$\boxed{\text{QUANTUM MECHANICS 3}}$ (Measurement, wave function, uncertainty principle)

1) $\lambda = \frac{h}{p}$ let $p = m_e v$ (classical physics)

(i) $v = \frac{1}{137} c$ $\therefore \lambda = \frac{h}{m_e c} \times 137$

$$\lambda = \frac{6.63 \times 10^{-34} \times 137}{9.109 \times 10^{-31} \times 2.998 \times 10^{8}} \quad (m)$$

$$= \boxed{3.3 \times 10^{-10} \, m}$$ i.e. atom sized!

(unsurprising, since in Bohr model, $2\pi r = n\lambda$ where r is the 'atomic radius').

If using Special Relativity: $p = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} m_e v$

$$p = \left(1 - \frac{1}{137^2}\right)^{-\frac{1}{2}} m_e \, c/137$$

$$\Rightarrow \boxed{\lambda = 3.3 \times 10^{-10} \, m}$$ (so to 2 s.f. S.R effects make no difference)

(ii) If $\lambda_e = \lambda_n \Rightarrow \frac{h}{p_e} = \frac{h}{p_n} \Rightarrow m_e v_e = m_n v_n$

$$\Rightarrow \boxed{\frac{v_e}{v_n} = \frac{m_n}{m_e}} \quad \therefore \frac{v_e}{v_n} = \frac{1.6749 \times 10^{-27}}{9.109 \times 10^{-31}}$$

$$\frac{v_e}{v_n} \simeq \boxed{1837}$$

Now $E = p^2/2m$ $p = \frac{h}{\lambda}$

So $\boxed{E = \frac{h^2}{2\lambda^2 m}}$ So $\boxed{\frac{E_e}{E_n} = \frac{m_n}{m_e}} = \boxed{1837}$

↑ if $\lambda_e = \lambda_n$

So if $\lambda_e = \lambda_n$, ratio of speeds <u>and</u> KE is the ratio of the masses.

(iii) $\Delta x \Delta p \simeq \frac{\hbar}{2}$ $\quad \Delta E \Delta t \simeq \frac{\hbar}{2}$ $\quad \Delta p \simeq mc$ $\quad \Delta E \simeq mc^2$

$$\therefore \boxed{\Delta x \simeq \frac{\hbar}{2mc}} \quad \text{and} \quad \boxed{\Delta t \simeq \frac{\hbar}{2mc^2}}$$

Z boson: $m = 91.2\,\text{GeV}/c^2 = \dfrac{91.2 \times 10^9 \times 1.602 \times 10^{-19}}{(2.998 \times 10^8)^2}$ kg

$$= \boxed{1.626 \times 10^{-25}\ \text{kg}}$$

$$\therefore \Delta x = \frac{6.626 \times 10^{-34}}{2\pi \times 2 \times 1.626 \times 10^{-25} \times 2.998 \times 10^{+8}} = \boxed{1.08 \times 10^{-18}\ \text{m}}$$

$$\Delta t = \Delta x / c = \boxed{3.61 \times 10^{-27}\ \text{s}}$$

i.e. Sub nuclear length scales $(\ll 10^{-15}\ \text{m})$ and very short times. $\{$ if $c \simeq \frac{\Delta x}{\Delta t}$ then like $c = \lambda/T$ for a very high energy γ ray $\}$

(iv)

$\psi(x,t) = A\sin(kx - \omega t)$

$\dfrac{\partial \psi}{\partial x} = kA\cos(kx - \omega t)$

$\dfrac{\partial^2 \psi}{\partial x^2} = -k^2 A\sin(kx - \omega t)$

$\therefore \dfrac{\partial^2 \psi}{\partial x^2} = -k^2 \psi$

$\dfrac{\partial \psi}{\partial t} = -\omega A\cos(kx - \omega t)$

$\dfrac{\partial^2 \psi}{\partial t^2} = -\omega A(-\sin(kx - \omega t))(-\omega)$

