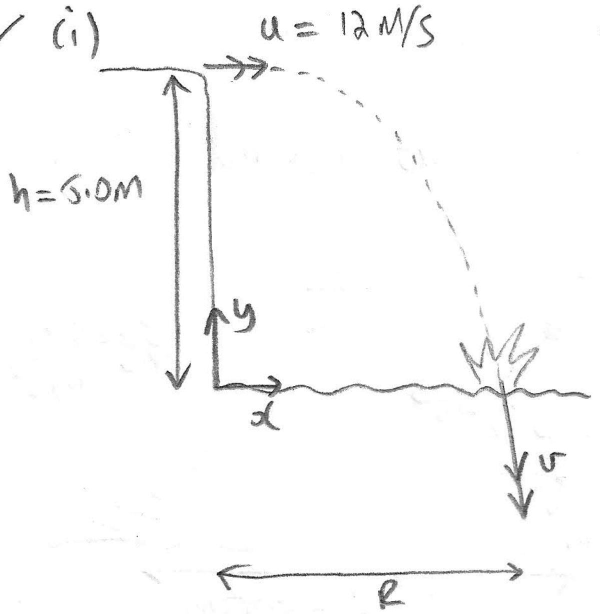
Projectile Motion

1/ (i)



$g = 9.81\ \text{m/s}^2$

Projectile motion with no air resistance

$$\boxed{\begin{aligned} x &= ut \\ y &= -\tfrac{1}{2}gt^2 + h \\ v_x &= u \qquad v_y = -gt \end{aligned}}$$

a) $y = 0$ when $h = \frac{1}{2}gt^2$

$\therefore$ Time of flight $t = \sqrt{\dfrac{2h}{g}}$

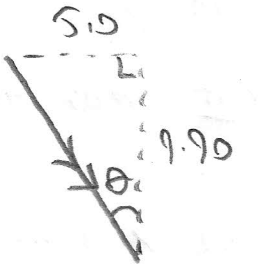
$$t = \sqrt{\frac{2 \times 5.0}{9.81}} = \boxed{1.01\ \text{s}}$$

b) $x = R$ when $t = 1.01\text{s}$

so $R = 5.0 \times \sqrt{\dfrac{2 \times 5.0}{9.81}} = \boxed{5.05\ \text{m}}$

d) $v_x = 5.0$ m/s $\quad v_y = -9.81 \times \sqrt{\dfrac{2 \times 5.0}{9.81}} = \boxed{-9.90\ \text{m/s}}$

(At impact with sea after 1.01s)

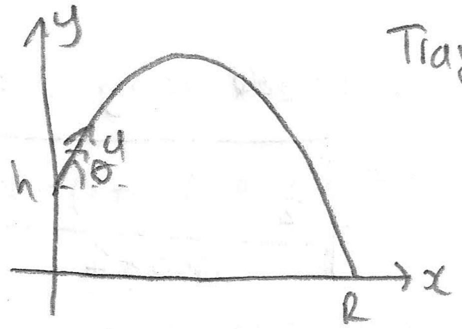


$\theta = \tan^{-1}\left(\dfrac{5.0}{9.90}\right) = \boxed{26.8^\circ}$ (from vertical)

c) The impact speed is

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{5.0^2 + 9.90^2} = \boxed{11.1\ \text{m/s}}$$

(ii) a)



Trajectory equation: $\boxed{y = h + x\tan\theta - \dfrac{g}{2u^2}(1+\tan^2\theta)x^2}$

when $y = 0$, $x = R$

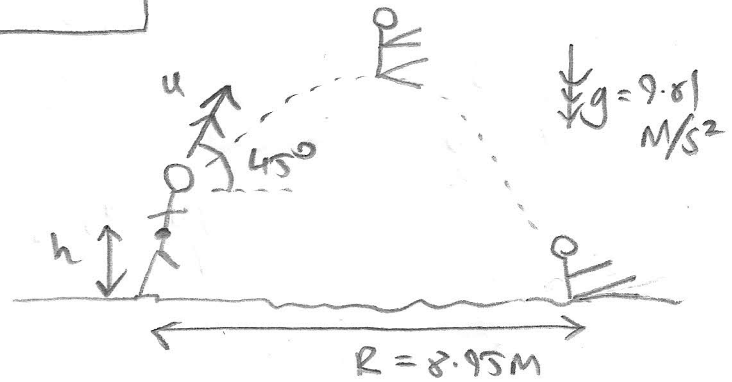
$$\therefore \frac{g}{2u^2}(1+\tan^2\theta)R^2 = h + R\tan\theta$$

$$\therefore \quad u^2 = \frac{g}{2} \frac{(1+\tan^2\theta) R^2}{h + R\tan\theta}$$

$$\therefore \quad \boxed{u = \sqrt{\frac{1}{2} \frac{g(1+\tan^2\theta)R^2}{h + R\tan\theta}}}$$ as required.

(b) World Long Jump record:

$h = 0.94M$
(Mike Powell's CoM)



Using the formula for u above: $u = \sqrt{\dfrac{\frac{1}{2}\, 9.81\,(1+1)\, 8.95^2}{0.94 + 8.95 \times 1}}$

$\tan 45° = 1$

$= \boxed{8.91\ M/s}$

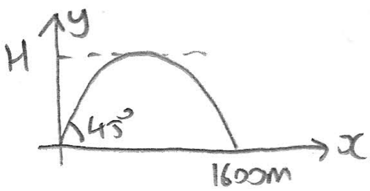
At this speed he would run 100M in

$t = \dfrac{100M}{8.91M/s} = \boxed{11.2s}$ ie not that fast.

