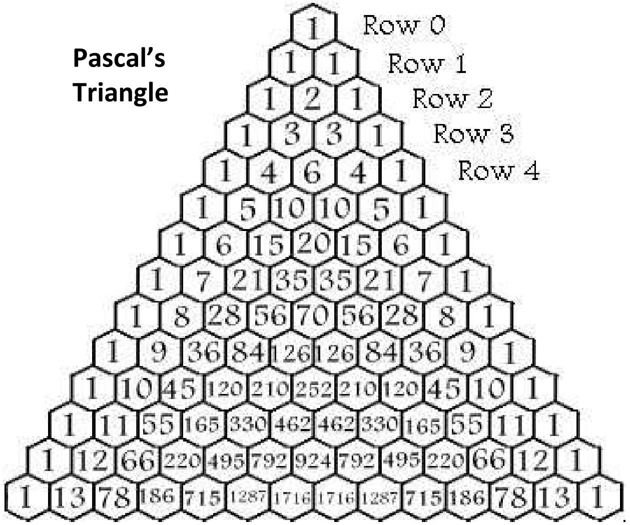
Combinatorics, nCr, nPr & Pascal's Triangle

Number of arrangements	Explanation	Example
$n!$	Number of distinct ways of arranging letters A,B,C are ABC, ACB, BAC, BCA, CAB, CBA i.e. **6** = 3 x 2 x 1 = **3!** Number of distinct ways of arranging *n* letters A,B,C ..... are ***n*!** These are the number of ***permutations*** of a set of *n* distinct objects.	How many permutations are there of ABCDEFG? 7! = **5040**
$\dfrac{n!}{r!}$	If out of *n* objects, *r* are **repeated**, we must divide the total number of permutations by *r*! Why? Write out *n*! permutations of *n* objects, with *r* marked, into a *r! x n! /r!* grid. Each *row* corresponds to a different permutation of the marked objects, and each *column* corresponds to a different permutation of the unmarked objects. If *r* marked objects are the same, each row is identical. Hence the total number of distinct permutations is *n!/r!*	Permutations of VOODOO are 6!/4! = 6 x 5 = **30**
$\dfrac{n!}{r!k!....}$	Consider *n* objects with *r* repeats of one object, *k* repeats of another etc. Following the same 'gridding' ideas as above, we must successively divide the number of permutations by the factorial of the repeats	Permutations of **MISSISSIPPI** (M x 1, I x 4, S x 4, P x 2, total 11 letters) are 11! / (4! x 4! x 2!) = **34,650** Permutations of **RADAR** are 5!/(2!2!) = **30** Permutations of **BAZOOKA** are 7!/(2!2!) = **1260**
$\dfrac{n!}{(n-r)!r!}$	Consider *n* objects comprising of *r* repetitions of one object and *n-r* repetitions of another object. i.e. only two types of object. **NOTE THIS IS ALSO ELEMENT *(n,r)* OF PASCAL'S TRIANGLE**	How many ways can a tennis player win three matches out of ten fixtures? Assume no draws are allowed. (So 3 wins and 7 losses). 10!/(7! x 3!) = 120
${}^nP_r = \dfrac{n!}{(n-r)!}$	**C**onsider the number of permutations of a ***subset*** of *r* letters from an alphabet of *n*. The number of ways of marking *r* letters for inclusion in the subset is $n!/(n-r)!r!$ The number of permutations of *r* letters is *r!* so the total number of permutations is $\dfrac{n!}{(n-r)!r!} \times r! = \dfrac{n!}{(n-r)!}$	Ginger Twos has 16 flavours of ice cream. How many *permutations* of three *distinct* flavours can we have? 16! / (13! x 3!) = 16 x 15 x 14 = **3360**
${}^nC_r = \dfrac{n!}{(n-r)!r!}$	The number of ***combinations*** of *r* objects from the list of *n* is when the order doesn't matter. i.e. A,B,C is the same combination as B,A,C as it contains the letters A, B and C. In this case we have *r*! permutations for each set of *r* objects and therefore the total number of combinations is ${}^nP_r / r! = n!/(n-r)!r!$	How many *combinations* of two *distinct* flavours are there to be tried at Ginger Twos? 16!/ / (14! x 2!) = 16 x 15 / 2 = **120** The number of combinations of four letter 'words' from an alphabet of 26 letters is 26!/(22! x 4!) = 26 x 25 x 24 x 23 / (4 x 3 x 2) = **14,950**