$= -\omega^2 A\sin(kx - \omega t)$

$\therefore \dfrac{\partial^2 \psi}{\partial t^2} = -\omega^2 \psi$

Now $k = \frac{2\pi}{\lambda}$ and $\omega = 2\pi f$ and $c = f\lambda$

so $\omega = \frac{2\pi c}{\lambda} \Rightarrow \boxed{\omega = ck}$ so $\omega^2 = c^2 k^2$

$$\therefore \frac{\partial^2 \psi}{\partial x^2} = -k^2 \psi \qquad \frac{\partial^2 \psi}{\partial t^2} = -c^2 k^2 \psi$$

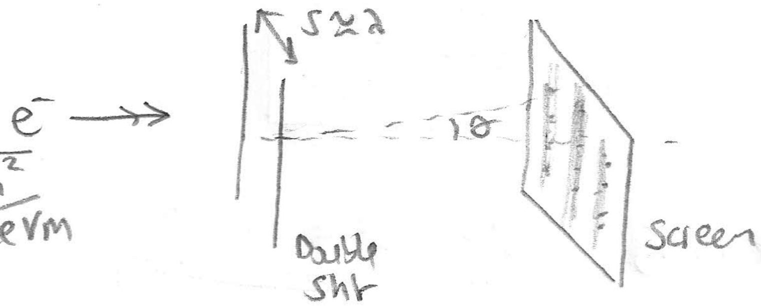
$$\therefore \quad \frac{\partial^2 \psi}{\partial t^2} = c^2 \left(\frac{\partial^2 \psi}{\partial x^2} \right)$$

or $$\boxed{\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}}$$ wave equation

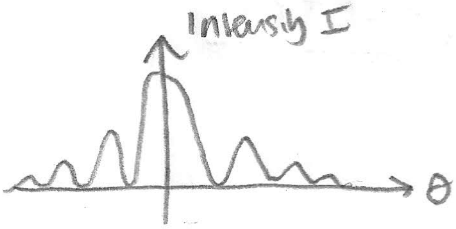
(v) * Fire electrons at a $\boxed{\text{double slit}}$, with spacing $\simeq$ de Broglie wavelength of electrons.

[$eV = \underbrace{\frac{h^2}{2\lambda^2 m}}_{\text{from (ii)}}$ so can use this to work out electron accelerating voltage V]

$$\lambda = \sqrt{\frac{h^2}{2eVm}}$$



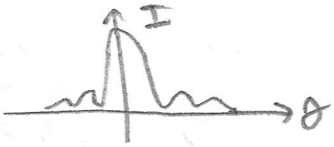
* On a screen, electrons that pass through the double slit(s) will, over time, build up the classic intensity pattern equivalent to diffracted light of wavelength λ.



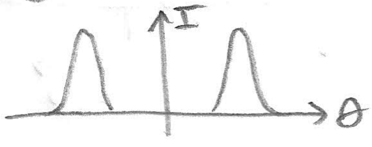
Maxima at $\boxed{S\sin\theta = n\lambda}$

ie e^- are WAVES that pass through BOTH SLITS

* Now for the strange bit! If you are able to determine the slit which the electron is heading towards you DON'T observe



, but



ie consistent with a $\boxed{\text{particle}}$ with random momenta that has an equal chance of passing through either slit ... but can't pass through both and 'interfere with itself'.

$\Rightarrow$ THIS EXPERIMENT DEMONSTRATES THE WAVE PARTICLE DUALITY OF ELECTRONS. BUT IMPORTANTLY,

THE WAVEFUNCTION (WHICH ENABLES THE DIFFRACTION PROPERTY) COLLAPSES WHEN THE ELECTRON MOMENTUM IS MEASURED. IT NOW HAS A KNOWN TRAJECTORY ("A MOMENTUM EIGENSTATE") AND HENCE THE SLIT IT PASSES THROUGH IS ABLE TO BE DEDUCED.