Usain Bolt's top speed $\approx 12M/s$

So could the long jump record be > 9.0m?
[Air resistance likely to be significant here, as is difficulty of turning → speed into ↗ velocity]

(ii)



"Basilica"
(1453)

a) Muzzle velocity is:

$u = \sqrt{\dfrac{\frac{1}{2}\, 9.81(1+1) \times 1600^2}{1600 \times 1}}$

$\boxed{u = 125\ M/s}$

[So initial KE of 272 kg ball is $\frac{1}{2}Mu^2 = \boxed{2.13 \times 10^6 \text{ J}}$]

b) At apogee : $\boxed{\begin{aligned} x_a &= \frac{1}{2g} u^2 \sin 2\theta \\ y_a &= \frac{1}{2g} u^2 \sin^2\theta \end{aligned}}$

iff $v_y = 0$

$$x_a = \frac{1}{2 \times 9.81} \times 125^2 \sin 90^\circ = \underset{(m)}{800} \checkmark \quad \left(\text{iff } \frac{1600}{2}\right)$$

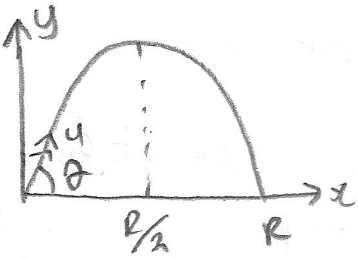
$$y_a = H = \frac{1}{2 \times 9.81} \times 125^2 \sin^2 45 = \boxed{400\text{M}}$$

[Note in the recent Netflix ottoman docudrama, the Basilica appears to be mounted horizontally, or on a ridge.

innacurate or deliberate?]

cannon. Constantinople

(iv) a) $\uparrow g$



$x = ut\cos\theta \quad \therefore t = \frac{x}{u\cos\theta}$

$y = ut\sin\theta - \frac{1}{2}gt^2$

$y = 0,\ x = R$

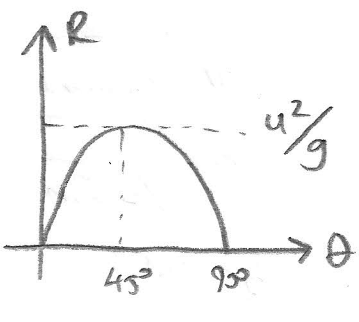
so $\quad 0 = t\left(u\sin\theta - gt/2\right)$

$$0 = \left(\frac{R}{u\cos\theta}\right)\left(u\sin\theta - \frac{g}{2}\frac{R}{u\cos\theta}\right)$$

Since $R \neq 0 \Rightarrow \quad R = \frac{2u^2\cos\theta\sin\theta}{g}$.

and $\sin(90^\circ) = 1$

$$\therefore \boxed{R = \frac{u^2}{g}\sin 2\theta}$$



By symmetry, $R = R_{max} = u^2/g$ when $\theta = 45^\circ$.

b) So maximum range of the Paris Gun is $R_{max} = \frac{u^2}{g}$

$= \frac{1640^2}{9.81} = 2.74 \times 10^5 \text{ m}$ or $\boxed{274 \text{ km}}$ (!)

Max height is $\boxed{y_a = \frac{1}{2g} u^2 \sin^2\theta}$

So if $\theta = 45°$ to maximize range

$$y_a = \frac{1}{2 \times 9.81} \times 1640^2 \times \sin^2 45°$$

$$= \boxed{68.5 \text{ km}}$$

$\frac{1}{2}$ since $\sin 45° = \frac{1}{\sqrt{2}}$

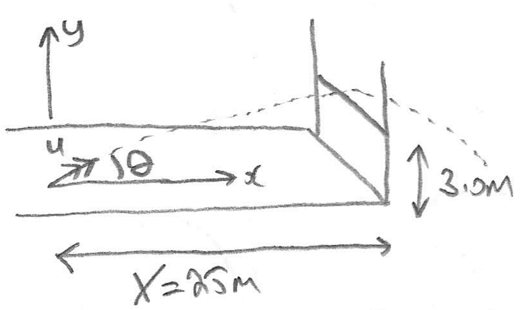
Note air resistance will have a significant effect at this speed, but nonetheless, the 106 kg shells would descend from the $\boxed{\text{STRATOSPHERE}}$ (ie altitude > 10 km)

The gunners would also have to consider the curvature of the Earth, reduction in g and air density ρ with height for an accurate calculation of the target position.

c) Time of flight is found from $R = ut\cos\theta$

$$\therefore \quad t = \frac{274 \times 10^3}{1640 \times \cos 45°} = \boxed{236 \text{ s}}$$ ie $\approx$ 3.9 minutes (!)

v)



$\theta = 30°$

Trajectory equation (note $h=0$)

$$y = x\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)x^2$$

when $x = X$, $y > Y$ $\quad X = 25\text{m}$, $Y = 3.0\text{m}$

So $X\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)X^2 > Y$

$$\therefore \quad x\tan\theta - Y > \frac{g}{2u^2}(1+\tan^2\theta)x^2$$

$$\therefore \quad u^2 > \frac{g(1+\tan^2\theta)x^2}{2(x\tan\theta - Y)}$$

$$\therefore \quad \boxed{u > \sqrt{\frac{g(1+\tan^2\theta)x^2}{2(x\tan\theta - Y)}}}$$

Same idea as in (i), but not $(R,0)$.

