta = n\lambda/d = n/5$
ie $0°, \pm 11.5°, \pm 23.6°, \pm 36.9°, \pm 53.1°$

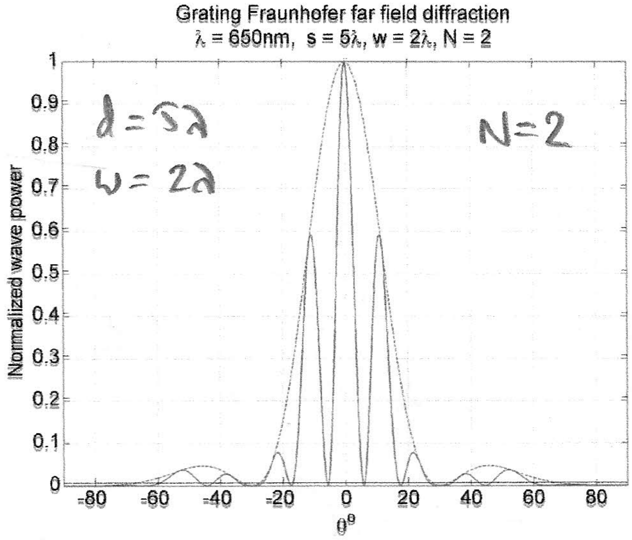
EFFECT OF MORE SLITS

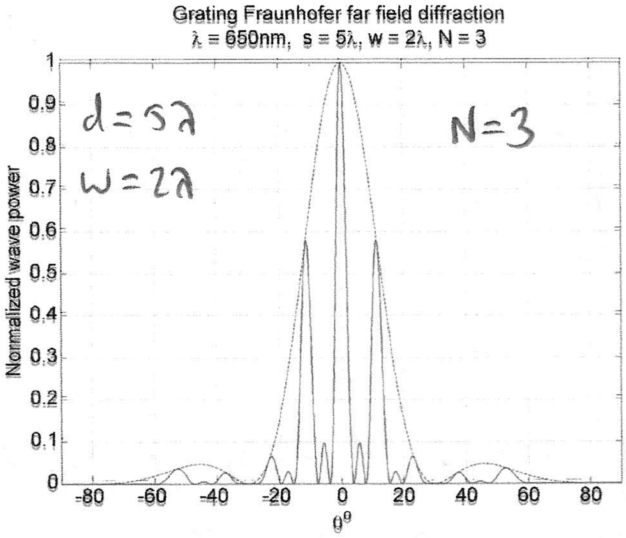
This adds fine structure (more subsidiary peaks) and reduces the peak widths.

