Probability of an event = number of desired outcomes / total possible number of outcomes

+	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Outcome	Freq	Probability
2	1	1/36 = 0.028
3	2	2/36 = 0.056
4	3	3/36 = 0.083
5	4	4/36 = 0.111
6	5	5/36 = 0.139
7	6	6/36 = 0.167
8	5	5/36 = 0.139
9	4	4/36 = 0.111
10	3	3/36 = 0.083
11	2	2/36 = 0.056
12	1	1/36 = 0.028

-	1	2	3	4	5	6
1	0	1	2	3	4	5
2	-1	0	1	2	3	4
3	-2	-1	0	1	2	3
4	-3	-2	-1	0	1	2
5	-4	-3	-2	-1	0	1
6	-5	-4	-3	-2	-1	0

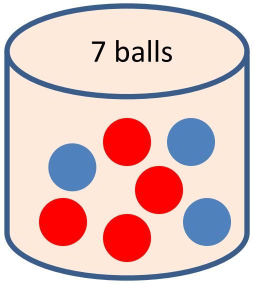
Outcome	Freq	Probability
-5	1	1/36 = 0.028
-4	2	2/36 = 0.056
-3	3	3/36 = 0.083
-2	4	4/36 = 0.111
-1	5	5/36 = 0.139
0	6	6/36 = 0.167
1	5	5/36 = 0.139
2	4	4/36 = 0.111
3	3	3/36 = 0.083
4	2	2/36 = 0.056
5	1	1/36 = 0.028

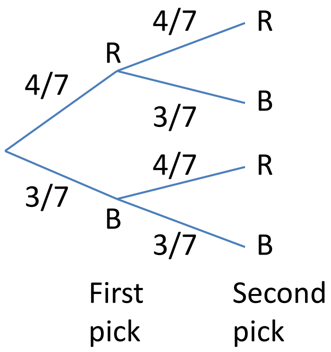
x	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

Outcome	Freq	Probability
1	1	1/36 = 0.028
2	2	2/36 = 0.056
3	2	2/36 = 0.056
4	3	3/36 = 0.083
5	2	2/36 = 0.056
6	4	4/36 = 0.111
8	2	2/36 = 0.056
9	1	1/36 = 0.028
10	2	2/36 = 0.056
12	4	4/36 = 0.111
15	2	2/36 = 0.056
16	1	1/36 = 0.028
18	2	2/36 = 0.056
20	2	2/36 = 0.056
24	2	2/36 = 0.056
25	1	1/36 = 0.028
30	2	2/36 = 0.056
36	1	1/36 = 0.028

Tabulated outcomes of two fair dice under +, - , x

Balls in bag style problems – *with replacement*





P(RR) = (4/7)(4/7) = **16/49** = 0.327

P(RB) = (4/7)(3/7) = **12/49** = 0.245

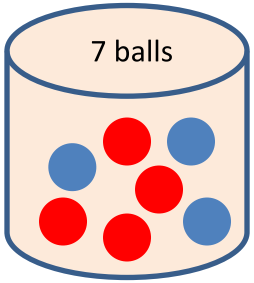
P(BR) = (3/7)(4/7) = **12/49** = 0.245

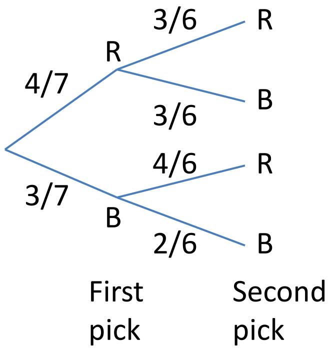
P(BB) = (3/7)(3/7) = **9/49** = 0.184

P(same colour) = P(RR) + P(BB) = **25/49** = 0.510
P(not RR) = 1 – P(RR) = **33/49** = 0.673