So $$u > \sqrt{\frac{9.81(1+\tan^2 30^\circ)25^2}{2(25\tan 30^\circ - 3.0)}}$$ $\left[\tan 30^\circ = \frac{1}{\sqrt{3}}\right]$

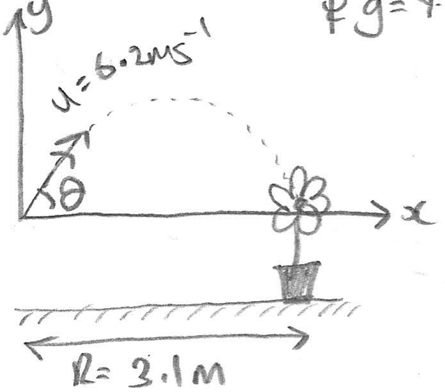
$$\boxed{u > 18.9 \text{ m/s}}$$

{probably much faster as air resistance will be significant}

$g = 9.81 \text{ m/s}^2$

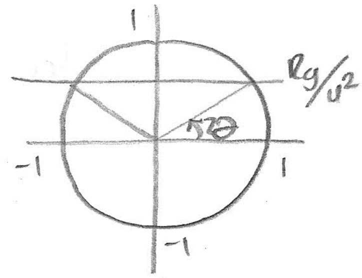
(vi)

a)



$$R = \frac{u^2}{g}\sin 2\theta$$

So $$\sin 2\theta = \frac{Rg}{u^2}$$



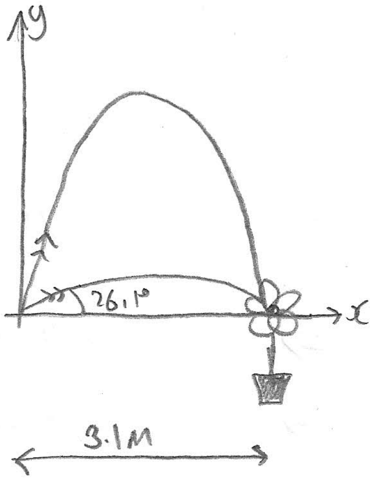
So $$\boxed{\theta_1 = \frac{1}{2}\sin^{-1}(Rg/u^2)}$$

or $$2\theta = 180^\circ - \sin^{-1}(Rg/u^2)$$

$$\Rightarrow \boxed{\theta_2 = 90^\circ - \frac{1}{2}\sin^{-1}(Rg/u^2)}$$

$$\therefore \quad \theta_1 = \frac{1}{2}\sin^{-1}\left(\frac{3.1 \times 9.81}{6.2^2}\right) = \boxed{26.1^\circ}$$

$$\theta_2 = \boxed{63.9^\circ}$$



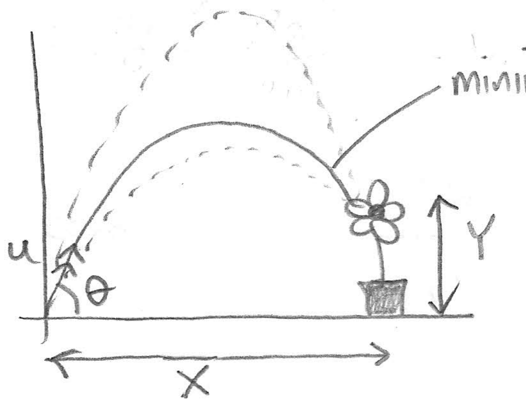
Now $\boxed{R = \frac{u^2}{g}\sin 2\theta}$ (since $h=0$)

Since R is fixed, we want the smallest u which satisfies this equation.

Now the maximum value of $\sin 2\theta$ is 1 when $\boxed{\theta = 45^\circ}$

So minimum value of u is $\sqrt{Rg}$ ie $\sqrt{3.1\times 9.81} = \boxed{5.5\,m/s}$

c) Now consider this situation



$g = 9.81\, m/s^2$

Water trajectory must intersect (X, Y).

Trajectory equation $Y = X\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)X^2$

So $\tan^2\theta\left(\frac{gX^2}{2u^2}\right) - \tan\theta(X) + Y + \frac{gX^2}{2u^2} = 0$

$$\tan^2\theta - \tan\theta\,\frac{2u^2}{gX} + \frac{2u^2Y}{gX^2} + 1 = 0$$

$$\left(\tan\theta - \frac{u^2}{gX}\right)^2 - \frac{u^4}{g^2X^2} + 1 + \frac{2u^2Y}{gX^2} = 0$$

$$\therefore \boxed{\tan\theta = \frac{u^2}{gX} \pm \sqrt{\frac{u^4}{g^2X^2} - 1 - \frac{2u^2Y}{gX^2}}}$$

Minimum velocity is when $\frac{u^4}{g^2X^2} - 1 - \frac{2u^2Y}{gX^2} = 0$

$\Rightarrow u^4 - 2u^2Yg - g^2X^2 = 0$

$\Rightarrow (u^2 - Yg)^2 - Y^2g^2 - g^2X^2 = 0$

$u^2 = Yg \pm \sqrt{g^2(X^2+Y^2)}$

$$\therefore \boxed{u = \sqrt{g}\left(Y + \sqrt{x^2+Y^2}\right)^{\frac{1}{2}}}$$ (1st answer to Q4)

At this point $\boxed{\tan\theta = \dfrac{u^2}{gX}}$

so if $u = 6.2\,m/s$, $X = 3.1\,m$, $Y = 0.5\,m$

$$\tan\theta = \frac{6.2^2}{9.81\times3.1} \pm \sqrt{\frac{6.2^4}{9.81^2\times3.1^2} - 1 - \frac{2\times6.2^2\times0.5}{9.81\times3.1^2}}$$

$$\tan\theta = 1.264 \pm 0.436$$

$$\therefore \theta_1 = \tan^{-1}(1.264 - 0.436) = \boxed{39.6°}$$

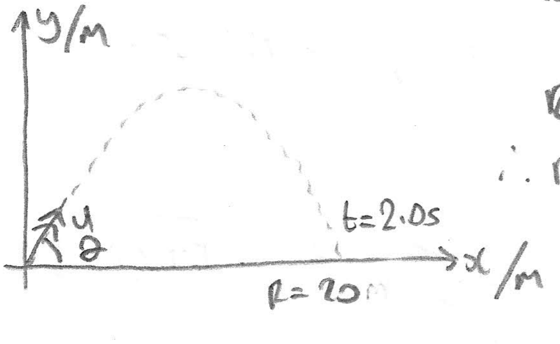
$$\theta_2 = \tan^{-1}(1.264 + 0.436) = \boxed{59.5°}$$

