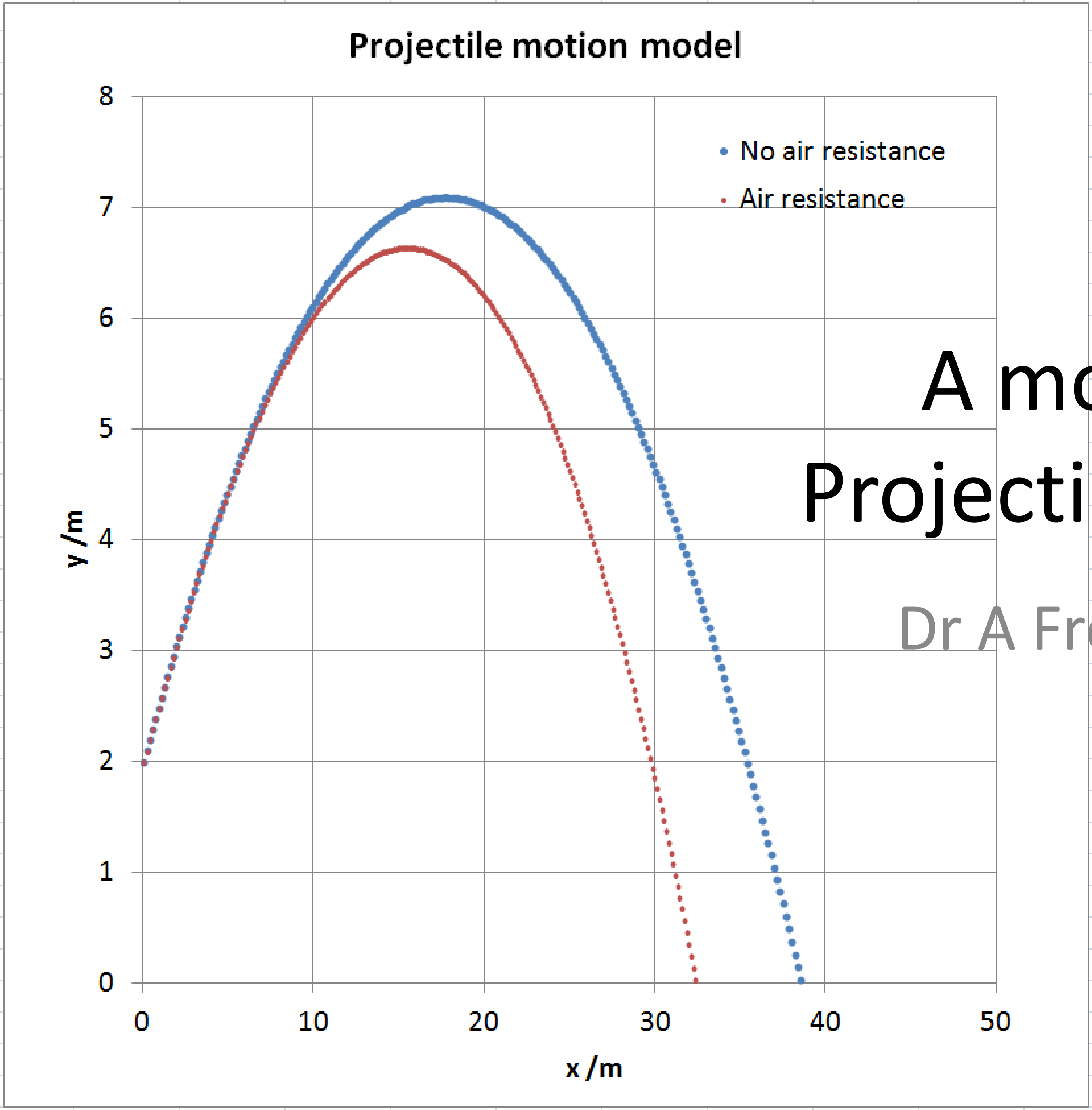
A model of Projectile Motion

Dr A French 2017



Exact model (no air resistance) using constant acceleration motion

The *only* acceleration is g downwards!