P(RRRR) = $(4/7)^4$ = 0.107

Balls in bag style problems – *no replacement*





P(RR) = (4/7)(3/6) = 12/42 = **2/7** = 0.286

P(RB) = (4/7)(3/6) = 12/42 = **2/47** = 0.286

P(BR) = (3/7)(4/6) = 12/42 = **2/7** = 0.286

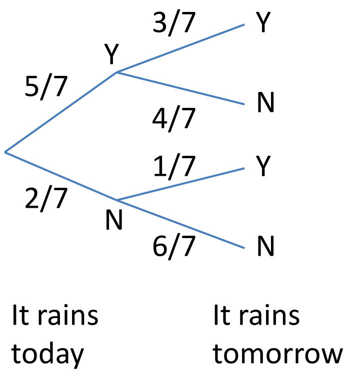
P(BB) = (3/7)(2/6) = 6/42 = **1/7** = 0.143

P(same colour) = P(RR) + P(BB) = 28/42 = **2/3**
P(not RR) = 1 – P(RR) = 30/42 = **5/7** = 0.714

Cards
2,3,4,5,6,7,8,9,10,J,K,Q,A (13 cards)
4 suits (diamonds, hearts, clubs, spades)
ignoring jokers (2 typically) 13 x 4 = **52 cards**

P(A of spades) = **1/52** = 0.019
P(A) = 4/52 = **1/13** = 0.077
P(Two A, without replacement)
= (4/52) x (3/51) = 0.00452 (0.452%)

The probability of events can be ***conditional*** upon what happened previously. P(rain tomorrow | rain today) means the probability of rain tomorrow ***given*** it is raining today.



P(Y|Y) = **3/7**
P(Y|N) = **1/7**

P(raining tomorrow) = P(Y|Y)P(Y) + P(Y|N)P(N)
= (3/7)(5/7) + (1/7)(2/7)
= **17/49** = 0.347

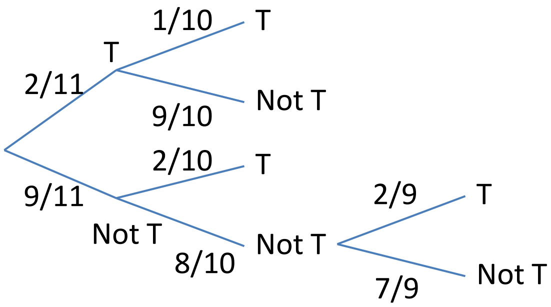
Probability with words
Consider the word MATHEMATICS, cut up in to letters and placed into a box

11 letters
M x 2, A x 2, T x 2, H,E,I,C,S x 1
4 vowels (A,A,E,I) and 7 consonants

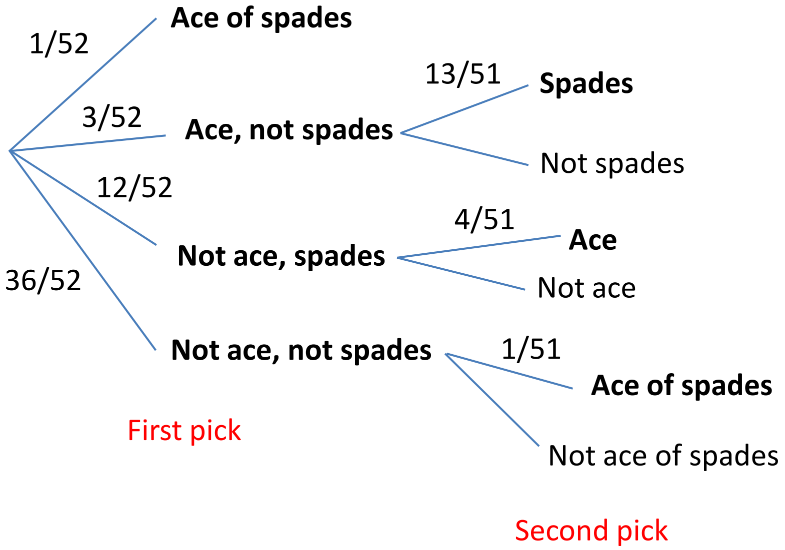
If letters are chosen from the box at random:

P(A) = **2/11**
P(vowel) = **4/11**
P(consonant, consonant) = (7/11)(6/10)

P(no T's in three picks) = (9/11)(8/10)(7/9)
= **28/55** = 0.509



For a pack of cards without jokers, what is the probability of obtaining an ace and a spade in two draws, without replacement?