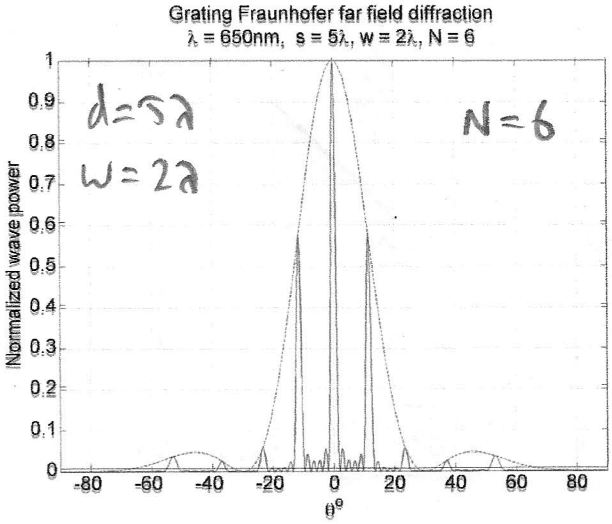
$N = 7, d = 5\lambda$
$w = 2\lambda$

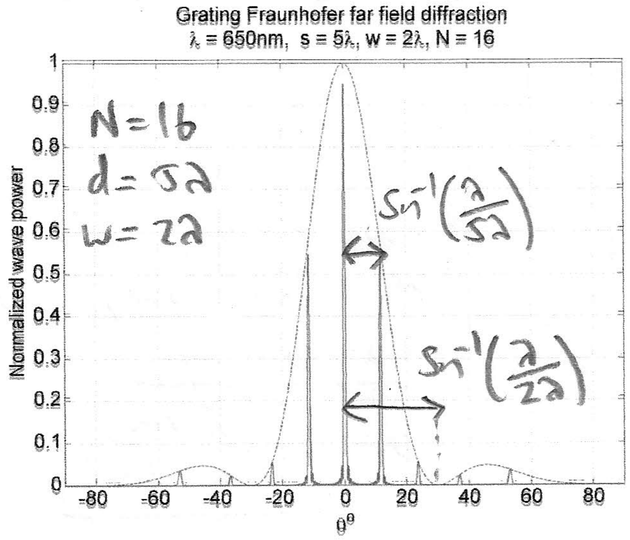




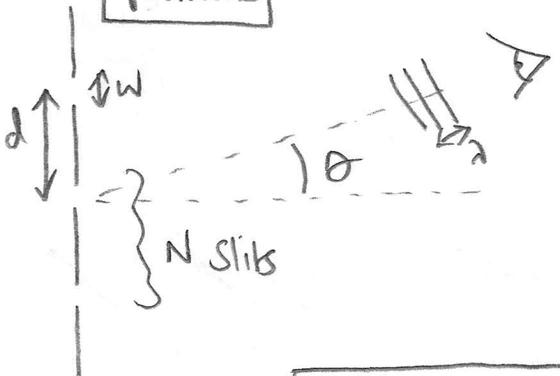








Summary of far-field "Fraunhofer" diffraction intensity patterns



Envelope due to finite slit width : $\boxed{\text{Zeros}}$ at $\boxed{\theta = \sin^{-1}\left(\frac{m\lambda}{w}\right) \quad m \neq 0}$

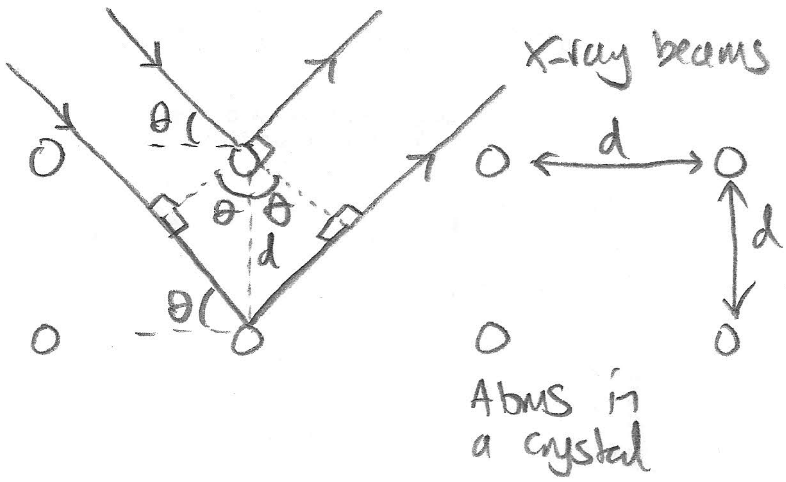
Maxima due to slit spacing : $\boxed{\text{Maxima}}$ at $\boxed{\theta = \sin^{-1}\left(\frac{n\lambda}{d}\right)}$

Fine structure due to # of slits $\boxed{\text{Zeros}}$ at $\boxed{\theta = \sin^{-1}\left(\frac{p\lambda}{Nd}\right)}$

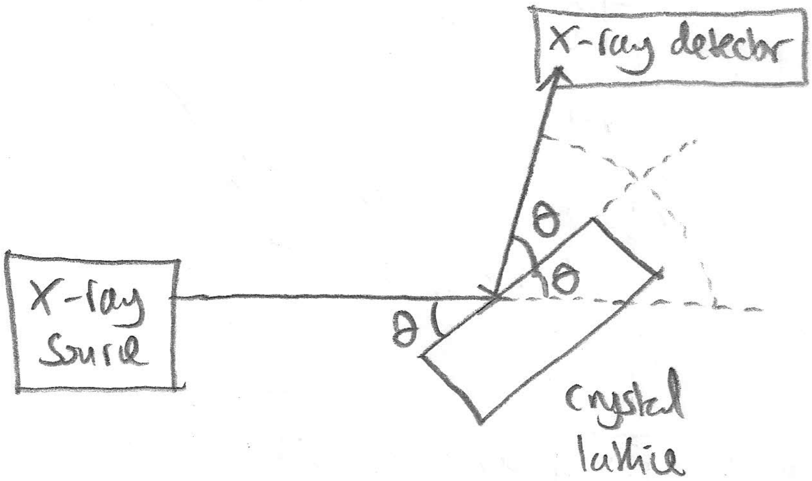
But $\boxed{\text{maxima}}$ when p/N = integer (ie slit spacing maxima)

Diffraction can be used to $\boxed{\text{image}}$ small things - since diffraction pattern maxima angular spacings are $\propto \frac{1}{d}$ so the smaller d, the <u>larger</u> the diffraction pattern spacing of lines.

For example $\boxed{\text{"Bragg's law of X-ray diffraction}}$ from atoms in a crystal lattice"