$$x = u_x t$$

$$y = h + u_y t - \tfrac{1}{2} g t^2$$

$$v_x = u_x$$

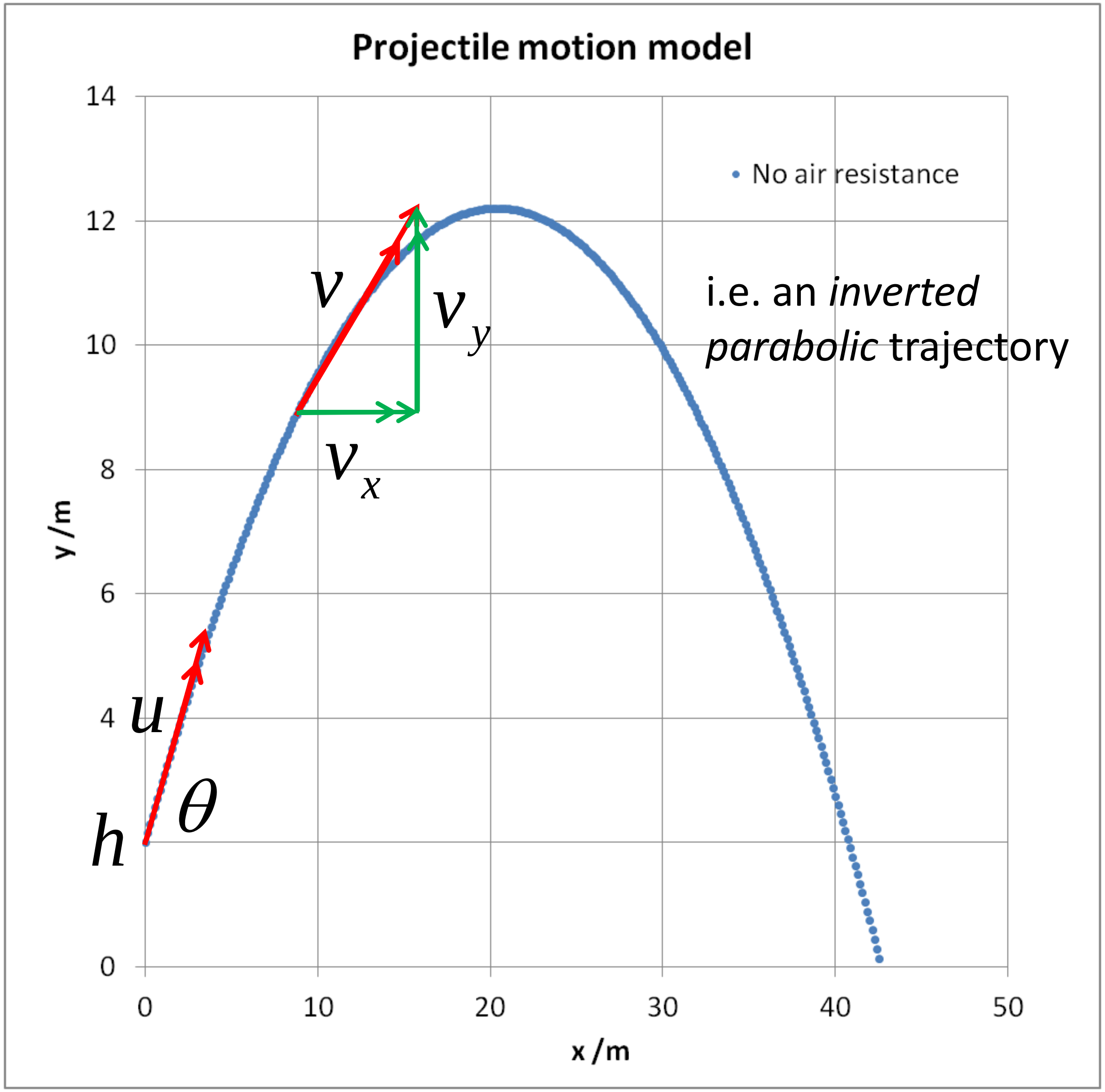
$$v_y = u_y - gt$$

$$v = \sqrt{v_x^2 + v_y^2}$$

Initial x and y velocities

$$u_x = u\cos\theta$$

$$u_y = u\sin\theta$$



i.e. an *inverted parabolic* trajectory

$$v_x = u_x$$

`= $c$9*cos( $c$8 * pi()/180 )`

Need to convert angles into **radians**

$$v_y = u_y - gt$$

`= $c$9*sin( $c$8 * pi()/180 ) - $c$11*e5`

$$t_{n+1} = t_n + \Delta t$$

`= e5 + $c$13`

$$v = \sqrt{v_x^2 + v_y^2}$$

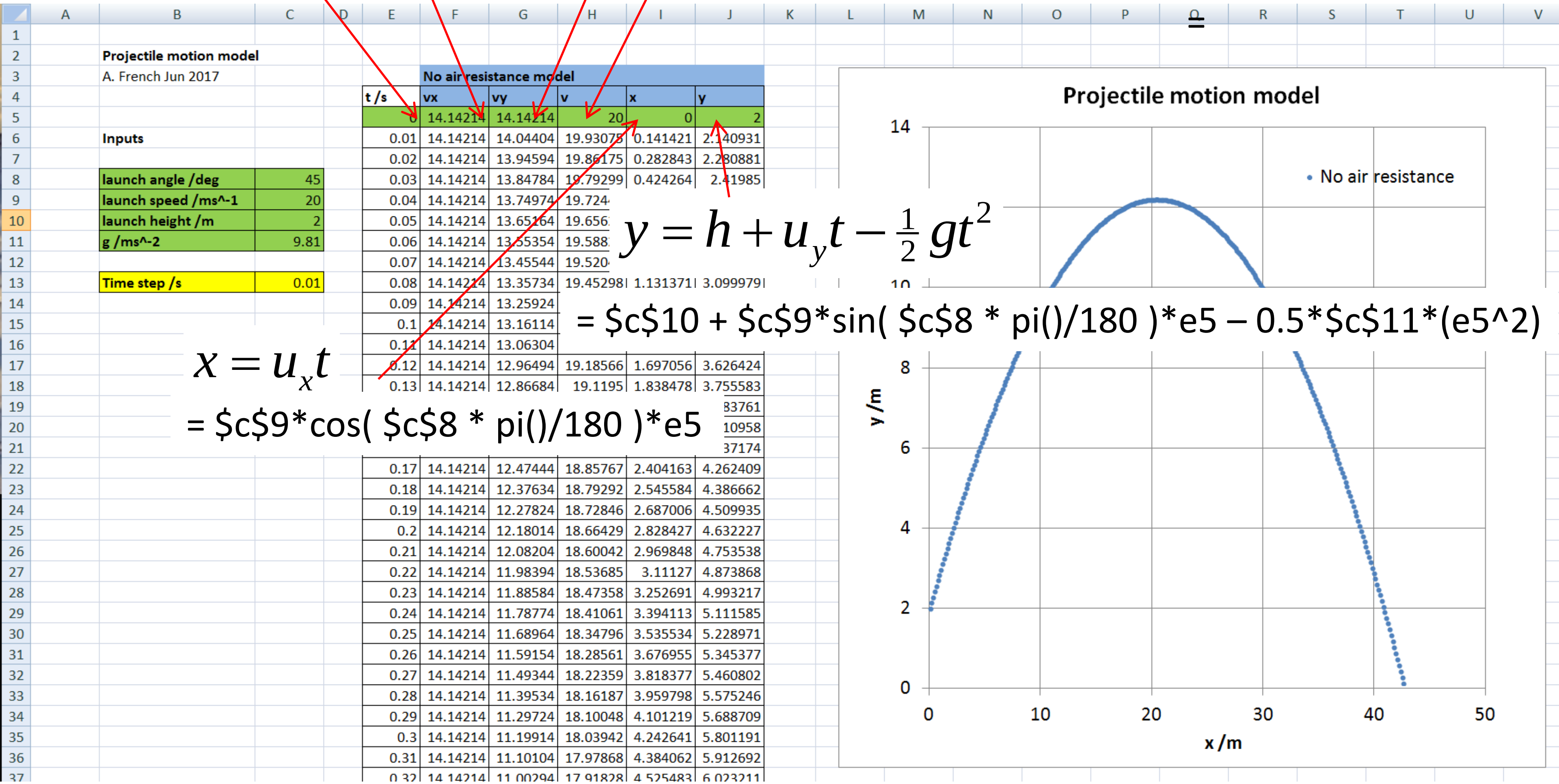
`= sqrt(f5^2 + g5^2)`

$$y = h + u_y t - \tfrac{1}{2} g t^2$$

`= $c$10 + $c$9*sin( $c$8 * pi()/180 )*e5 – 0.5*$c$11*(e5^2)`

$$x = u_x t$$

`= $c$9*cos( $c$8 * pi()/180 )*e5`



Projectile motion model

A. French Jun 2017

Inputs

launch angle /deg	45
launch speed /ms^-1	20
launch height /m	2
g /ms^-2	9.81
Time step /s	0.01