Minimum u is $\sqrt{9.81}\left(0.5 + \sqrt{3.1^2+0.5^2}\right)^{\frac{1}{2}}$

$$= \boxed{5.98\,m/s}$$

and $\theta = \tan^{-1}\left(\dfrac{5.98^2}{9.81\times3.1}\right) = \boxed{49.6°}$

(vi)



$$R = \frac{u^2}{g}\sin2\theta = \frac{2u^2}{g}\sin\theta\cos\theta \quad (1)$$

$$R = ut\cos\theta$$

$$\therefore R^2 = u^2t^2\cos^2\theta \quad (2)$$

$\frac{(2)}{(1)}$: $R = \dfrac{u^2t^2\cos^2\theta}{\frac{2u^2}{g}\sin\theta\cos\theta}$

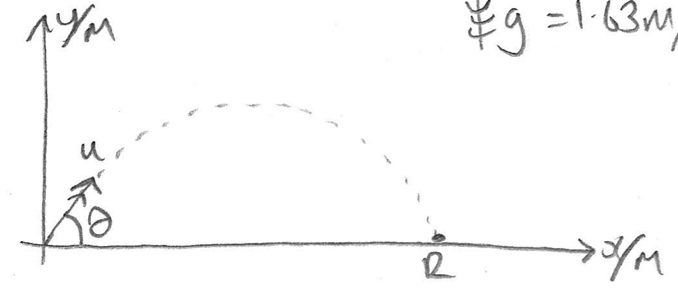
$$\therefore R = \frac{g}{2}t^2\frac{\cos\theta}{\sin\theta} \Rightarrow \tan\theta = \frac{gt^2}{2R} \quad \therefore \boxed{\theta = \tan^{-1}\left(\frac{gt^2}{2R}\right)}$$

$$\therefore \quad \theta = \tan^{-1}\left(\frac{9.81 \times 2.0^2}{2 \times 20}\right) = \boxed{44.5^\circ}$$

$R = ut\cos\theta \quad \therefore \quad u = \frac{R}{t\cos\theta}$

$$u = \frac{20}{2.0 \times \cos\theta} = \boxed{14.0\,m/s}$$

2/



$g = 1.63\,m/s^2$

Alan Shepard's golf shot on the moon

(a,b)

Using the same analysis as in (vi)

$u = R/t\cos\theta \qquad \theta = \tan^{-1}\left(\frac{gt^2}{2R}\right) \qquad R = 4023\,m, \quad t = 70\,s$

$$\Rightarrow \theta = \tan^{-1}\left(\frac{1.63 \times 70^2}{2 \times 4023}\right) = \boxed{44.8^\circ}$$

(ie $\simeq 45^\circ$). optimal angle for max R

You could guess 45° for this Question.

$$u = \frac{4023}{70 \times \cos\theta} = \boxed{81\,m/s}$$

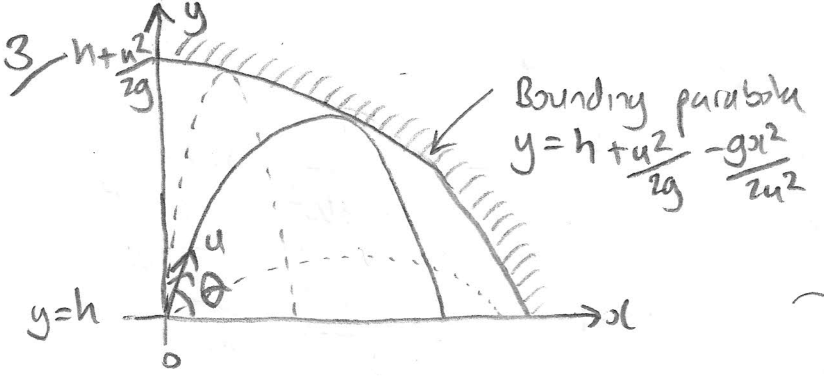
(Note $R_{max} = u^2/g \simeq 81^2/1.63 \simeq 4025\,m$ ie calc makes sense).

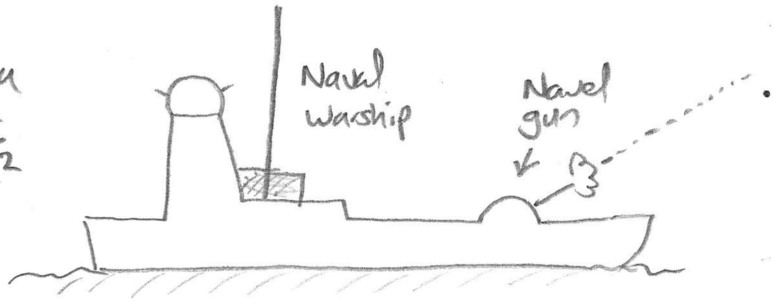
(c) Assuming $\theta = 45^\circ$, $u = 81\,m/s$ but $g = 3.71\,m/s^2$ ie Mars.

Max range is $R = u^2/g = \frac{81^2}{3.71} = \boxed{1770\,m}$

ie just over a mile!

3





Trajectory equation: $\boxed{y = h + x\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)x^2}$

let's assume a specified value of u.