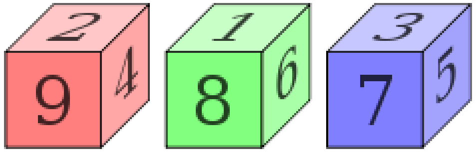
P(ace and a spade) =

P(ace of spades) + ...
P(ace, not spades then spades) + ...
P(not ace, spades then ace) + ...
P(not ace, not spades then ace of spades)

= 1/52 + (3/52)(13/51) + (12/52)(4/51) + (36/52)(1/51)
= **29/442**
= 6.56%

Note: 3 x 13 = 39 cards are not spades. 3 of these will be aces so there are 36 cards that are not spades and not aces

Non-transitive dice



Consider three fair dice with numbers: **Dice A:** {2,2,4,4,9,9} ; **Dice B:** {1,1,6,6,8,8} ; **Dice C:** {3,3,5,5,7,7}
Player 1 chooses one of the dice, and *Player 2* then chooses one from the remaining pair.
The chosen dice are then rolled by each player and the highest score wins.

On average, A has higher scores than B, C has higher scores than A, but, counter to intuition, *C* has on average higher *lower* scores than B. This scenario is equivalent to the 'rock, paper, scissors' game. i.e. paper beats rock, scissors beats paper, but scissors *loses* to rock. If the dice were *transitive* then scissors would beat rock since scissors beats paper and paper beats rock. The dice are *non-transitive* because this 'hierarchy of winners' logic does not hold, just like rock, paper, scissors.

Expectations: These are the *same* for each dice

$$\bar{x}_A = \tfrac{1}{6}\left(2+2+4+4+9+9\right) = \tfrac{30}{6} = 5$$

$$\bar{x}_B = \tfrac{1}{6}\left(1+1+6+6+8+8\right) = \tfrac{30}{6} = 5$$

$$\bar{x}_C = \tfrac{1}{6}\left(3+3+5+5+7+7\right) = \tfrac{30}{6} = 5$$

Results of Dice X vs Dice Y: Winners are marked. Note a draw is impossible!

		Dice A					
		2	2	4	4	9	9
	1	A	A	A	A	A	A
	1	A	A	A	A	A	A
Dice B	6	B	B	B	B	A	A
	6	B	B	B	B	A	A
	8	B	B	B	B	A	A
	8	B	B	B	B	A	A

		Dice A					
		2	2	4	4	9	9
	3	C	C	A	A	A	A
	3	C	C	A	A	A	A
Dice C	5	C	C	C	C	A	A
	5	C	C	C	C	A	A
	7	C	C	C	C	A	A
	7	C	C	C	C	A	A

		Dice B					
		1	1	6	6	8	8
	3	C	C	B	B	B	B
	3	C	C	B	B	B	B
Dice C	5	C	C	B	B	B	B
	5	C	C	B	B	B	B
	7	C	C	C	C	B	B
	7	C	C	C	C	B	B