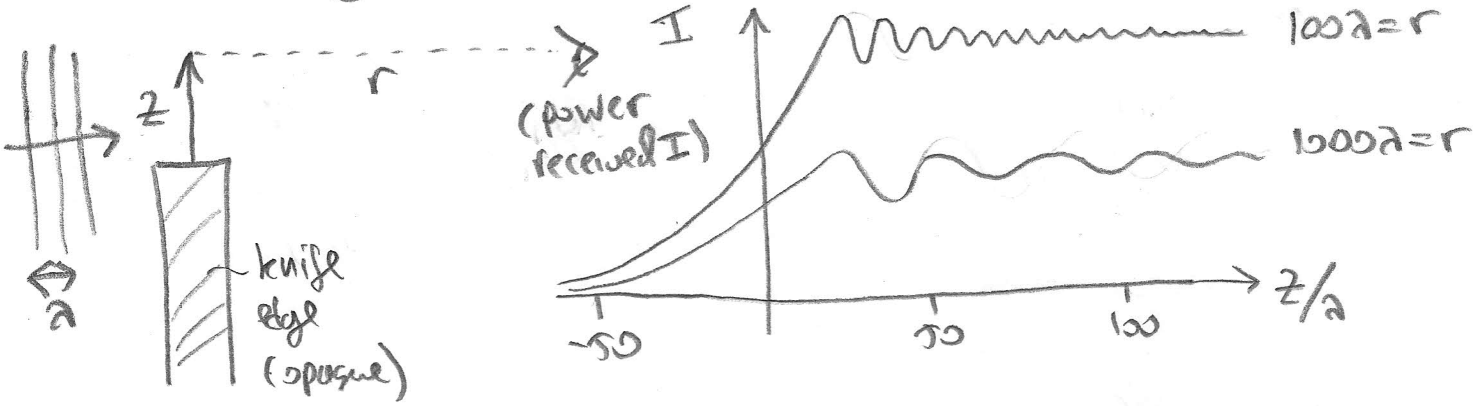
$\boxed{\text{path difference}}$ between X-ray beams reflecting off adjacent layers is $2d\sin\theta$. So get constructive interference when $\boxed{2d\sin\theta = n\lambda}$



* Move X-ray detector on a turn-table, get maxima when $\boxed{\theta = \sin^{-1}\left(\frac{n\lambda}{2d}\right)}$

KNIFE EDGE DIFFRACTION

This is an example of $\boxed{\text{Fresnel diffraction}}$ ie where phase does not vary linearly close to an aperture. (ie 'plane waves' no longer valid)



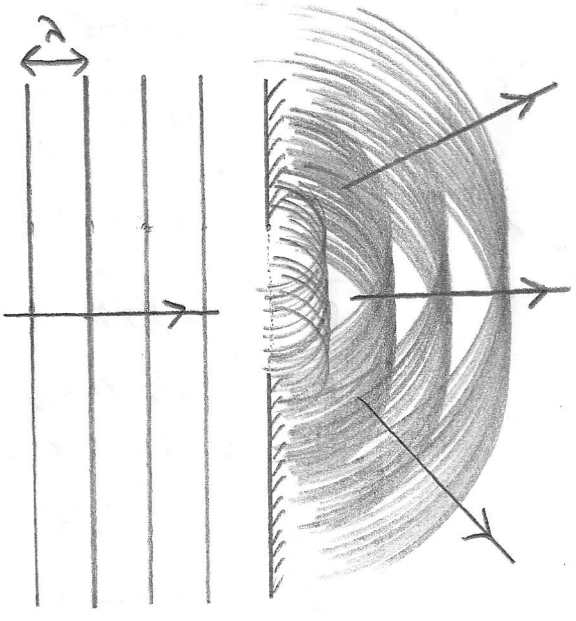
As z ↑ the oscillations in power diminish, and as r ↑ the received power diminishes (but interestingly the oscillations with z have longer 'wavelengths')

GEOMETRIC OPTICS

Light travelling in straight line rays is a useful paradigm in explaining optical phenomena geometrically.

Hugens' principle: "Every point on a wave field is a source of spherical waves"

→ use a ruler & compass to describe diffraction.

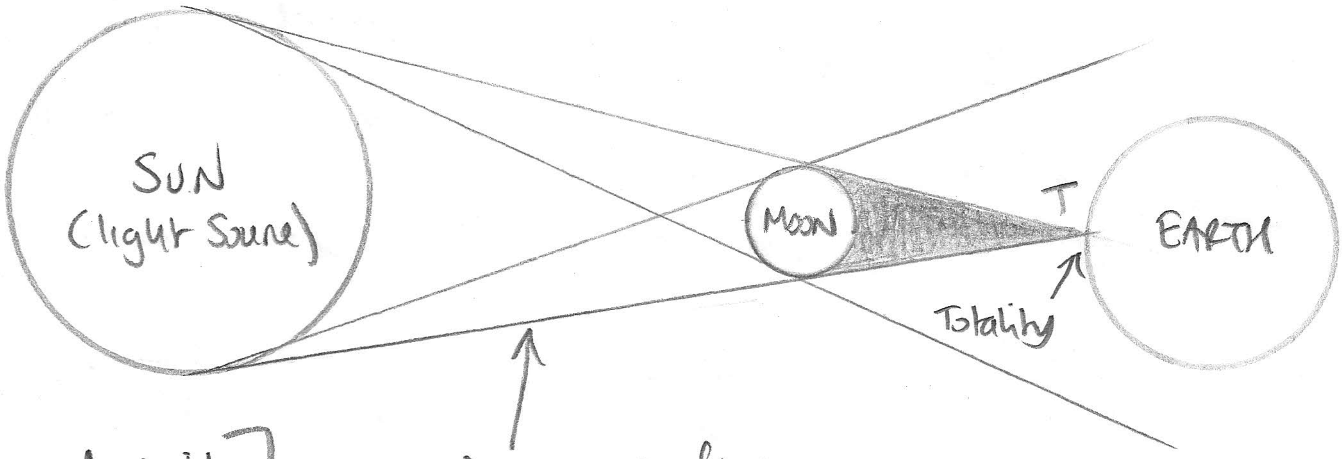


← in this case we draw circles with centres all within the gap with radii λ, 2λ, 3λ etc.