If this is valid for a given (x,y) coordinate, $\tan\theta$ must be real.

$$\tan^2\theta\left(\frac{x^2 g}{2u^2}\right) - \tan\theta\,(x) + y - h + \frac{gx^2}{2u^2} = 0$$

$$\tan^2\theta - \tan\theta \times \frac{2u^2}{gx} + 1 + \frac{2u^2}{gx^2}(y-h) = 0$$

$$\left(\tan\theta - \frac{u^2}{gx}\right)^2 - \frac{u^4}{g^2x^2} + 1 + \frac{2u^2}{gx^2}(y-h) = 0$$

$$\therefore \tan\theta = \frac{u^2}{gx} \pm \sqrt{\frac{u^4}{g^2x^2} - 1 - \frac{2u^2}{gx^2}(y-h)}$$

so for $\tan\theta$ to be real:

$$\frac{u^4}{g^2x^2} - 1 - \frac{2u^2(y-h)}{gx^2} \geq 0$$

$$\frac{u^4}{g} - gx^2 \geq 2u^2(y-h)$$

$$\frac{u^2}{2g} - \frac{gx^2}{2u^2} \geq y - h$$

$$\therefore \boxed{y \leq h + \frac{u^2}{2g} - \frac{gx^2}{2u^2}}$$

This is the bounding parabola.
(well, the upper limit of the inequality)

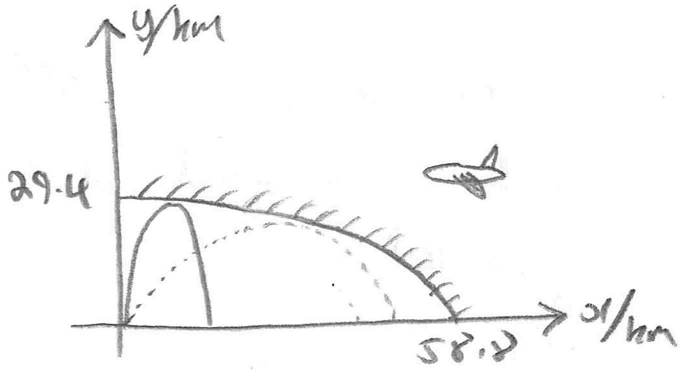
So if $h=0$, $u = 760\,m/s$ in metres

Bounding parabola is: $y = \frac{760^2}{2 \times 9.81} - \frac{9.81}{2 \times 760^2}\left(1000 \times \frac{x}{km}\right)^2$

$$\therefore \boxed{\frac{y}{km} = 29.4 - \frac{8.49}{1000}\left(\frac{x}{km}\right)^2}$$

So if directly overheard, aircraft must be at 29.4 km

when the projectile hits



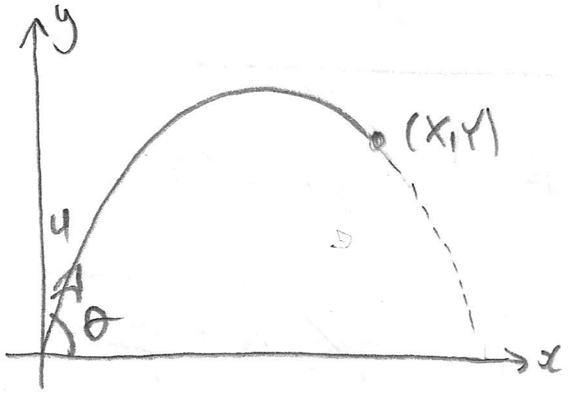
Note problem is somewhat more complicated because the target is moving! (For the gunner)

However, outside the bounding parabola the aircraft cannot be reached, so this is a defined 'safe zone'.

.... problem is, most Naval ships have missiles these days!

4/

a)



Trajectory equation:

$$\boxed{y = x\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)x^2}$$

if $y = Y$, $x = X$

$\therefore$ (see above): $\tan\theta = \frac{u^2}{gX} \pm \sqrt{\frac{u^4}{g^2X^2} - 1 - \frac{2u^2}{gX^2}Y}$

h=0

for $\tan\theta$ to be real, $\sqrt{\cdots}$ must be

(b) s.t $\cdots \geq 0$.

$$\therefore \quad \frac{u^4}{g^2X^2} - 1 - \frac{2u^2Y}{gX^2} \geqslant 0$$

$$u^4 - g^2X^2 - 2u^2gY \geqslant 0$$

$$(u^2 - gY)^2 - g^2Y^2 - g^2X^2 \geqslant 0$$

$$\therefore \quad u^2 \geqslant gY \pm \underbrace{\sqrt{g^2(X^2+Y^2)}}$$

This is $> gY$

$$\therefore \quad \boxed{u \geqslant \sqrt{g}\left(Y + \sqrt{X^2+Y^2}\right)^{1/2}}$$ as required.

b) So when $u = \sqrt{g}\left(Y + \sqrt{X^2+Y^2}\right)^{1/2}$

$$\Rightarrow \quad \boxed{\tan\theta = \frac{u^2}{gX}} \qquad (\text{cu } \sqrt{\ldots} = 0)$$

$$\therefore \quad \tan\theta = \frac{g\left(Y + \sqrt{X^2+Y^2}\right)}{gX} = \frac{Y}{X} + \frac{\sqrt{X^2+Y^2}}{X}$$

$$\therefore \quad \boxed{\theta = \tan^{-1}\left(\frac{Y + \sqrt{X^2+Y^2}}{X}\right)}$$

{ Note this <u>doesn't</u> depend on g! }

Minimum u trajectory is: $\quad y = x\tan\theta - \frac{g}{2u^2}(1+\tan^2\theta)x^2$

$\boxed{\tan\theta = \alpha + \sqrt{1+\alpha^2}}$ if $\alpha = Y/X$.

$$\therefore \quad y = x\left(\alpha + \sqrt{1+\alpha^2}\right) - \frac{g}{2u^2}\left(1 + \alpha^2 + 1 + \alpha^2 + 2\alpha\sqrt{1+\alpha^2}\right)x^2$$

Now $\quad g/u^2 = \frac{1}{X\tan\theta} = \frac{1}{X\left(\alpha + \sqrt{1+\alpha^2}\right)}$