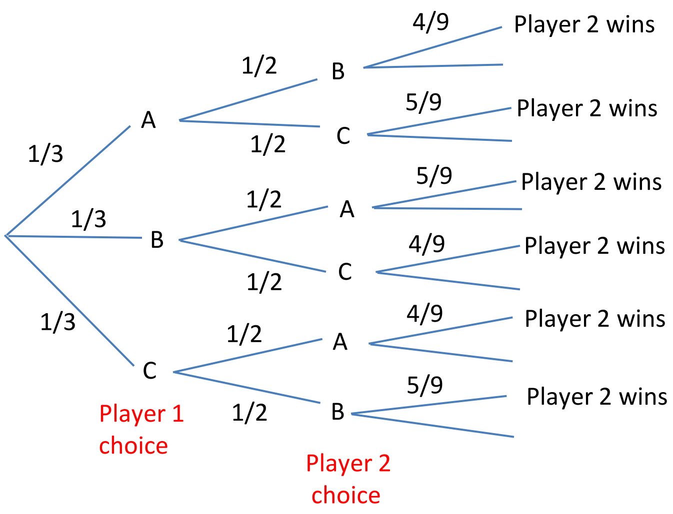
$$P(A>B) = \frac{20}{36} = \frac{5}{9} \qquad P(B>A) = \frac{16}{36} = \frac{4}{9}$$

$$P(C>A) = \frac{20}{36} = \frac{5}{9} \qquad P(A>C) = \frac{16}{36} = \frac{4}{9}$$

$$P(B>C) = \frac{20}{36} = \frac{5}{9} \qquad P(C>B) = \frac{16}{36} = \frac{4}{9}$$

So, on average, A beats B, C beats A but C *loses* to B

From the win tables we can construct a tree diagram to determine the **probability of winning**, if *both players choose randomly*



$$\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{4}{9}\right) = \frac{2}{27}$$

$$\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{5}{9}\right) = \frac{5}{54}$$

$$\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{5}{9}\right) = \frac{5}{54}$$

$$\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{4}{9}\right) = \frac{2}{27}$$

$$\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{4}{9}\right) = \frac{2}{27}$$

$$\left(\frac{1}{3}\right)\left(\frac{1}{2}\right)\left(\frac{5}{9}\right) = \frac{5}{54}$$

Player 2 win probability

Hence the probability of Player 2 winning, based on *random choice* and Player 1 going first is

$$P(\text{player 2 wins}) = \frac{4+5+5+4+4+5}{54} = \frac{27}{54} = 0.5$$

However, if instead Player 2 *always* selects the dice which will have the highest probability (5/9) of beating the selection of Player 1:

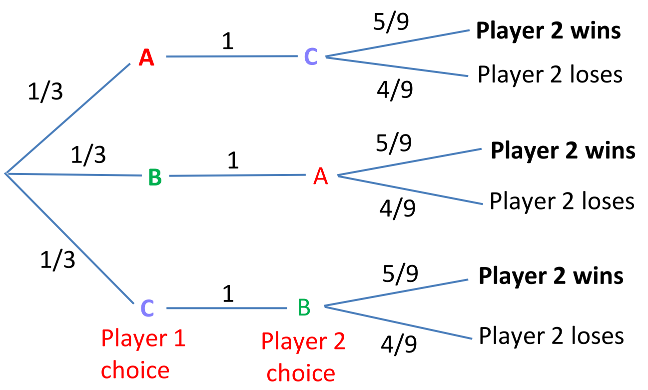
$$P(\text{player 2 wins}) = \frac{5}{9} \approx 0.556$$

So if choice is truly random, Player 2 has no advantage. However, if one is able to choose the optimal dice dependent upon what Player 1 has chosen , there is a *distinct advantage to choosing second.* (Continued on the next page ...)

http://en.wikipedia.org/wiki/Nontransitive_dice

Non-transitive dice continued ....