The lines overlap to illustrate the resulting diffraction wave pattern.

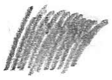
ECLIPSES

Solar eclipse (Not too scale!)

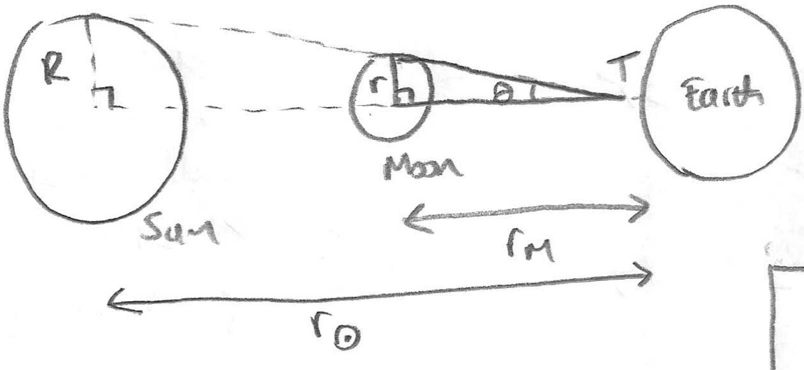


[This is a daylight phenomena, whereas the $\boxed{\text{lunar eclipse}}$ is at night]

Draw rays from 'Extreme' points on the solar disc that represent the boundary of illumination (ie lines which mark the shadow or $\boxed{\text{umbra}}$ due to the moon)

In the  umbra region, no light can reach from the sun - the moon blocks it all. ∴ on Earth the Sun will be $\boxed{\text{eclipsed}}$ , with only the solar corona visible.

Note if the rays converge on earth at T, this means the moon will exactly block the Sun, since they have the same angular width.



{ Similar triangles

So ∴ $\boxed{\dfrac{R}{r_\odot} = \dfrac{r}{r_M}}$

$r \approx 1.737 \times 10^6\,m$

$r_M \approx 63.8\,R_\oplus$

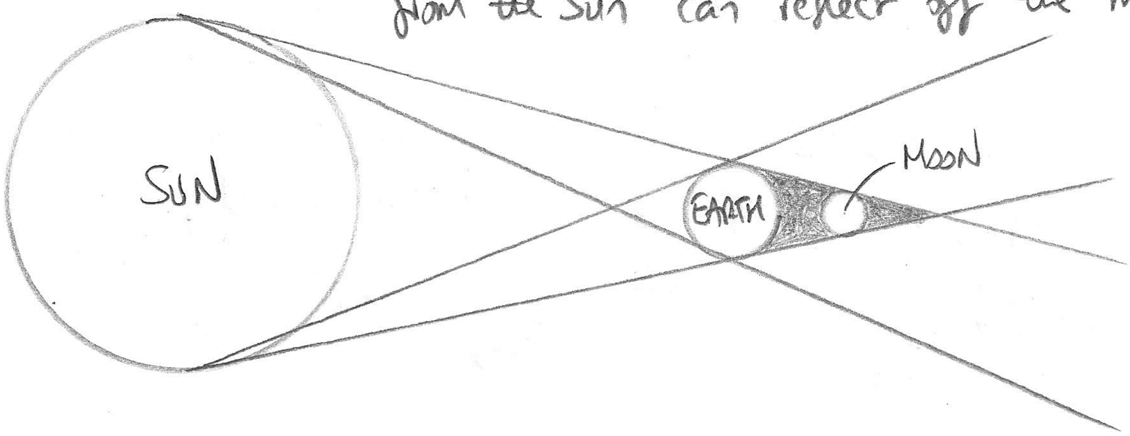
$\approx 63.8 \times 6.378 \times 10^6\,m$

$r_\odot \approx 1.496 \times 10^{11}\,m$ (IAU)