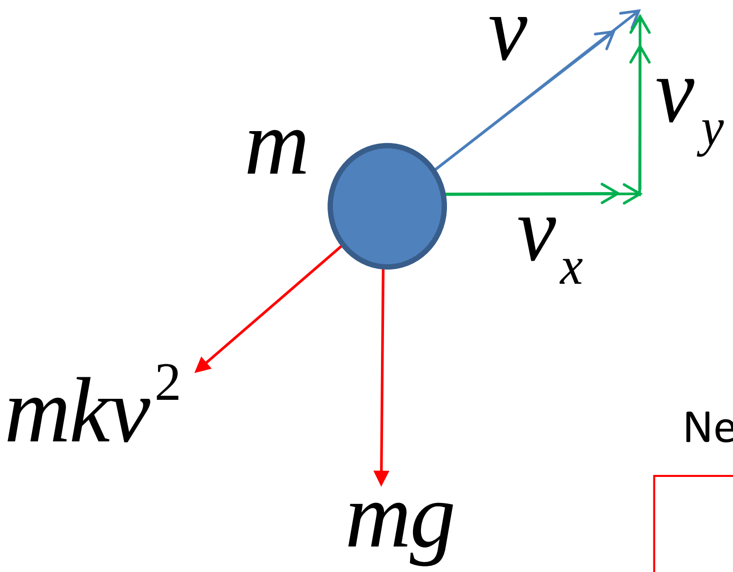
No air resistance model

t /s	**vx**	**vy**	**v**	**x**	**y**
0	14.14214	14.14214	20	0	2
0.01	14.14214	14.04404	19.93075	0.141421	2.140931
0.02	14.14214	13.94594	19.86175	0.282843	2.280881
0.03	14.14214	13.84784	19.79299	0.424264	2.41985
0.04	14.14214	13.74974	19.724		
0.05	14.14214	13.65164	19.656		
0.06	14.14214	13.55354	19.588		
0.07	14.14214	13.45544	19.520		
0.08	14.14214	13.35734	19.45298	1.131371	3.099979
0.09	14.14214	13.25924			
0.1	14.14214	13.16114			
0.11	14.14214	13.06304			
0.12	14.14214	12.96494	19.18566	1.697056	3.626424
0.13	14.14214	12.86684	19.1195	1.838478	3.755583
					83761
					10958
					37174
0.17	14.14214	12.47444	18.85767	2.404163	4.262409
0.18	14.14214	12.37634	18.79292	2.545584	4.386662
0.19	14.14214	12.27824	18.72846	2.687006	4.509935
0.2	14.14214	12.18014	18.66429	2.828427	4.632227
0.21	14.14214	12.08204	18.60042	2.969848	4.753538
0.22	14.14214	11.98394	18.53685	3.11127	4.873868
0.23	14.14214	11.88584	18.47358	3.252691	4.993217
0.24	14.14214	11.78774	18.41061	3.394113	5.111585
0.25	14.14214	11.68964	18.34796	3.535534	5.228971
0.26	14.14214	11.59154	18.28561	3.676955	5.345377
0.27	14.14214	11.49344	18.22359	3.818377	5.460802
0.28	14.14214	11.39534	18.16187	3.959798	5.575246
0.29	14.14214	11.29724	18.10048	4.101219	5.688709
0.3	14.14214	11.19914	18.03942	4.242641	5.801191
0.31	14.14214	11.10104	17.97868	4.384062	5.912692
0.32	14.14214	11.00294	17.91828	4.525483	6.023211

Projectile motion model

No air resistance model in Excel: XUVAT equations in both x and y directions!

Model which incorporates air resistance



Cross sectional area

$$k = \frac{\frac{1}{2} c_D \rho A}{m}$$

Drag coefficient

Mass

Air density

Newton II

$$x: \qquad ma_x = -\frac{v_x}{v} mkv^2$$

$$y: \qquad ma_y = -mg - \frac{v_y}{v} mkv^2$$

Air resistance always *opposes* the direction of velocity

Model which incorporates air resistance

$$a_x = -\frac{v_x}{v} k v^2$$

$$a_y = -g - \frac{v_y}{v} k v^2$$

x and y accelerations

$$k = \frac{\frac{1}{2} c_D \rho A}{m}$$

Cross sectional area

Drag coefficient

Mass

Air density

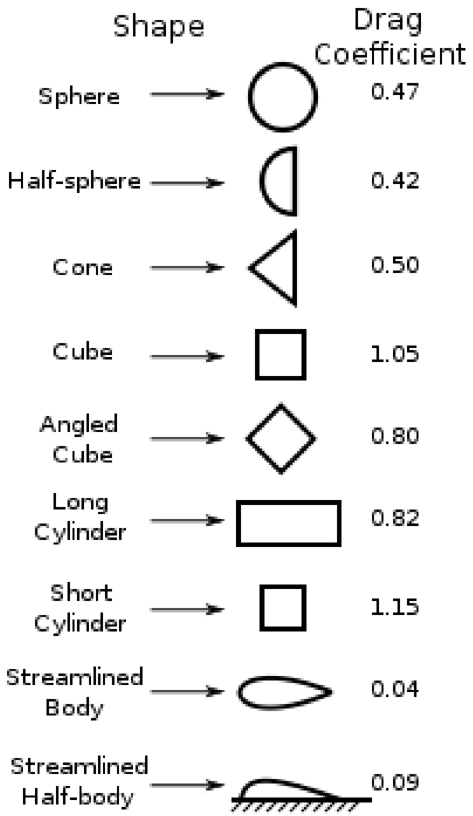
$$\frac{\Delta v_x}{\Delta t} = a_x, \quad \frac{\Delta v_y}{\Delta t} = a_y$$