(vi) Classical physics $E = \frac{1}{2}mv^2$ kinetic energy

$p = mv$ momentum

$\therefore \frac{p^2}{m^2} = v^2$

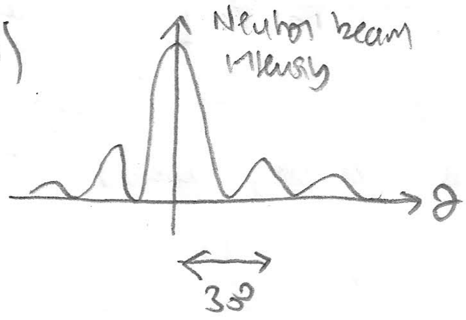
$\therefore E = \frac{1}{2}m \frac{p^2}{m^2}$

$\therefore \boxed{E = \frac{p^2}{2m}}$

Note in Special Relativity $E^2 - p^2c^2 = m^2c^4$

So relativistic version is: $\boxed{E = \sqrt{m^2c^4 + p^2c^2}}$

which for photons reduces to $\boxed{E = pc}$

(vii) 

$S\sin\theta = n\lambda$ for diffraction pattern

let $\theta = 30°$ and $n = 1$

So $S\sin 30° = \lambda$

$\Rightarrow \boxed{\lambda = \frac{S}{2}}$ $S = 2.15 \times 10^{-10}\,m$

So $\boxed{\lambda = 1.08 \times 10^{-10}\,m}$ de Broglie wavelength of neutrons.

From above: $\boxed{E = \frac{h^2}{2\lambda^2 m}}$ and since $p = \frac{h}{\lambda}$ and $p = mv$

$\Rightarrow \boxed{v = \frac{h}{m\lambda}}$

So $E = \dfrac{(6.626\times10^{-34})^2}{2\times(\frac{1}{2}\times2.15\times10^{-10})^2\times1.6749\times10^{-27}} \dfrac{1}{1.602\times10^{-19}}$ eV

$= \boxed{0.071\,eV}$ (ie not much!)

$v = \dfrac{6.626\times10^{-34}}{1.6749\times10^{-27}\times(\frac{1}{2}\times2.15\times10^{-10})} = \boxed{3680\ m/s}$

So we need a 'slow neutron source' for neutron diffraction imaging. Actually, this is not too far off $\boxed{\text{thermal}}$ $\boxed{\text{neutrons}}$. $\boxed{U = \frac{3}{2}k_BT}$ where $T = 298k$ (room temp

is $\frac{3}{2}\times1.38\times10^{-23}\times298 \Big/ 1.602\times10^{-19} \approx \boxed{0.04\,eV}$

(viii) photoelectric effect revision! $\boxed{E = \frac{hc}{\lambda} - \phi}$ work function

max KE of photoelectrons

energy of UV photons

$E = 5.1\,eV$
$\phi = 4.3\,eV$ (Silver)

So $\boxed{\lambda = \dfrac{hc}{E+\phi}}$

$\therefore \lambda = \dfrac{6.626\times10^{-34}\times2.998\times10^{8}}{(5.1+4.3)\times1.602\times10^{-19}}$ (m)

$= 1.32\times10^{-7}$ m

$= \boxed{131.9\ nm}$

Now for an electron beam of the same λ

$\boxed{E_e = \dfrac{h^2}{2\lambda^2m}}$ so $E_e = \dfrac{h^2}{2m}\left(\dfrac{E+\phi}{hc}\right)^2 = \boxed{\dfrac{(E+\phi)^2}{2mc^2}}$

$$\therefore \quad E_e = \frac{\left((5.1+4.3)\times 1.602\times 10^{-19}\right)^2}{2\times 9.109\times 10^{-31}\times (2.998\times 10^{8})^2} \; \frac{1}{1.602\times 10^{-19}} \; eV$$

$$= \boxed{8.64\times 10^{-5}\ eV}$$

{ Perhaps this justifies why electrons can't spontaneously escape from a metal surface ie ϕ is about 49,740 time larger than this value. Energy is the unit of trade in the universe, not λ }.