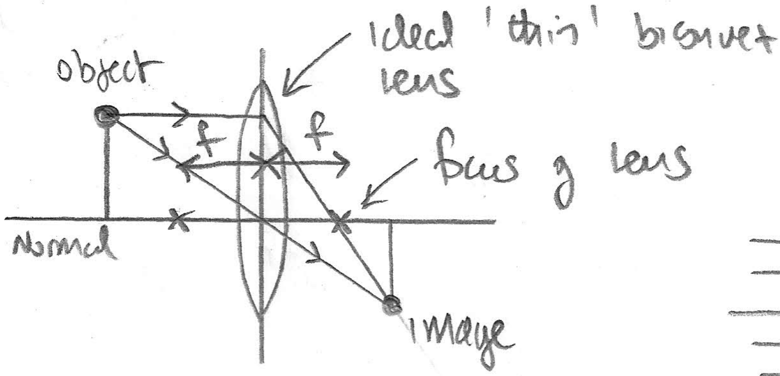
$R \approx 6.960 \times 10^8\,m$

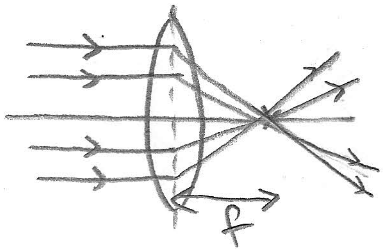
Lunar eclipse

In this case the Moon is in the umbra due to the earth. $\therefore$ no light from the Sun can reflect of the Moon.

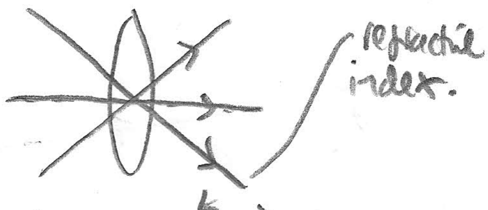


LENSES





Ideally a lens will focus all // rays to a single point, and rays towards the centre of the lens will pass through unrefracted.



In reality, the lenses have finite width, different λ will have different foci $\leftarrow$ due to $n(\lambda) \neq$ constant (chromatic aberration) and rays away from the normal to the lens won't focus at f either (spherical aberration).

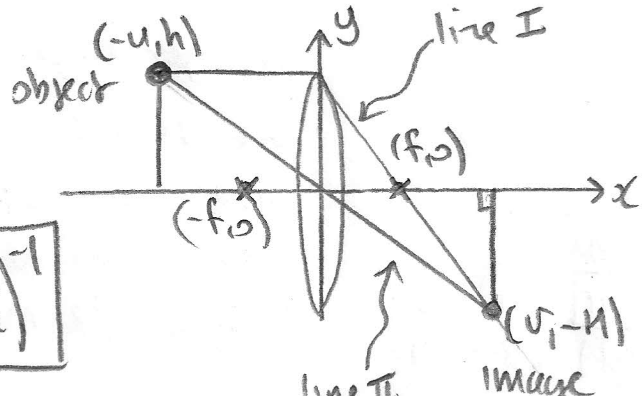


Image coordinates $(v,-H)$ are intersection of lines I, II

$$\boxed{\begin{aligned} \text{I}: \quad & y = h - \frac{h}{f}x \\ \text{II}: \quad & y = -\frac{h}{u}x \end{aligned}}$$

so $h - \frac{h}{f}v = -\frac{h}{u}v$

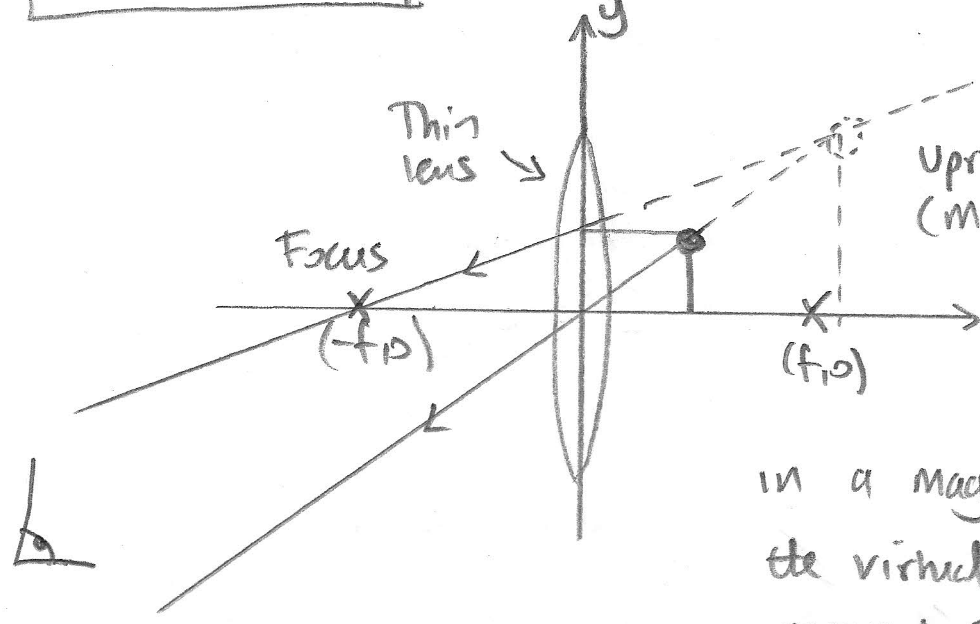
$\therefore \quad 1 = v\left(-\frac{1}{u} + \frac{1}{f}\right)$