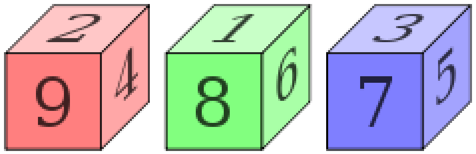
The tree diagram below describes the situation where a *rule* is used to maximize the odds of Player 2 winning. i.e. Player 2 chooses the dice from the remaining pair based upon the choice of Player 1.



$$P(\text{player 2 wins}) = \left(\tfrac{1}{3}\right)\left(\tfrac{5}{9}\right) + \left(\tfrac{1}{3}\right)\left(\tfrac{5}{9}\right) + \left(\tfrac{1}{3}\right)\left(\tfrac{5}{9}\right)$$

$$P(\text{player 2 wins}) = \tfrac{5}{9} \approx 0.55555...$$

So applying the rule gives Player 2 about a 5.6% *advantage* over both players randomly choosing the dice



Dice A: {2,2,4,4,9,9}
Dice B: {1,1,6,6,8,8}
Dice C: {3,3,5,5,7,7}

$$P(C > A) = \frac{5}{9} \qquad P(A > C) = \frac{4}{9}$$

$$P(A > B) = \frac{5}{9} \qquad P(B > A) = \frac{4}{9}$$

$$P(B > C) = \frac{5}{9} \qquad P(C > B) = \frac{4}{9}$$

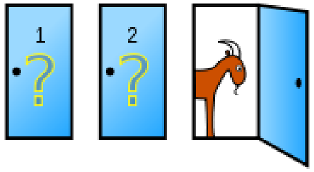
The Monty Hall problem

In a game show there are three doors. Behind two doors are *goats* and behind The remaining door is a *sports car.* The idea is that you win the prize behind the door you choose.

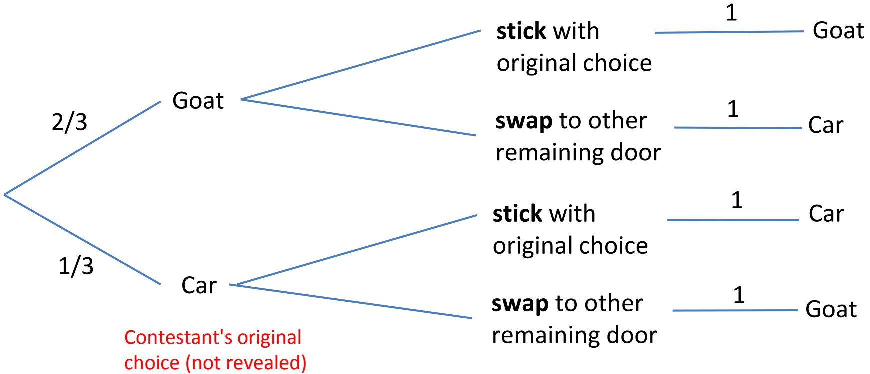
A contestant *chooses a door at random*, then the presenter (who *knows* what is behind each door) *reveals a goat* from behind one of the two doors which *have not been chosen.*

The contestant is given the choice whether to **stick** with the original choice or **swap** to the remaining closed door.

Which option (**stick** or **swap**) gives the highest probability of winning the car?



The *Monty Hall problem* is a brain teaser, in the form of a probability puzzle (Gruber, Krauss and others), loosely based on the American television game show *Let's Make a Deal* and named after its original host, Monty Hall. The problem was originally posed in a letter by Steve Selvin to the *American Statistician* in 1975. It became famous as a question from a reader's letter quoted in Marilyn vos Savant's "Ask Marilyn" column in *Parade* magazine in 1990. Amazingly, many very eminent scientists, mathematicians and other heavyweight academics claimed Marilyn was wrong at the time. Much to their embarrassment, her solution (that the contestant should **swap**), was most certainly correct!



Contestant's original choice (not revealed)

Contestant's choice after a goat has been revealed (leaving two doors remaining)

$$P(\text{Car given stick}) = \left(\tfrac{1}{3}\right) \times 1 = \tfrac{1}{3}$$

$$P(\text{Car given swap}) = \left(\tfrac{2}{3}\right) \times 1 = \tfrac{2}{3}$$

So you **are twice as likely to win the car if you adopt the swap strategy.**

You are *twice as likely to choose the goat on the first choice*. If you choose a goat and swap, you get a car. So therefore swapping is the logical choice.