PTO

$$\therefore \quad y = x\left(\alpha + \sqrt{1+\alpha^2}\right) - \frac{\left(1 + \left(\alpha + \sqrt{1+\alpha^2}\right)^2\right)x^2}{2X\left(\alpha + \sqrt{1+\alpha^2}\right)}$$

$$y = x\left(\alpha + \sqrt{1+\alpha^2} - \frac{1 + \left(\alpha + \sqrt{1+\alpha^2}\right)^2}{2\left(\alpha + \sqrt{1+\alpha^2}\right)} \frac{x}{X}\right)$$

$$y = x\left(\alpha + \sqrt{1+\alpha^2}\left(1 - \frac{1 + \left(\alpha + \sqrt{1+\alpha^2}\right)^2}{2\left(\alpha + \sqrt{1+\alpha^2}\right)\sqrt{1+\alpha^2}} \frac{x}{X}\right)\right)$$

$$= \frac{1 + \left(\alpha + \sqrt{1+\alpha^2}\right)^2}{2\left(\alpha\sqrt{1+\alpha^2} + 1 + \alpha^2\right)} = \frac{1 + \alpha^2 + 2\alpha\sqrt{1+\alpha^2} + 1 + \alpha^2}{2\alpha^2 + 2\alpha\sqrt{1+\alpha^2} + 2}$$

$$= \frac{2\alpha^2 + 2\alpha\sqrt{1+\alpha^2} + 2}{2\alpha^2 + 2\alpha\sqrt{1+\alpha^2} + 2} = 1$$

$$\therefore \quad \boxed{y = x\left(\alpha + \sqrt{1+\alpha^2}\left(1 - \frac{x}{X}\right)\right)}$$ as required

c) $X = 144, Y = 17$ $\quad\therefore$

$[\alpha = Y/X]$

$$u \geq \sqrt{g}\left(17 + \sqrt{144^2 + 17^2}\right)^{1/2}$$

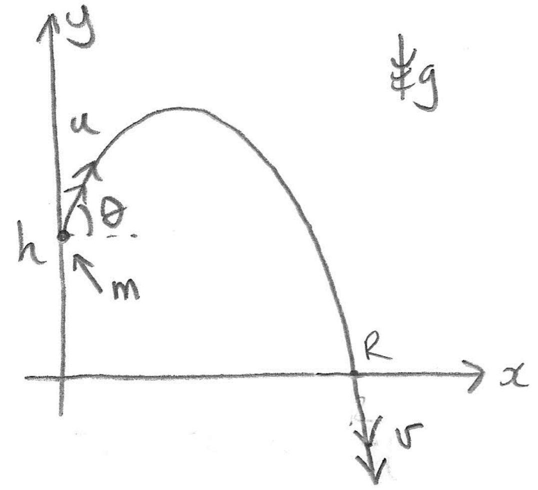
$$u \geq \sqrt{g}\left(17 + 145\right)^{1/2}$$

$$u \geq \sqrt{162g}$$

$$\boxed{u \geq 39.9 \text{ m/s}}$$

$$y = x\left(0.018 + 1.007\left(1 - x/144\right)\right)$$ $\longrightarrow$ See Excel sheet.

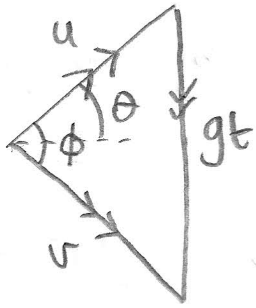
5/



Velocity of particle m is

$$\boxed{\underline{v} = \underline{u} + \underline{g}t}$$

vector triangle



a) Now area of triangle is $A = \frac{1}{2} v \times u \sin\phi$.

Alternatively: $A = \frac{1}{2} gt \times u\cos\theta$ ("Half base × height")

$\therefore$ $\boxed{vu\sin\phi = gut\cos\theta}$

b) Conservation of energy: $mgh + \frac{1}{2}mu^2 = \frac{1}{2}mv^2$

$\therefore$ $\boxed{v^2 = u^2 + 2gh}$ $\Rightarrow$ $\boxed{v = \sqrt{u^2+2gh}}$

Range $\boxed{R = ut\cos\theta}$ So using above result:

$vu\sin\phi = gR$

Since constant speed in x direction

$\therefore$ $R = \dfrac{vu}{g}\sin\phi$

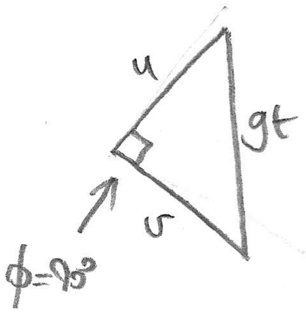
$\therefore$ $R = \dfrac{\sqrt{u^2+2gh}}{g} u\sin\phi$

$\therefore$ $\boxed{R = \dfrac{u^2}{g}\sin\phi\sqrt{1 + \dfrac{2gh}{u^2}}}$

c) Maximum range given fixed u means $\phi = 90^\circ$

ie $\sin\phi = 1$

$$\therefore \boxed{R_{max} = \frac{u^2}{g}\sqrt{1+\frac{2gh}{u^2}}}$$



$\phi = 90^\circ$

So by Pythagoras: $g^2t^2 = u^2 + v^2$

Now $v^2 = u^2 + 2gh$

$$\Rightarrow \quad g^2t^2 = u^2 + u^2 + 2gh$$

$$t^2 = \frac{2(u^2+gh)}{g^2}$$

$$t = \frac{\sqrt{2}\,(u^2+gh)^{1/2}}{g}$$

$$t = \frac{u}{g}\left(2+\frac{2gh}{u^2}\right)^{1/2}$$

let $\eta = \dfrac{mgh}{\frac{1}{2}mu^2} = \dfrac{2gh}{u^2}$ $\quad \therefore \boxed{t = \dfrac{u}{g}\sqrt{2+\eta}}$