(ix) let resolution of an electron microscope be $\simeq \lambda$

$$\boxed{E = \frac{h^2}{2\lambda^2 m}}$$

So if $\lambda \simeq 0.42\,nm$

$$E = \frac{(6.626\times 10^{-34})^2}{2\times (0.42\times 10^{-9})^2 \times 9.109\times 10^{-31}} \; \frac{1}{1.602\times 10^{-19}} \; eV$$

$$\boxed{E = 8.53\ eV}$$ (which is surprisingly low).

Now $$\boxed{\lambda = \frac{h}{\sqrt{2mE}}}$$ So if $E = eV$

then $$\boxed{\lambda/nm = \frac{1.23}{\sqrt{V}}}$$ where V is in Volts.

For modest accelerating voltages we need to apply a relativistic factor

$$\boxed{\frac{\lambda}{nm} = \frac{1.23}{\sqrt{V}} \; \frac{1}{\sqrt{1 + 9.784\times 10^{-7}\,V}}}$$

where the factor $9.784 \times 10^{-7} = \frac{e}{2mc^2}$

$$= \frac{1.602 \times 10^{-19}}{2 \times 9.109 \times 10^{-31} \times (2.998 \times 10^{8})^2}$$

so for $V = 300$ kV

$$\Rightarrow \quad \lambda = \frac{1.23\,\text{nm}}{\sqrt{300 \times 10^3}} \quad \frac{1}{\sqrt{1 + 9.784 \times 10^{-7} \times 300 \times 10^3}}$$

$$= 1.97 \times 10^{-3}\,\text{nm}$$

$$= \boxed{1.97\,\text{pm}} \qquad [\,1\,\text{pm} = 10^{-12}\,\text{m}\,].$$

(x) Uncertainty principle : $\Delta E \Delta t \geq \hbar/2$ $\quad \Delta p \Delta x \geq \hbar/2$

so $\boxed{\Delta t \geq \frac{\hbar}{2\Delta E}}$ and $\boxed{\Delta x \geq \frac{\hbar}{2\Delta p}}$

Now electron in β decay has energy $\Delta E = 5$ MeV.

Assume $\frac{\Delta p^2}{2m} \simeq \Delta E$ $\quad$ so $\Delta p \approx \sqrt{2m\Delta E}$

$$\therefore \quad \Delta t \geq \frac{6.626 \times 10^{-34}}{2\pi \times 2 \times (5 \times 10^{6} \times 1.602 \times 10^{-19})}$$

$$\boxed{\Delta t \geq 6.58 \times 10^{-23}\,\text{s}}$$

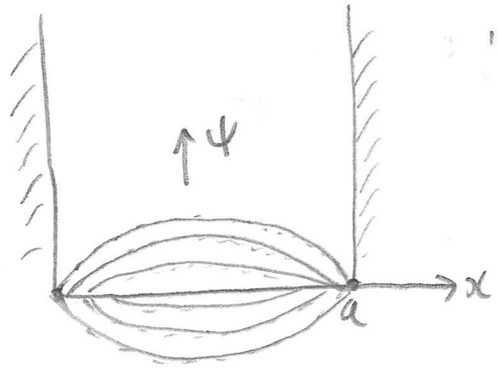
$$\Delta x \geq \frac{\hbar}{2\sqrt{2m\Delta E}} \quad \Rightarrow \Delta x \geq \frac{6.626 \times 10^{-34}}{2\pi \times 2\sqrt{2 \times 9.109 \times 10^{-31} \times 5 \times 10^{6} \times 1.602 \times 10^{-19}}}$$

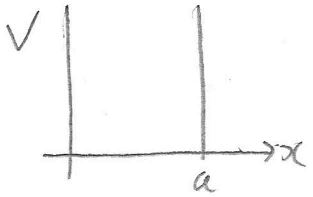
$$\boxed{\Delta x \geq 4.36 \times 10^{-14}\,\text{m}}$$

2/ (i)

"particle in a box"