Pascal's Triangle



Row n
Column r $\quad {}^{n}C_{r} = \binom{n}{r} = \dfrac{n!}{(n-r)!\,r!}$

The Binomial Distribution
What is the *probability* of r successes and n-r fails
out of n independent trials, if the probability of success at each trial
is p and failure is 1-p ?

$$P(r\text{ successes}) = \binom{n}{r} p^{r}(1-p)^{n-r}$$

This is the ***Binomial Distribution.*** We say $r \sim B(n, p)$

This is because there are $\binom{n}{r}$ possible *permutations* of r successes and n-r fails in n events

e.g. consider the season record of the Junior Colts D soccer team
The (average) win probability for each game is p = 0.4. What is the probability of winning
6 out of 10 games?

$$P(6\text{ wins}) = \binom{10}{6}(0.4)^{6}(0.6)^{4} = \boxed{0.111}$$

$\dfrac{n!}{r!}$

If out of n objects, r are **repeated**, we must divide the total number of permutations by r!
Why? Write out n! permutations of n objects, with r marked, into a $r!$ x $n!/r!$ grid.
Each row corresponds to a different permutation of the marked objects, and each column corresponds to a different permutation of the unmarked objects.
If r marked objects are the same, each row is identical. Hence the total number of distinct permutations is $n!/r!$

Permutations of VOODOO are 6!/4! = 6 x 5 = **30**

$\dfrac{n!}{r!k!....}$

Consider n objects with r repeats of one object, k repeats of another etc.
Following the same 'gridding' ideas as above, we must successively divide the number of permutations by the factorial of the repeats

Permutations of **MISSISSIPPI** (M x 1, I x 4, S x 4, P x 2, total 11 letters) are 11! / (4! x 4! x 2!) = **34,650**
Permutations of **RADAR** are 5!/(2!2!) = **30**
Permutations of **BAZOOKA** are 7!/(2!2!) = **1260**

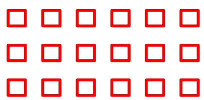
$n!$

....

$\rightarrow$

a

$r!$

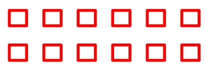


....

$\rightarrow$

b

$k!$



....

$n! = r! \times a$

$\therefore a = \dfrac{n!}{r!}$

$\dfrac{n!}{r!} = k! \times b$

$\therefore b = \dfrac{n!}{r!k!}$

e.g. n letters with r repeated letters of one type
and k repeated letters of another type
such as R,A,D,A,R

Constantinople and adjacent vowels

Consider the word **C O N S T A N T I N O P L E**

There are 14 letters, 2 x O, 3 x N and 2 x T

Hence there are 14!/(2! x 3! x 2!) = **3,632,428,800** permutations of letters.

By considering just the vowels (V) and consonants (C), how many permutations are there where no vowels are next to each other?

Consider, separately, letters VC and C. There are 5 x VC and 4 x C hence 9!/(5! x 4!) =126 permutations. CV and V also has 126 permutations, totalling 252.

Now there are 5!/2! permutations of vowels and 9!/(3! x 2!) permutations of consonants.

Hence there are 252 x 5! x 9! / (2! x 3! x 2!) = **457,228,800** permutations where no vowels are next to each other.

Birthday problem

What is the probability P(N) of two or more people sharing the same birthday in a room of N people?