x and y accelerations

$$\frac{\Delta x}{\Delta t} = v_x, \quad \frac{\Delta y}{\Delta t} = v_y$$

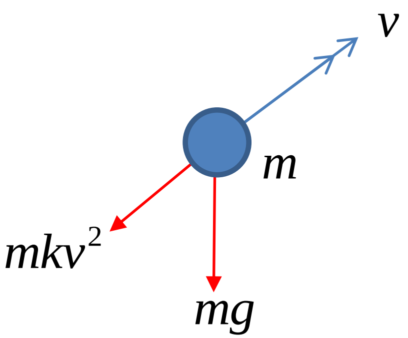
x and y velocities

For *no* air resistance: $a_x = 0 \quad a_y = -g$



Measured Drag Coefficients

Model which incorporates air resistance



$$k = \frac{\frac{1}{2}c_D\rho A}{m}$$

Air resistance factor

$$t = 0$$

$$u_x = u\cos\theta$$

$$u_y = u\cos\theta$$

$$x = 0$$

$$y = h$$

Initial conditions

$$t_{n+1} = t_n + \Delta t$$

Finite time step (e.g. 0.01s)

$$a_x = -\frac{v_x}{v}kv^2$$

x Acceleration

$$a_y = -g - \frac{v_y}{v}kv^2$$

y Acceleration

$$x_{n+1} = x_n + v_x\Delta t + \tfrac{1}{2}a_x\Delta t^2$$

$$y_{n+1} = y_n + v_y\Delta t + \tfrac{1}{2}a_y\Delta t^2$$

Constant acceleration motion between the time steps (the “Verlet” method)

i.e. how x,y, v_x, v_y change between time steps

$$v_x^{(n+1)} = v_x^{(n)} + a_x\Delta t$$

$$v_y^{(n+1)} = v_y^{(n)} + a_y\Delta t$$

$$v = \sqrt{v_x^2 + v_y^2}$$

$$a_y = -g - \frac{v_y}{v} k v^2$$

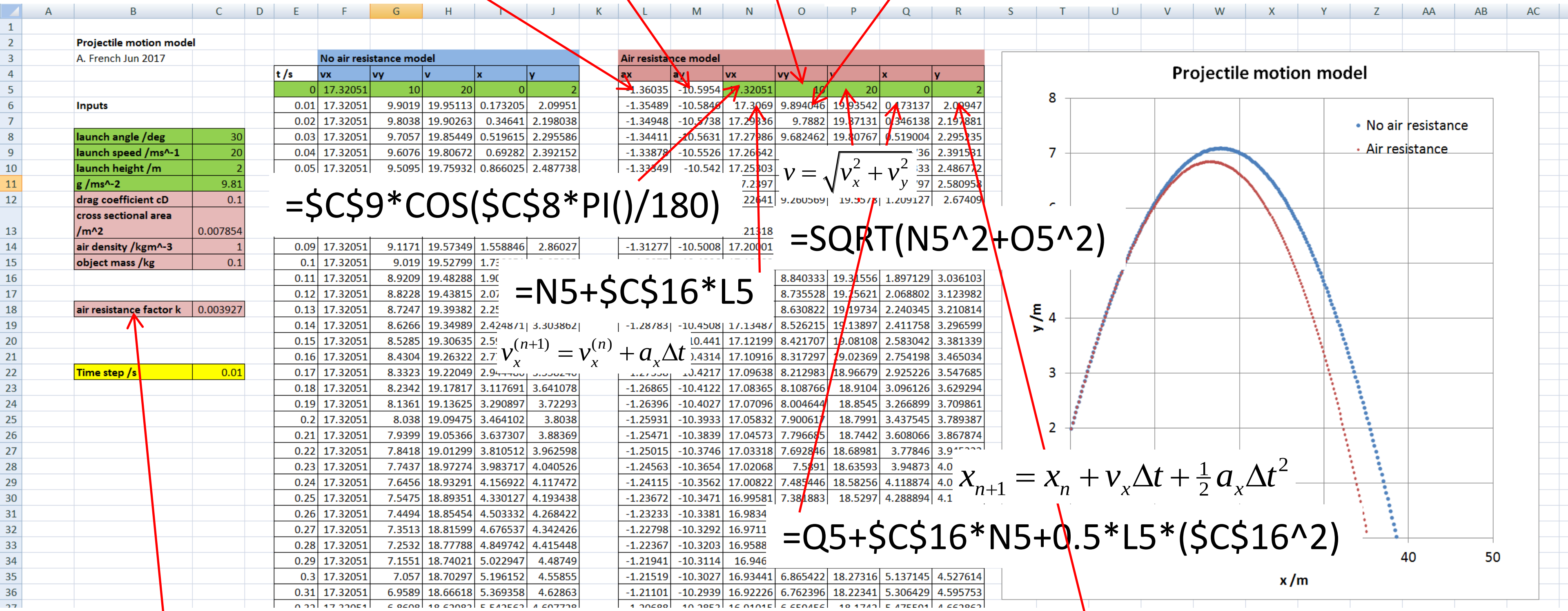
=-$C$11 -(O5/P5)*$C$18*(P5^2)

$$a_x = -\frac{v_x}{v} k v^2$$

=$C$9*SIN($C$8*PI()/180)

=-(N5/P5)*$C$18*(P5^2)

=O5+$C$16*M5 $v_y^{(n+1)} = v_y^{(n)} + a_y \Delta t$



Projectile motion model

A. French Jun 2017

Inputs

launch angle /deg	30
launch speed /ms^-1	20
launch height /m	2
g /ms^-2	9.81
drag coefficient cD	0.1
cross sectional area /m^2	0.007854
air density /kgm^-3	1
object mass /kg	0.1
air resistance factor k	0.003927
Time step /s	0.01

