The house buying problem

I have n houses to choose from, following viewings. All match my requirements for multiple bathrooms, open-plan kitchens, secluded gardens with mega-shed potential, houses that are not to too close to the M27, but within cycling distance of my workplace. However, the market is so competitive that once I reject a choice, I cannot put in an offer later since the house gets immediately snapped up!
What is my optimum strategy to have the maximum chance of choosing the *best house overall*?

The strategy is, as n increases, to ***reject* the first k houses** and **then select the first house which is better than the best of the first** k. The optimum value for k is: $\boxed{k = \frac{n-1}{e}}$

Assume we reject the first k houses, the probability that pick j of the *remaining* n-k houses **is the best overall** is:

$$\boxed{p(n,k,j) = P(\text{house } j \text{ of } n\text{-}k \text{ is the best overall}) = P(\text{house } j \text{ of } n\text{-}k \text{ is the best}) \times P(\text{the previous } j\text{-1 houses are } \textit{not} \text{ better than the best in the first } k)}$$

$$\therefore p(n,k,j) = \frac{1}{n} \times \frac{k}{k+j-1} \longleftarrow P(\text{the previous } j\text{-1 houses are } \textit{not} \text{ better than the best in the first } k)$$

i.e. the *same probability* that the **best in the first** k+j -1 **houses** is *within* the first k, which were rejected.

$P(\text{house } j \text{ of } n\text{-}k \text{ is the best overall})$

Assume no prior knowledge and a randomized viewing list so every house has the same chance of being the best.

Hence the probability of choosing the best house using this strategy is:

$$q(n,k) = \sum_{j=1}^{n-k} p(n,k,j) = \frac{1}{n} \times \sum_{j=1}^{n-k} \frac{k}{k+j-1}$$
$$= \tfrac{1}{n}\left(1 + k\left(\tfrac{1}{k+1} + \tfrac{1}{k+2} + \ldots + \tfrac{1}{n-1}\right)\right)$$

Now: $$\underbrace{1 + \tfrac{1}{2} + \tfrac{1}{3} + \ldots + \tfrac{1}{n-1}}_{H_{n-1}} = \underbrace{1 + \tfrac{1}{2} + \ldots + \tfrac{1}{k}}_{H_k} + \tfrac{1}{k+1} + \tfrac{1}{k+2} + \ldots + \tfrac{1}{n-1}$$
$$\therefore \tfrac{1}{k+1} + \tfrac{1}{k+2} + \ldots + \tfrac{1}{n-1} = H_{n-1} - H_k$$

$$\boxed{\therefore q(n,k) = \tfrac{1}{n}\left(1 + k\left(H_{n-1} - H_k\right)\right)}$$

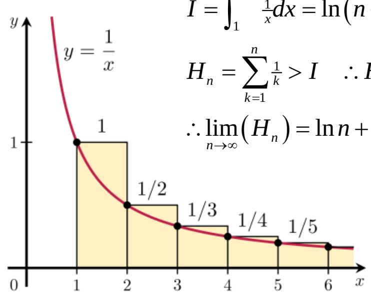
Harmonic series **sum**

$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$

$\lim_{n\to\infty}(H_n) = \ln n + \gamma$

$\gamma \approx 0.5772156649$

Euler–Mascheroni constant



$$I = \int_1^{n+1} \tfrac{1}{x}dx = \ln(n+1)$$

$$H_n = \sum_{k=1}^{n} \tfrac{1}{k} > I \quad \therefore H_n > \ln(n+1)$$

$$\therefore \lim_{n\to\infty}(H_n) = \ln n + \gamma$$

The difference between the harmonic series sum and the integral of $1/x$ tends to a *constant* as n increases.

For large n : $\therefore q(n,k) \approx \tfrac{1}{n}\left(1 + k\ln(n-1) - k\ln k\right) = \tfrac{1}{n} + \tfrac{k}{n}\ln\left(\tfrac{n-1}{k}\right)$

$H_n = 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \approx \ln n + \gamma$

$$\frac{\partial}{\partial k} q(n,k) = \tfrac{1}{n}\left\{\ln(n-1) - \ln k - 1\right\}$$

$$\therefore \frac{\partial}{\partial k} q(n,k) = 0 \quad \text{when} \quad \ln\left(\frac{n-1}{k}\right) = 1$$

i.e. what k gives the maximum q?

$$\therefore \frac{n-1}{k} = e \quad \Rightarrow \boxed{k = \frac{n-1}{e}}$$

$$\therefore q_{\max}(n) = q\left(n, k = \frac{n-1}{e}\right)$$
$$= \tfrac{1}{n} + \tfrac{1}{n}\frac{n-1}{e}\ln(e)$$
$$= \tfrac{1}{n} + \tfrac{1}{e}\left(1 - \tfrac{1}{n}\right)$$
$$\to \boxed{\frac{1}{e}} \quad \text{for large } n$$

$1/e \approx \frac{1}{2.7183} \approx 0.3679$

so one should reject about 37% of the n houses to choose from before picking the best so far.

Dashed -- lines are the approximate result
$q(n,k) = \tfrac{1}{n} + \tfrac{k}{n}\ln\left(\tfrac{n-1}{k}\right)$

Solid lines with **.** are the exact calculation
$q(n,k) = \tfrac{1}{n}\left(1 + k\left(H_{n-1} - H_k\right)\right)$