Answer *P(N)* is 1 – probability of nobody sharing a birthday = 1 – *P'(N).* Ignore leap years and twins for the ensuing analysis!

The probability of two people *not* sharing a birthday is P'(2) = 364/365

The probability of three people not sharing a birthday is P'(3) = (364/365) x (363/365) since the first two people have birthdays on two days out of 365, leaving 363 to choose from.

The probability of four people not sharing a birthday is therefore P'(4) = (364/365) x (363/365) x (362/365)

The probability of N people not sharing a birthday is:

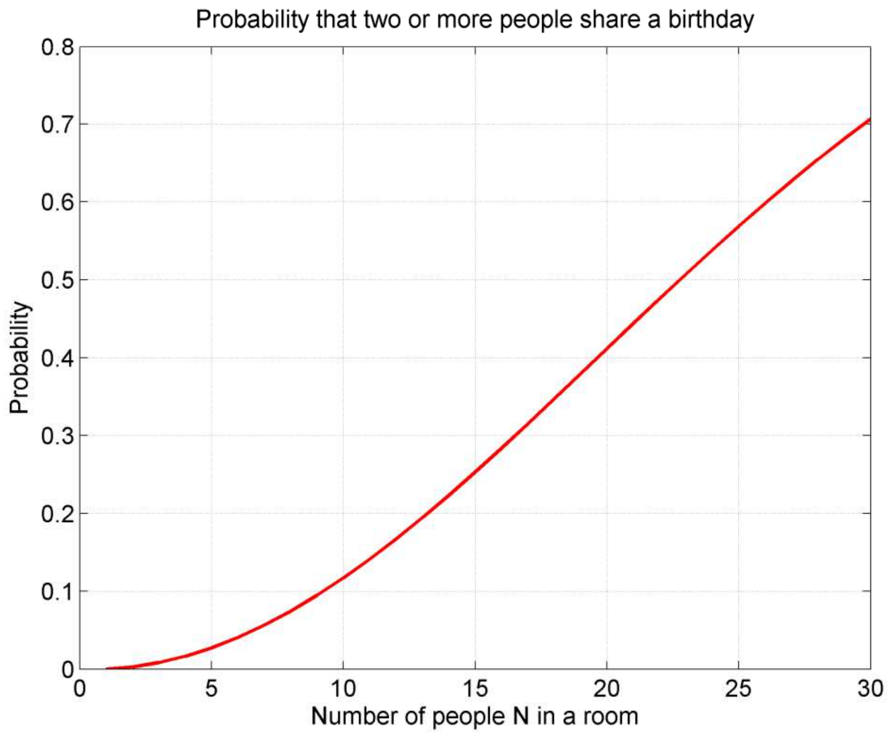
$$P'(N) = \frac{364 \times 363 \times 362 \times ....(365 - N + 1)}{365^N}$$

$$P'(N) = \frac{364 \times 363 \times 362 \times ....(365 - N + 1) \times (365 - N)!}{365^N (365 - N)!}$$

$$P'(N) = \frac{365!}{365^N (365 - N)!}$$

Hence $$\boxed{P(N) = 1 - \frac{365!}{365^N (365 - N)!}}$$

If $N > 23$ then $P(23) > 50\%$ i.e. better odds than flipping a coin!



UK National Lottery

There are 49 numbers in the UK National Lottery. Six numbers plus a 'bonus ball' are picked randomly each time. Let the 6 winning numbers be denoted G ('good') and the 43 numbers which are not picked be labelled B ('Bad'). So 6 G and 43 B. Out of the 43 B, one is the 'bonus ball' which we will denote by a *

Winning the Jackpot

To win the Jackpot you need to choose all the 6 G numbers.