No air resistance model

t /s	vx	vy	v	x	y
0	17.32051	10	20	0	2
0.01	17.32051	9.9019	19.95113	0.173205	2.09951
0.02	17.32051	9.8038	19.90263	0.34641	2.198038
0.03	17.32051	9.7057	19.85449	0.519615	2.295586
0.04	17.32051	9.6076	19.80672	0.69282	2.392152
0.05	17.32051	9.5095	19.75932	0.866025	2.487738
0.09	17.32051	9.1171	19.57349	1.558846	2.86027
0.1	17.32051	9.019	19.52799	1.73	
0.11	17.32051	8.9209	19.48288	1.90	
0.12	17.32051	8.8228	19.43815	2.07	
0.13	17.32051	8.7247	19.39382	2.25	
0.14	17.32051	8.6266	19.34989	2.424871	3.303862
0.15	17.32051	8.5285	19.30635	2.5	
0.16	17.32051	8.4304	19.26322	2.7	
0.17	17.32051	8.3323	19.22049	2.9	
0.18	17.32051	8.2342	19.17817	3.117691	3.641078
0.19	17.32051	8.1361	19.13625	3.290897	3.72293
0.2	17.32051	8.038	19.09475	3.464102	3.8038
0.21	17.32051	7.9399	19.05366	3.637307	3.88369
0.22	17.32051	7.8418	19.01299	3.810512	3.962598
0.23	17.32051	7.7437	18.97274	3.983717	4.040526
0.24	17.32051	7.6456	18.93291	4.156922	4.117472
0.25	17.32051	7.5475	18.89351	4.330127	4.193438
0.26	17.32051	7.4494	18.85454	4.503332	4.268422
0.27	17.32051	7.3513	18.81599	4.676537	4.342426
0.28	17.32051	7.2532	18.77788	4.849742	4.415448
0.29	17.32051	7.1551	18.74021	5.022947	4.48749
0.3	17.32051	7.057	18.70297	5.196152	4.55855
0.31	17.32051	6.9589	18.66618	5.369358	4.62863

Air resistance model

ax	ay	vx	vy	v	x	y
-1.36035	-10.5954	17.32051	10	20	0	2
-1.35489	-10.5846	17.3069	9.894046	19.93542	0.173137	2.09947
-1.34948	-10.5738	17.29336	9.7882	19.87131	0.346138	2.197881
-1.34411	-10.5631	17.27986	9.682462	19.80767	0.519004	2.295235
-1.33878	-10.5526	17.26642			36	2.391531
-1.33349	-10.542	17.25303			33	2.486772
		7.2397			97	2.580958
		22641	9.260569	19.5578	1.209127	2.67409
		21318				
-1.31277	-10.5008	17.20001				
			8.840333	19.31556	1.897129	3.036103
			8.735528	19.25621	2.068802	3.123982
			8.630822	19.19734	2.240345	3.210814
-1.28783	-10.4508	17.13487	8.526215	19.13897	2.411758	3.296599
	10.441	17.12199	8.421707	19.08108	2.583042	3.381339
	.4314	17.10916	8.317297	19.02369	2.754198	3.465034
	10.4217	17.09638	8.212983	18.96679	2.925226	3.547685
-1.26865	-10.4122	17.08365	8.108766	18.9104	3.096126	3.629294
-1.26396	-10.4027	17.07096	8.004644	18.8545	3.266899	3.709861
-1.25931	-10.3933	17.05832	7.900617	18.7991	3.437545	3.789387
-1.25471	-10.3839	17.04573	7.796685	18.7442	3.608066	3.867874
-1.25015	-10.3746	17.03318	7.692846	18.68981	3.77846	3.9
-1.24563	-10.3654	17.02068	7.5891	18.63593	3.94873	4.0
-1.24115	-10.3562	17.00822	7.485446	18.58256	4.118874	4.0
-1.23672	-10.3471	16.99581	7.381883	18.5297	4.288894	4.1
-1.23233	-10.3381	16.9834				
-1.22798	-10.3292	16.9711				
-1.22367	-10.3203	16.9588				
-1.21941	-10.3114	16.946				
-1.21519	-10.3027	16.93441	6.865422	18.27316	5.137145	4.527614
-1.21101	-10.2939	16.92226	6.762396	18.22341	5.306429	4.595753

=$C$9*COS($C$8*PI()/180)

$v = \sqrt{v_x^2 + v_y^2}$

=SQRT(N5^2+O5^2)

=N5+$C$16*L5

$v_x^{(n+1)} = v_x^{(n)} + a_x \Delta t$

$x_{n+1} = x_n + v_x \Delta t + \frac{1}{2} a_x \Delta t^2$

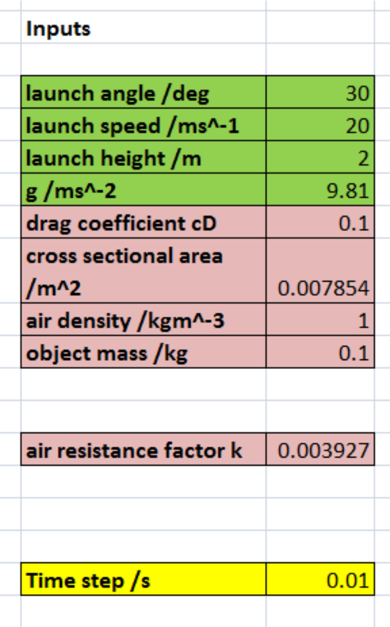
=Q5+$C$16*N5+0.5*L5*($C$16^2)

=R5+$C$16*O5+0.5*M5*($C$16^2)