Now $R_{max} = ut\cos\theta$

$$R_{max}^2 = u^2t^2\cos^2\theta$$

$$R_{max}^2 = u^2t^2(1-\sin^2\theta)$$

$$\frac{u^4}{g^2}\left(1+\frac{2gh}{u^2}\right) = \frac{u^4}{g^2}\left(2+\frac{2gh}{u^2}\right) \times (1-\sin^2\theta)$$

$$1+\eta = (2+\eta)(1-\sin^2\theta)$$

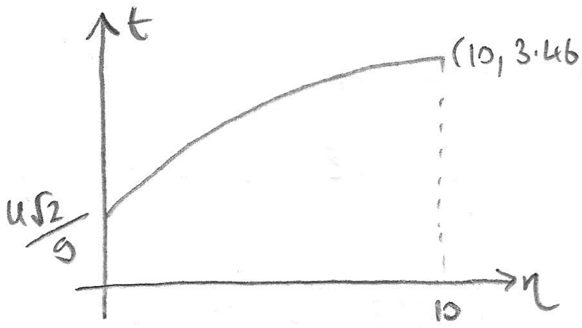
$$\sin^2\theta = 1 - \frac{1+\eta}{2+\eta}$$

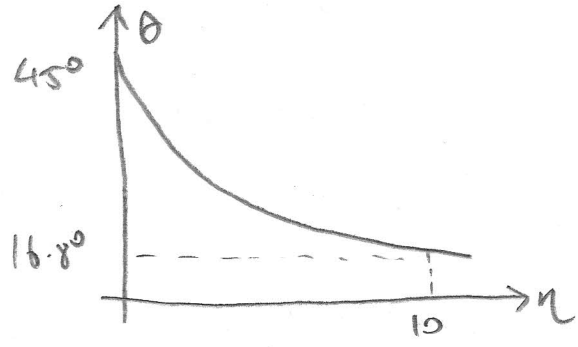
$$\sin^2\theta = \frac{2+\eta-1-\eta}{2+\eta}$$

$$\sin^2\theta = \frac{1}{2+\eta}$$

$$\therefore \boxed{\theta = \sin^{-1}\left(\frac{1}{\sqrt{2+\eta}}\right)}$$

So $t = \frac{u}{g}\sqrt{2+\eta}$ $\qquad \theta = \sin^{-1}\left(\frac{1}{\sqrt{2+\eta}}\right)$





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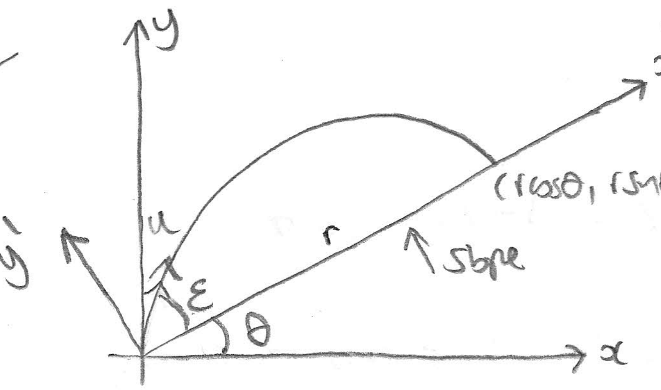
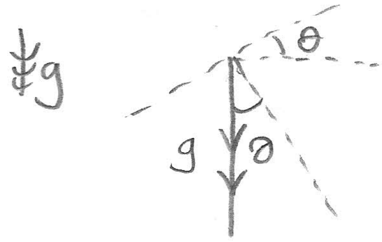




This problem is best solved using a rotated coordinate system, where g has components in both.

Constant acceleration motion:

$$\boxed{\begin{aligned} x' &= ut\cos\varepsilon - \tfrac{1}{2}g\sin\theta\, t^2 \\ y' &= ut\sin\varepsilon - \tfrac{1}{2}g\cos\theta\, t^2 \end{aligned}}$$

when projectile hits slope, $y' = 0$; $x' = r$

$\therefore \quad ut\sin\varepsilon = \frac{1}{2}g\cos\theta\, t^2$

$\therefore \quad \boxed{\frac{2u\sin\varepsilon}{g\cos\theta} = t}$ $\qquad$ ($t>0$ is the solution we are after)

$\therefore \quad r = u\cos\varepsilon\left(\frac{2u\sin\varepsilon}{g\cos\theta}\right) - \frac{1}{2}g\sin\theta\left(\frac{2u\sin\varepsilon}{g\cos\theta}\right)^2$

$r = \frac{2u^2}{g}\frac{\cos\varepsilon\sin\varepsilon}{\cos\theta} - \frac{2u^2}{g}\frac{\sin\theta}{\cos^2\theta}\sin^2\varepsilon$

$$\therefore \quad \boxed{r = \frac{2u^2}{g}\,\frac{1}{\cos\theta}\left(\cos\varepsilon\sin\varepsilon - \tan\theta\,\sin^2\varepsilon\right)}$$

Consider: $\sin(2\varepsilon+\theta) = \sin 2\varepsilon\cos\theta + \cos 2\varepsilon\sin\theta$

$= 2\sin\varepsilon\cos\varepsilon\cos\theta + (\cos\varepsilon\cos\varepsilon - \sin\varepsilon\sin\varepsilon)\sin\theta$

$$\therefore \quad \frac{\sin(2\varepsilon+\theta)}{\cos\theta} = 2\sin\varepsilon\cos\varepsilon - (\cos^2\varepsilon - \sin^2\varepsilon)\tan\theta$$

$\cos^2\varepsilon = 1 - \sin^2\varepsilon$

$$\therefore \quad \frac{\sin(2\varepsilon+\theta)}{\cos\theta} = 2\sin\varepsilon\cos\varepsilon - (1 - 2\sin^2\varepsilon)\tan\theta$$

$$\therefore \quad \frac{\frac{1}{2}\sin(2\varepsilon+\theta)}{\cos\theta} = \sin\varepsilon\cos\varepsilon + \sin^2\varepsilon\tan\theta - \tan\theta/2$$

if this was -ve, could use to simplify! i.e. Apply this trick if projectile launched down a slope ...?

