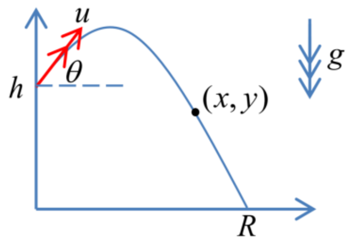
Projectile motion (ignoring air resistance) implies a *constant acceleration* of g *vertically downwards*, and a *constant velocity horizontally*. g is the *strength of gravity* in Nkg^{-1}, or ms^{-2} (since by Newton's Second Law: mass × acceleration = vector sum of force, and the only force in air resistance free projectile motion is *weight*, of magnitude mg acting downwards). Near the Surface of Earth $g = 9.81\mathrm{ms}^{-2}$. On the moon $g = 1.63\mathrm{ms}^{-2}$ and on Mars $g = 3.71\mathrm{ms}^{-2}$.

Let x denote the horizontal displacement from a coordinate origin, and y the vertical displacement. At $t = 0$ a projectile is launched from $(x, y) = (0, h)$ at velocity $v_x = u\cos\theta, v_y = u\sin\theta$. Angle θ is the initial *elevation* from the horizontal and u is the magnitude of the *launch speed.* Since *acceleration is constant*:



$x = ut\cos\theta,\ y = h + ut\sin\theta - \frac{1}{2}gt^2$, and x, y velocity components are:
$v_x = u\cos\theta,\ v_y = u\sin\theta - gt$.

Note the speed at any point in the trajectory is: $v = \sqrt{v_x^2 + v_y^2}$.

Hence: $t = \dfrac{x}{u\cos\theta} \Rightarrow y = h + x\tan\theta - \frac{g}{2u^2}\left(1+\tan^2\theta\right)x^2$, since $\dfrac{1}{\cos^2\theta} = 1+\tan^2\theta$

The trajectory of a projectile is therefore an *inverted parabola*. The *apogee* (highest point) at (x_a, y_a) is when $v_y = 0$.
$\therefore u\sin\theta - gt_a = 0 \Rightarrow t_a = \frac{1}{g}u\sin\theta$ and hence: $x_a = \frac{1}{2g}u^2\sin 2\theta,\ y_a = h + \frac{1}{2g}u^2\sin^2\theta$ [Note: $2\sin\theta\cos\theta = \sin 2\theta$].
The projectile crosses the x axis at $(R, 0)$ where R is the *maximum range* of the projectile.

Question 1

(i) Sophie throws a stone horizontally at a speed of 5.0m/s off the end of a harbour wall, 5.0m above the sea. Calculate (a) the time of flight; (b) the horizontal distance R just before the stone hits the sea; (c) the speed the stone hits the water; (d) The angle from the vertical that the stone hits the water.

(ii) (a) Use the trajectory equation $y = h + x\tan\theta - \frac{g}{2u^2}\left(1+\tan^2\theta\right)x^2$ to show that when $y = 0, x = R$

$$u = \sqrt{\frac{1}{2}\frac{g\left(1+\tan^2\theta\right)R^2}{h + R\tan\theta}}.$$

(b) As of 2020, the World Long Jump record stands at 8.95m (Mike Powell USA, 1991). Mike was 1.88m tall, so assume his centre of mass takes off at $h = 0.94\mathrm{m}$, and hits the sand at $x = 8.95\mathrm{m}, y = 0$. If his initial launch elevation was $\theta = 45^\circ$, calculate his launch speed u in m/s. What time would he run the 100m at this speed?

(iii) Orban the Hungarian constructed a 27 foot long 'super cannon' for Sultan Mehmet II in preparation for the siege of Constantinople in 1453. *Basilica* could fire a 272kg stone ball about 1.6km.

(a) Use the equation in (i) to determine the muzzle velocity u assuming $h = 0,\ \theta = 45^\circ, R = 1{,}600\mathrm{m}$.

(b) Hence determine the maximum height of the stone ball (in m) and the time of flight (in s).

(iv) The *Paris Gun* was one of the largest artillery pieces in World War I. It was used by the Germans to bombard the French capital. The Paris Gun had a muzzle velocity of 1,640m/s.

(a) Show that if a projectile is launched from ground level ($h = 0$), the maximum range is achieved when elevation angle is $\theta = 45^\circ$.

(b) Hence calculate the maximum range R of the Paris Gun, if its 106kg shells were launched from ground level.

(c) Calculate the maximum height (in m), and also the time of flight (in s).