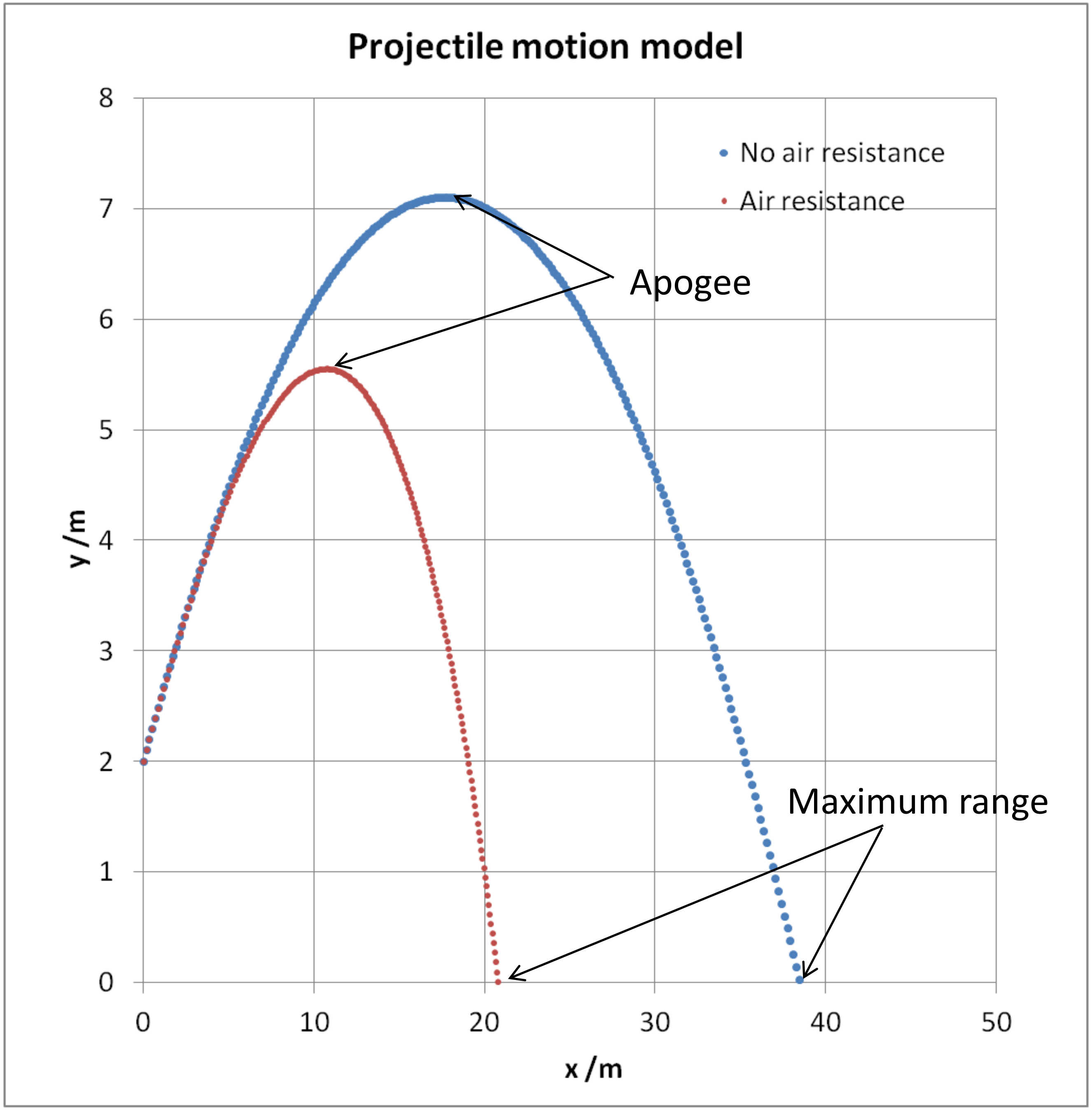
$y_{n+1} = y_n + v_y \Delta t + \frac{1}{2} a_y \Delta t^2$

$$k = \frac{\frac{1}{2} c_D \rho A}{m}$$ =0.5*c12*c13*c14/c15

Air resistance model in Excel – constant acceleration motion between each time step



Inputs	
launch angle /deg	30
launch speed /ms^-1	20
launch height /m	2
g /ms^-2	9.81
drag coefficient cD	0.1
cross sectional area /m^2	0.007854
air density /kgm^-3	1
object mass /kg	0.1
air resistance factor k	0.003927
Time step /s	0.01



Investigate the effect of air resistance using the model!