$$P(GGGGGG) = \left(\frac{6}{49}\right)\left(\frac{5}{48}\right)\left(\frac{4}{47}\right)\left(\frac{3}{46}\right)\left(\frac{2}{45}\right)\left(\frac{1}{44}\right) = \frac{6!43!}{49!} = \boxed{\frac{1}{{}^{49}C_6} = 1 \text{ in } 13{,}983{,}816}$$

Bonus prize

To win this prize you must have 5 of the G numbers and match the bonus ball.

$$P(G,G,G,G,G,B^*) = \frac{6!}{5!1!}\mathrm{P(GGGGGB)}P(B=*)$$

$$P(GGGGGB) = \left(\frac{6}{49}\right)\left(\frac{5}{48}\right)\left(\frac{4}{47}\right)\left(\frac{3}{46}\right)\left(\frac{2}{45}\right)\left(\frac{43}{44}\right) = \frac{6!43!}{49!}\times 43$$

$$P(B=*) = \left(\frac{1}{43}\right)$$

$$P(G,G,G,G,G,B^*) = 6\times\frac{6!43!}{49!}\times 43 = \boxed{\frac{6}{{}^{49}C_6} = 1 \text{ in } 2{,}330{,}636}$$

Match 4 prize

To win this prize you must have 4 of the G numbers

$$P(G,G,G,G,B,B) = \frac{6!}{4!2!}P(GGGGBB)$$

$$P(GGGGGB) = \left(\frac{6}{49}\right)\left(\frac{5}{48}\right)\left(\frac{4}{47}\right)\left(\frac{3}{46}\right)\left(\frac{43}{45}\right)\left(\frac{42}{44}\right) = \frac{6!43!}{49!}\times\frac{43\times 42}{2}$$

$$P(G,G,G,G,G,B,B) = 15\times\frac{6!43!}{49!}\times\frac{43\times 42}{2} = \boxed{\frac{13{,}545}{{}^{49}C_6} = 1 \text{ in } 1{,}032}$$

Match 5 prize

To win this prize you must have 5 of the G numbers, but don't match the bonus ball

$$P(G,G,G,G,G,B\,\mathrm{not}^*) = \frac{6!}{5!1!}P(GGGGGB)P(B\,\mathrm{not}^*)$$

$$P(GGGGGB) = \left(\frac{6}{49}\right)\left(\frac{5}{48}\right)\left(\frac{4}{47}\right)\left(\frac{3}{46}\right)\left(\frac{2}{45}\right)\left(\frac{43}{44}\right) = \frac{6!43!}{49!}\times 43$$

$$P(B\,\mathrm{not}^*) = \left(\frac{42}{43}\right)$$

$$P(G,G,G,G,G,B\,\mathrm{not}^*) = 6\times\frac{6!43!}{49!}\times 43\times\left(\frac{42}{43}\right) = \boxed{\frac{252}{{}^{49}C_6} = 1 \text{ in } 55{,}491}$$

Match 3 prize

To win this prize you must have 3 of the G numbers

$$P(G,G,G,B,B,B) = \frac{6!}{3!3!}P(GGGBBB)$$

$$P(GGGGGB) = \left(\frac{6}{49}\right)\left(\frac{5}{48}\right)\left(\frac{4}{47}\right)\left(\frac{43}{46}\right)\left(\frac{42}{45}\right)\left(\frac{41}{44}\right) = \frac{6!43!}{49!}\times\frac{43\times 42\times 41}{3\times 2}$$

$$P(G,G,G,G,G,B,B) = 20\times\frac{6!43!}{49!}\times\frac{43\times 42\times 41}{3\times 2} = \boxed{\frac{246{,}820}{{}^{49}C_6} = 1 \text{ in } 57}$$

Overall prize odds are

$$\frac{1+6+252+13{,}545+246{,}820}{{}^{49}C_6} = \boxed{\frac{260{,}624}{{}^{49}C_6} = 1 \text{ in } 54}$$

UK National Lottery Mathematics
http://plus.maths.org/content/os/issue29/features/haigh/index