Prior knowledge: know $2\varepsilon + \theta = \pi/2$ for max range

$$\frac{dr}{d\varepsilon} = \frac{2u^2}{g\cos\theta}\left(\cos\varepsilon\cos\varepsilon - \sin\varepsilon\sin\varepsilon - 2\tan\theta\sin\varepsilon\cos\varepsilon\right)$$

So $\frac{dr}{d\varepsilon} = 0$ when $\cos^2\varepsilon - \sin^2\varepsilon = 2\tan\theta\sin\varepsilon\cos\varepsilon$

$\cos 2\varepsilon = \tan\theta\sin 2\varepsilon$

$$\boxed{1 = \tan\theta\tan 2\varepsilon}$$

$\therefore$ $r = r_{max}$ when $\boxed{\varepsilon = \frac{1}{2}\tan^{-1}\left(\frac{1}{\tan\theta}\right)}$

But this simplifies further....

check: $\theta \to 0$, $\tan^{-1}\infty = 90^\circ$ $\therefore$ $\varepsilon \to 45^\circ$ ✓

Now consider $\boxed{2\varepsilon + \theta = \frac{\pi}{2}}$ ie " ε bisects the angle between the slope and the vertical".

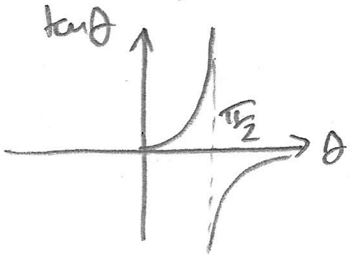
$$\therefore \quad 2\varepsilon = \frac{\pi}{2} - \theta$$

$$\tan 2\varepsilon = \tan\left(\frac{\pi}{2} - \theta\right)$$

$$= \frac{\tan\frac{\pi}{2} - \tan\theta}{1 + \tan\frac{\pi}{2}\tan\theta}$$

$$= \frac{1 - \tan\theta / \tan\pi/2}{\frac{1}{\tan\frac{\pi}{2}} + \tan\theta}$$

$\tan\frac{\pi}{2} \to \infty$



$$\therefore \quad \tan 2\varepsilon = \frac{1}{\tan\theta}$$

$$\therefore \quad \tan 2\varepsilon \tan\theta = 1 \quad \text{if} \quad 2\varepsilon + \theta = \frac{\pi}{2}$$

So $\boxed{\varepsilon = \frac{\pi}{4} - \theta/2}$ for max range r.

$$\cos\varepsilon\sin\varepsilon = \frac{1}{2}\sin 2\varepsilon \qquad \text{so for } \varepsilon = \frac{\pi}{4} - \theta/2$$

$$\cos\varepsilon\sin\varepsilon = \frac{1}{2}\sin\left(\frac{\pi}{2} - \theta\right) = \frac{1}{2}\left(\sin\frac{\pi}{2}\cos\theta - \cos\frac{\pi}{2}\sin\theta\right)$$

$$= \boxed{\frac{1}{2}\cos\theta}$$

$$\sin^2\varepsilon = \left(\sin\left(\frac{\pi}{4} - \frac{\theta}{2}\right)\right)^2$$

$$= \left[\sin\frac{\pi}{4}\cos\frac{\theta}{2} - \cos\frac{\pi}{4}\sin\frac{\theta}{2}\right]^2$$

$\sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$\cos\frac{\pi}{4} = \frac{1}{\sqrt{2}}$

$$= \frac{1}{2}\left(\cos\frac{\theta}{2} - \sin\frac{\theta}{2}\right)^2$$

$$= \frac{1}{2}\left(\cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} - 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\right) = \boxed{\frac{1}{2}(1 - \sin\theta)}$$

$$\therefore\ r_{max} = \frac{2u^2}{g}\frac{1}{\cos\theta}\left(\frac{1}{2}\cos\theta - \frac{\sin\theta}{\cos\theta}\frac{1}{2}(1-\sin\theta)\right)$$

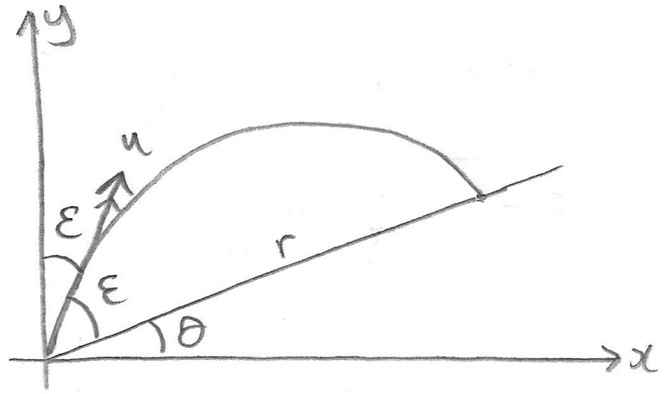
$$r_{max} = \frac{u^2}{g}\left(1 - \frac{\sin\theta}{\cos^2\theta}(1-\sin\theta)\right)$$

$$r_{max} = \frac{u^2}{g}\left(1 - \frac{\sin\theta}{\cos^2\theta} + \frac{\sin^2\theta}{\cos^2\theta}\right)$$

$$r_{max} = \frac{u^2}{g}\left(\frac{\cos^2\theta + \sin^2\theta - \sin\theta}{\cos^2\theta}\right)$$

$$\boxed{r_{max} = \frac{u^2}{g}\left(\frac{1-\sin\theta}{\cos^2\theta}\right)}$$

In Summary:



For maximum range r:

$$\boxed{\varepsilon = \frac{\pi}{4} - \frac{\theta}{2}}\qquad (\text{s.t. } 2\varepsilon + \theta = \tfrac{\pi}{2})$$

and $\boxed{r = \frac{u^2}{g}\left(\frac{1-\sin\theta}{\cos^2\theta}\right)}$ so $\boxed{r = \frac{u^2}{g}\frac{1}{1+\sin\theta}}$

$$\cos^2\theta = 1 - \sin^2\theta = (1+\sin\theta)(1-\sin\theta)$$