$$\boxed{v = \left(\frac{1}{f} - \frac{1}{u}\right)^{-1}}$$

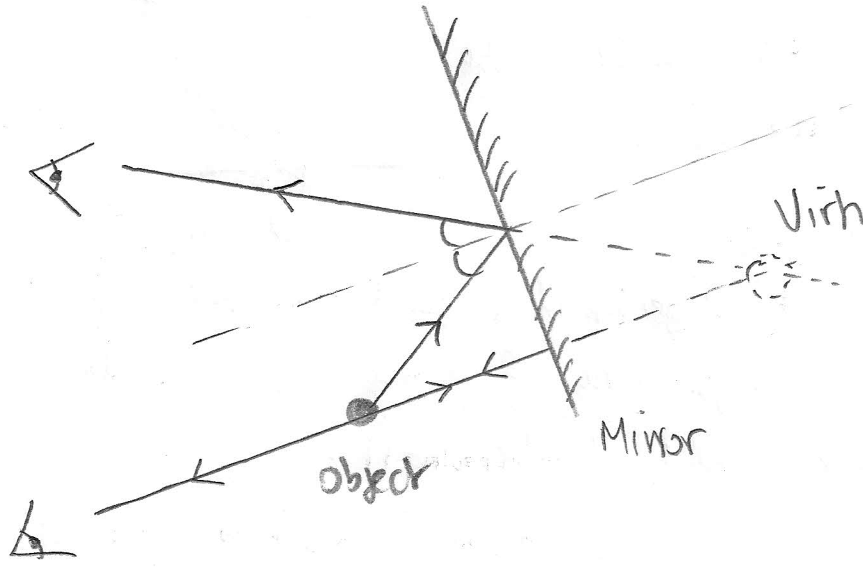
Thin lens formula $\rightarrow$ $\boxed{\frac{1}{v} + \frac{1}{u} = \frac{1}{f}}$

$$\boxed{H = \frac{h}{u}v = \frac{h}{u}\left(\frac{1}{f} - \frac{1}{u}\right)^{-1}}$$

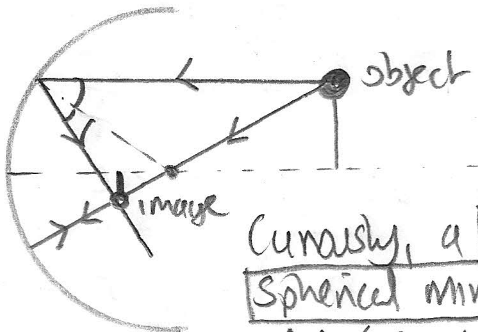
Virtual images



in a magnifying glass the virtual image is the <u>apparent source</u> of (diverging) light rays reaching the observer



For reflections in a planar mirror the apparent source of light rays is the virtual image, the same ⊥ distance of the object from the mirror, but 'inside' the mirror.

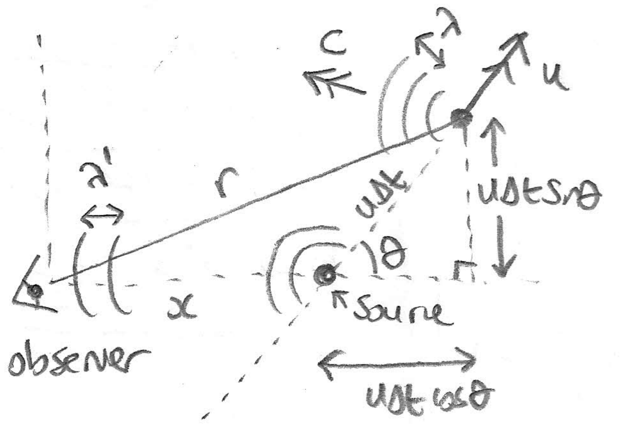


Concave spherical mirror

Curiously, a concave spherical mirror produces an (inverted) reflection which is a REAL image!

(see 'flying coin' demo)

$\boxed{\text{DOPPLER SHIFT}}$ is the shift in frequency of waves received by an observer if a source is in relative motion. If the source is $\boxed{\text{moving away}}$ the frequency $\boxed{\text{reduces}}$ and wavelength increases (REDSHIFT), if the source is approaching the frequency increases and wavelength reduces (BLUESHIFT).



(source is moving at speed u at θ from line from observer)

let wave source emit waves at speed c and wave length λ

let period of these waves be $\boxed{\Delta t = \frac{1}{f}}$ $\boxed{c = f\lambda}$

let waves arrive at observer with wavelength λ' and $\boxed{c = f'\lambda'}$

Time between wave crests at observer $\boxed{\Delta t' = \frac{1}{f'}}$ is:

$$\boxed{\Delta t' = \Delta t + \frac{r-x}{c}}$$

ie $\frac{r-x}{c}$ is extra time to correspond to distance travelled between wave crests emitted at source.