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launch height /m	2
g /ms^-2	9.81
drag coefficient cD	0.1
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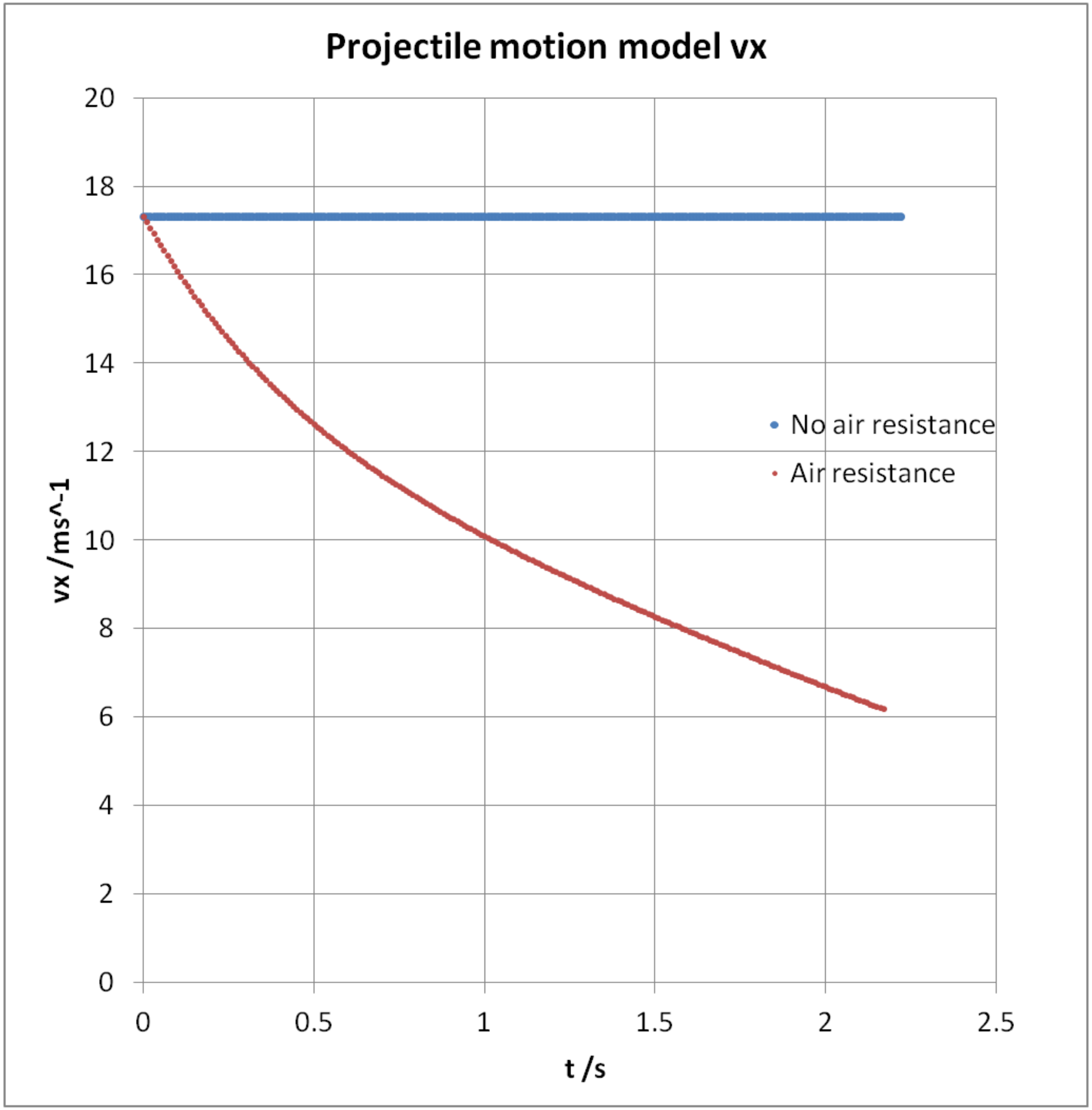
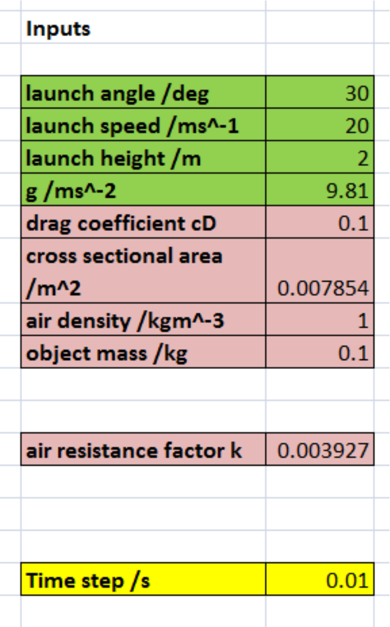
air density /kgm^-3	1
object mass /kg	0.1
air resistance factor k	0.003927
Time step /s	0.01





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g /ms^-2	9.81
drag coefficient cD	0.1
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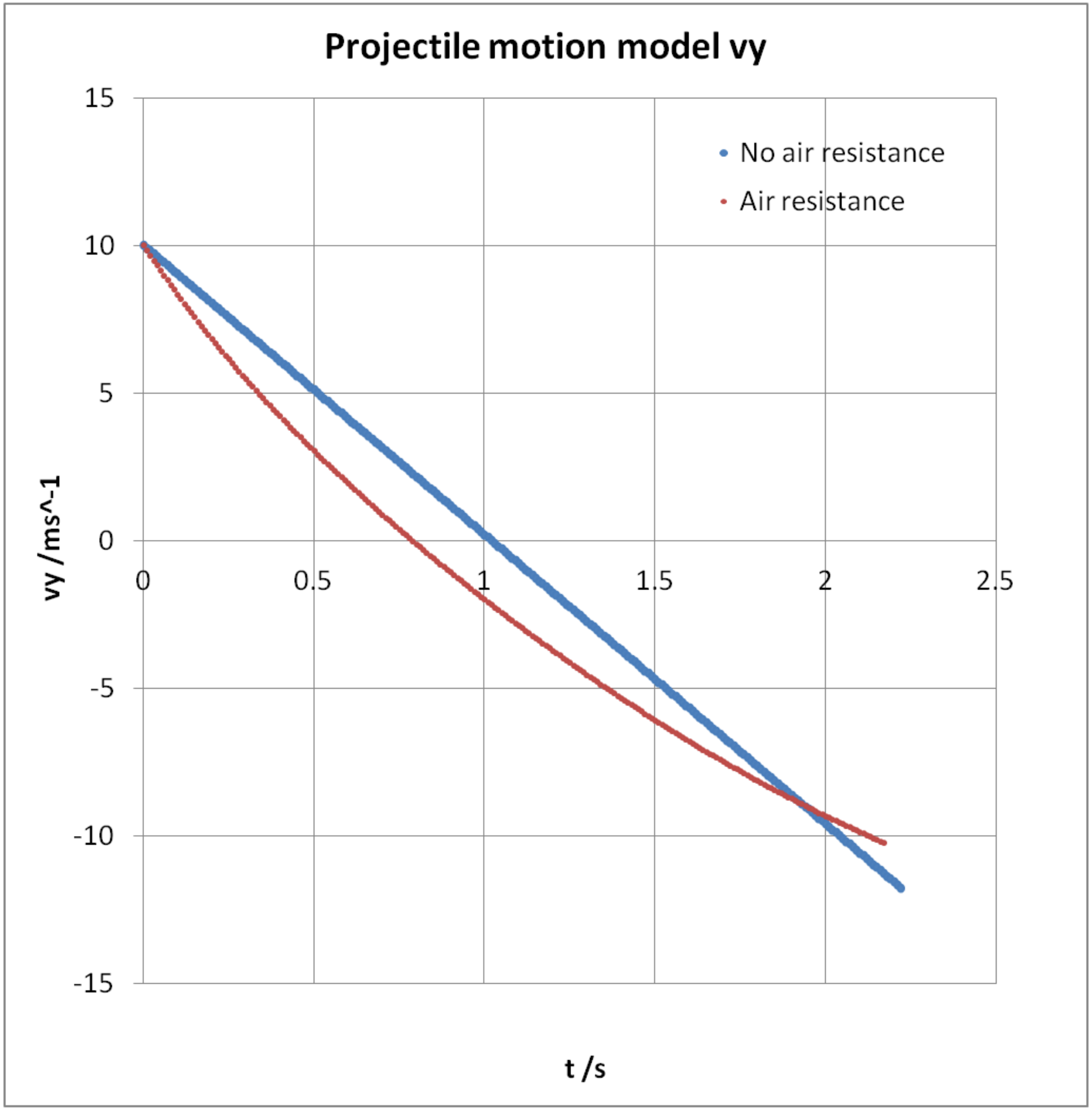
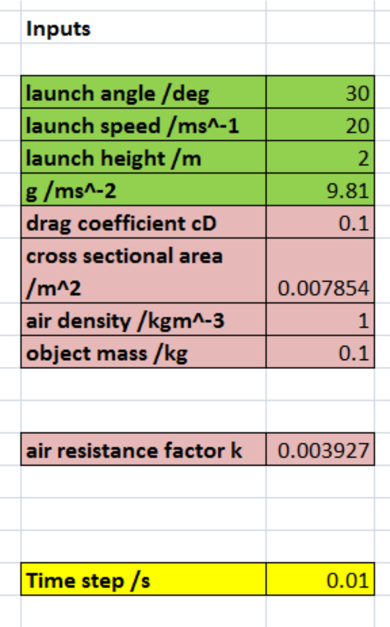
object mass /kg	0.1
air resistance factor k	0.003927
Time step /s	0.01