(v) In the 2003 Rugby Union World Cup final, Jonny Wilkinson scored the winning drop goal for England. If he struck the ball from ground level at an angle of thirty degrees, what velocity must he have kicked the ball, given he cleared the goals (height 3m) from a range of 25m? Ignore air resistance.

(vi) A water jet from a garden hose is initially $u = 6.2\text{ms}^{-1}$. Dr French sits in his garden and wishes to water a plant 3.1m away. The plant is at the same vertical level at the hose.

(a) Calculate the two possible elevation angles of the hose.

(b) For the hose to water the plant, what is the *minimum* water velocity? And what elevation angle does this correspond to?

(c) Repeat (a) and (b) if the plant is 0.5m above the hose nozzle. (Hint: this is essentially Q4).

(vii) A golfer performs an outrageous chip shot from ground level, where the ball lands directly into the hole without bouncing. The time of flight is 2.0s and the hole is 20m away. What was (a) the launch velocity of the ball in m/s and (b) what was the launch angle in degrees?

Question 2 The Apollo 14 astronaut Alan Shepard hit a golf ball on the Moon using a secretly attached head of a six-iron club to a piece of rock collecting equipment. The physicist Ethan Siegel suggested that Alan could have easily hit the ball for about 2.5miles (4023m), and the time of flight of the trajectory would be about 70s. Use this information to determine (a) the velocity the ball was struck and (b) the elevation angle. (c) Assuming a similar swing, how far would a gold ball travel on Mars before hitting the Martian surface?

Question 3 A naval gun has a muzzle velocity of u and can be raised or lowered to any angle of elevation θ. Show from the trajectory equation $y = h + x\tan\theta - \frac{g}{2u^2}\left(1+\tan^2\theta\right)x^2$ that it is possible for the gun to fire at a target within a *bounding parabola* $y = h + \frac{1}{2g}u^2 - \frac{g}{2u^2}x^2$. If a naval gun has a muzzle velocity of 760m/s, determine the altitude of an aircraft /km vs range in km such that it will never be hit by any projectiles fired from the gun. Assume $h = 0$.

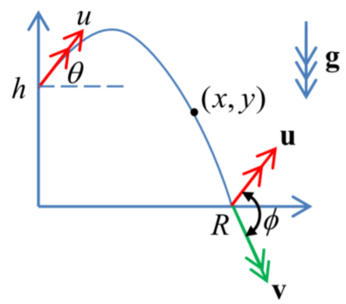
Question 4 A projectile is launched from (0,0) and must pass through coordinates (X, Y).

(a) Show that the launch velocity $u \geq \sqrt{g}\left(Y + \sqrt{X^2+Y^2}\right)^{\frac{1}{2}}$

(b) For the minimum possible u, show that the launch angle is $\theta = \tan^{-1}\left(\dfrac{Y+\sqrt{X^2+Y^2}}{X}\right)$ and the equation of the trajectory is $y = x\left(\alpha + \sqrt{1+\alpha^2}\,(1 - x/X)\right)$ where $\alpha = Y/X$.

(c) For $X = 144, Y = 17$ (m), calculate the minimum possible u (in m/s) and plot the trajectory using Excel.

Question 5 If a projectile of mass m is launched at speed u, at elevation θ and height h, it reaches $(R,0)$ after time t with speed v. Let angle ϕ separate the initial launch velocity vector $\mathbf{u}$ with the velocity $\mathbf{v}$ at $(R,0)$.



(a) Show from triangle area calculations that: $uv\sin\phi = gut\cos\theta$

(b) Hence show from projectile motion and *conservation of energy* that range

$$R = \frac{u^2}{g}\sin\phi\sqrt{1+\frac{2gh}{u^2}}.$$

(c) Hence show that in order to fire a projectile at fixed velocity u to *maximum range*, the elevation angle is: $\theta = \sin^{-1}\left(\dfrac{1}{\sqrt{2+\eta}}\right)$ and the corresponding time of flight is: $t = \dfrac{u}{g}\sqrt{2+\eta}$ where η is the ratio of initial Gravitational Potential Energy (GPE) to Kinetic Energy (KE): $\eta = \dfrac{mgh}{\frac{1}{2}mu^2}$.

(d) Sketch maximum range, elevation angle and time of flight vs α in the range $0 < \alpha < 10$.

Question 6 A projectile is fired from a slope of constant gradient $\tan\theta$. The launch speed is u and the elevation *measured from the slope* is ε. Show the maximum range reached up the slope is $r = \dfrac{u^2}{g}\left(\dfrac{1}{1+\sin\theta}\right)$ when $\varepsilon = \frac{\pi}{4} - \frac{1}{2}\theta$.