Pythagoras:

$$r^2 = (x + u\Delta t\cos\theta)^2 + u^2\Delta t^2\sin^2\theta$$

$$r^2 = x^2 + 2u\Delta t x\cos\theta + u^2\Delta t^2\cos^2\theta + u^2\Delta t\sin^2\theta$$

$$r^2 = x^2 + 2u\Delta t x\cos\theta + u^2\Delta t^2$$

$$r^2 = x^2\left(1 + \frac{2u\Delta t\cos\theta}{x} + \frac{u^2\Delta t^2}{x^2}\right)$$ Assume $u\Delta t \ll x$

So $r \approx x\left(1 + \frac{2u\Delta t\cos\theta}{x}\right)^{\frac{1}{2}} \approx x + u\Delta t\cos\theta$ (Binomial expansion)

[ie "plane wave approximation", extra distance 'horizontally' is $u\Delta t\cos\theta$]

$\therefore$ $\boxed{r - x \approx u\Delta t\cos\theta}$

So $\frac{1}{f'} = \frac{1}{f} + \frac{u\cos\theta}{c}\frac{1}{f} = \frac{1}{f}\left(1+\frac{u}{c}\cos\theta\right)$

$\therefore$ $\boxed{f' = f\left(1+\frac{u}{c}\cos\theta\right)^{-1}}$

$R = \frac{1}{2}c\Delta T$ ↑ There and back time

Note in Radar the target re-radiates back to the observer

so $\boxed{f' = f\left(1+\frac{u}{c}\cos\theta\right)^{-2}}$

and if $u \ll c$, $\theta = 0^\circ$

$\Rightarrow$ $\boxed{f' \simeq -\frac{2u}{c}f}$

Define Doppler shift $\boxed{\Delta f = f'-f}$

and Redshift $\boxed{z = \frac{f-f'}{f'}}$

So $\boxed{\Delta f = f\left[\left(1+\frac{u}{c}\cos\theta\right)^{-1} - 1\right]}$

If $\theta = 0^\circ$ and $u \ll c$ (e.g. Siren on a car, light emitted by an aircraft...)

$\Delta f \approx f\left(1 - \frac{u}{c} - 1\right)$

$\therefore$ $\boxed{\Delta f \approx -\frac{u}{c}f}$ which is the most 'popular' Doppler shift formula.

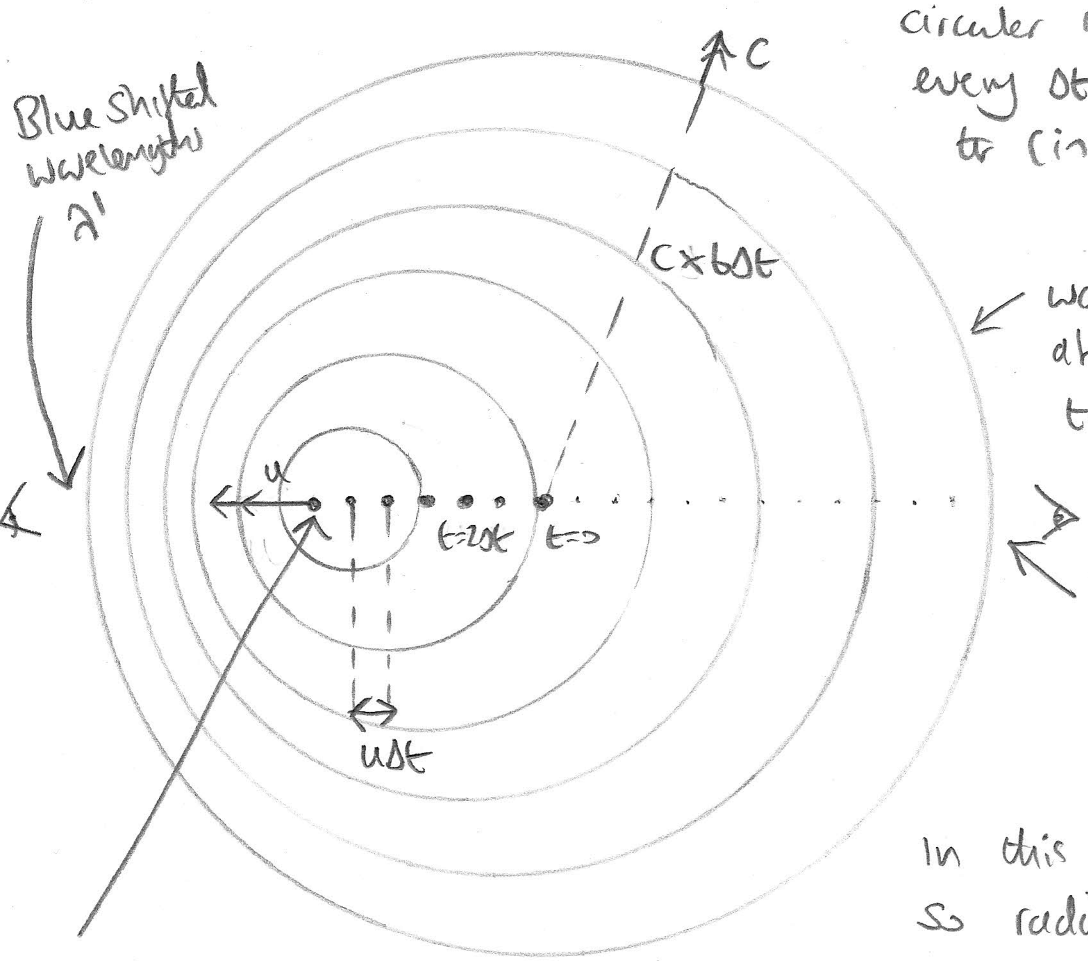
$z = \frac{f}{f'} - 1 = 1 + \frac{u}{c}\cos\theta - 1 = \frac{u}{c}\cos\theta$