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launch height /m	2
g /ms^-2	9.81
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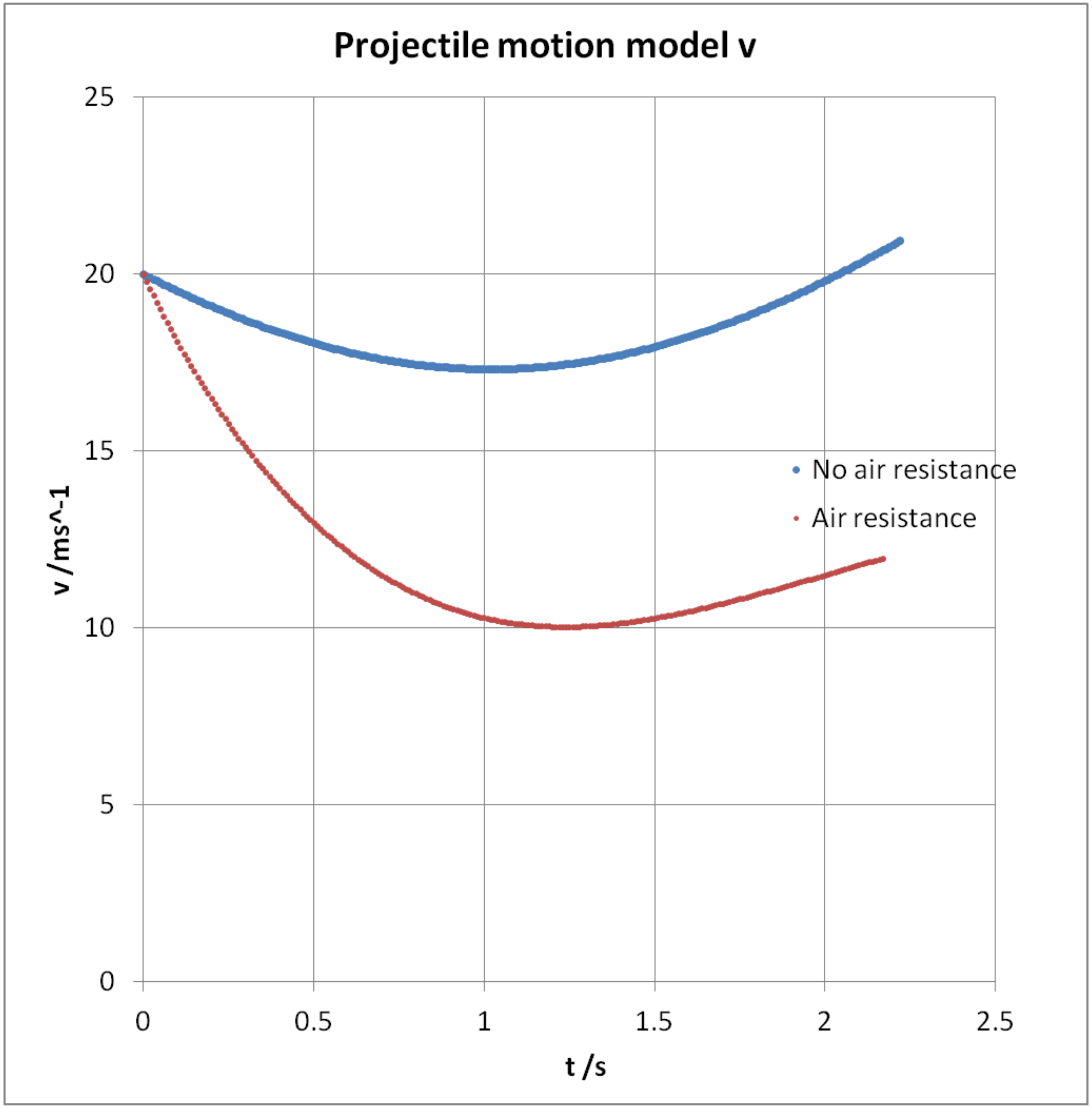

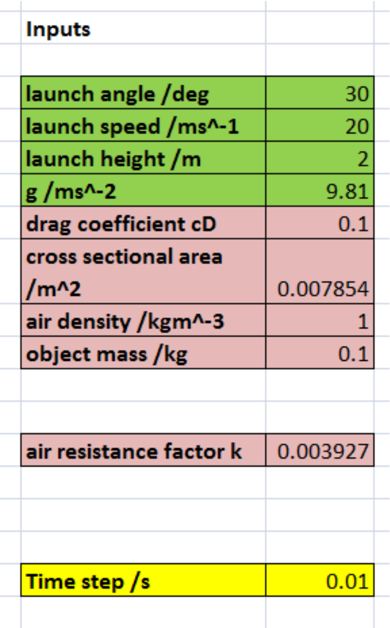

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launch height /m	2
g /ms^-2	9.81
drag coefficient cD	0.1
cross sectional area /m^2	0.007854
air density /kgm^-3	1
object mass /kg	0.1
air resistance factor k	0.003927
Time step /s	0.01