So $\boxed{z = \frac{u}{c}\cos\theta}$ is perhaps an easier (well more memorable) description of Doppler shift. [It is used in cosmology, where the general trend of galaxies to be $\boxed{\text{redshifted}}$ due to the expansion of the universe]

MACH'S CONSTRUCTION, SHOCK FRONTS AND A GRAPHICAL VISUALIZATION OF DOPPLER SHIFT

You can visualize the Doppler effect geometrically. Draw circular wavefronts produced every Δt seconds, from $t=0$ to (in this case $6\Delta t$)

Blue shifted wavelengths λ'



wavefront emmitted at $t=0$, at time $t=6\Delta t$

Red shifted wavelengths λ'

Wave source position at $t=6\Delta t$

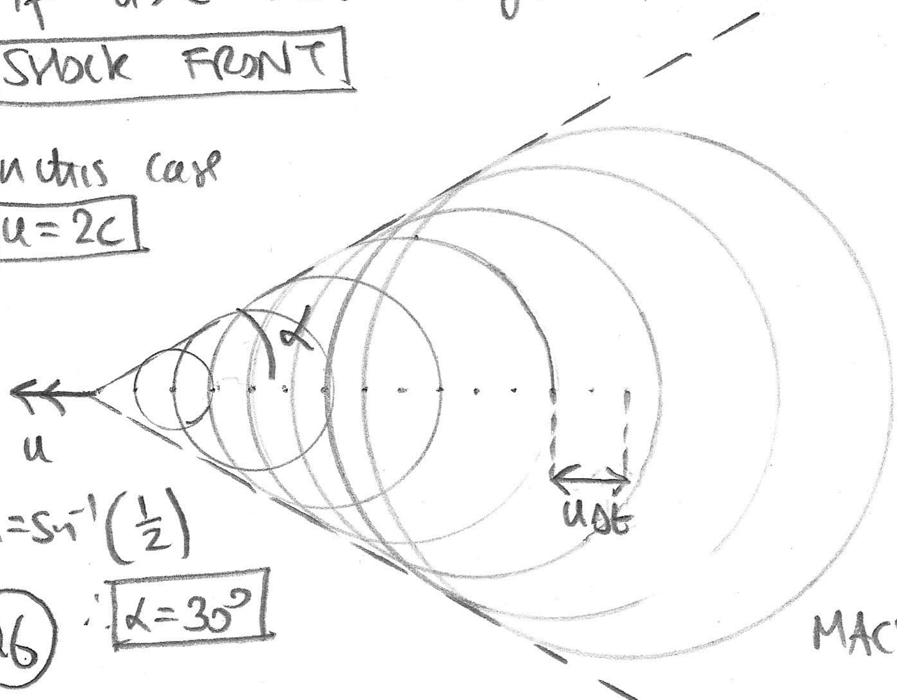
In this example $\boxed{c=2u}$ so radius of circle from $t=0$ location is $12\Delta t$

Doppler shift formula: $\boxed{\dfrac{\lambda'}{\lambda} = 1 + \dfrac{u}{c}\cos\theta}$

If $u>c$ then we get a

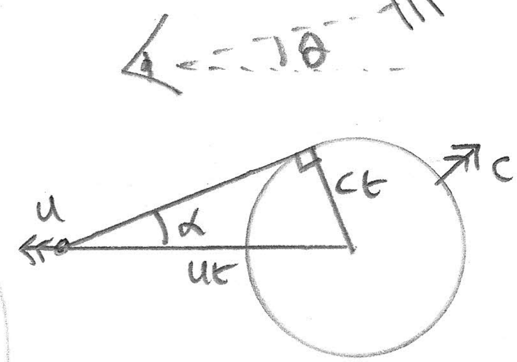
$\boxed{\text{SHOCK FRONT}}$

In this case $\boxed{u=2c}$



$\alpha = \sin^{-1}\left(\frac{1}{2}\right)$

$\therefore \boxed{\alpha = 30^\circ}$



$ut\sin\alpha = ct$

$\therefore \boxed{\alpha = \sin^{-1}\left(\frac{c}{u}\right)}$

MACH # $\boxed{M = \frac{u}{c}}$ $\boxed{\alpha = \sin^{-1}\left(\frac{1}{M}\right)}$