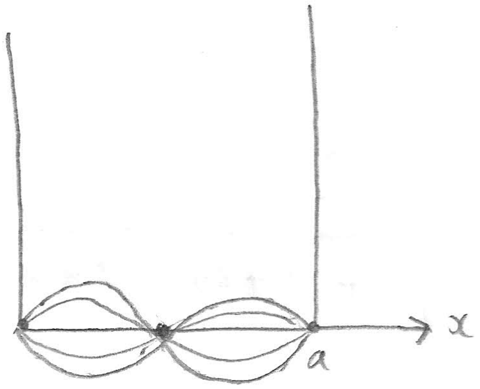
$n=1$



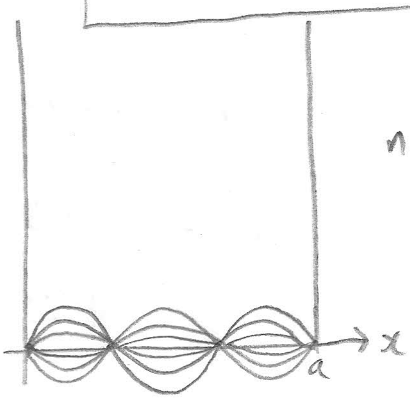


$$\boxed{V = \begin{cases} \infty & \text{otherwise} \\ 0 & 0<x<a \end{cases}}$$

$n=2$



$n=3$



Standing waves $\psi(x,t)$ describe the wavefunction of a particle of mass m within a 'box' of potential energy V.
'Walls' are effectively infinite, so no possibility of the particle existing beyond $0<x<a$. $\therefore$ ψ must have <u>nodes</u> at $x=0$ and $x=a$. So, like a guitar string, a whole # of $\frac{1}{2}$ wavelengths must fit into the box.

Standing waves $\Rightarrow$ nodes are fixed, and also no energy transferred from system. Hence:

Time variation of standing wave

$$\boxed{\psi(x,t) = A_n \sin\left(\frac{n\pi x}{a}\right) \cos \omega t} \quad \omega = 2\pi f$$

$\frac{2\pi x}{\lambda_n}$ and $\frac{\lambda_n n}{2} = a$

$\Rightarrow \lambda_n = 2a/n$

$\therefore \frac{2\pi x}{\lambda_n} = \frac{2\pi x}{2a} n = \boxed{\frac{n\pi x}{a}}$

So n must be a +ve integer

ie $\frac{\lambda}{2} \times n = a$.

(ii) Time independent Schrödinger equation:

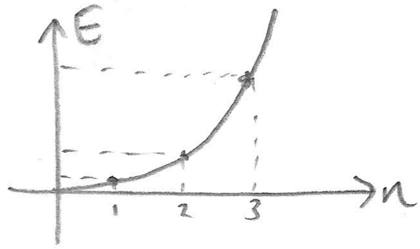
$$\boxed{-\frac{\hbar^2}{2m}\frac{\partial^2\psi}{\partial x^2} + V\psi = E\psi}$$

Inside box, $V=0$. if $\psi = A_n \sin\left(\frac{n\pi x}{a}\right)\cos\omega t$

$$\frac{\partial^2\psi}{\partial x^2} = -\left(\frac{n\pi}{a}\right)^2\psi$$

$$\therefore \; +\frac{\hbar^2}{2m}\left(\frac{n\pi}{a}\right)^2\psi = E\psi$$

$$\therefore \; \boxed{E = \frac{\hbar^2\pi^2}{2ma^2}n^2}$$



For a proton of mass $m = 1.6726\times10^{-27}$ kg
and $a = 10^{-15}$ m

$$\Rightarrow \frac{\hbar^2\pi^2}{2ma^2} = \frac{\left(\frac{6.626\times10^{-34}}{2\pi}\right)^2\pi^2}{2\times1.6726\times10^{-27}\times(10^{-15})^2} \quad (J)$$

[If $a = 0.47\times10^{-15}$ m
$\boxed{E = 927\text{ MeV}\times n^2}$]

$$= 3.281\times10^{-11}\text{ J} = \boxed{204.8\text{ MeV}}$$

Note rest mass of a proton is $\frac{1.6726\times10^{-27}\times(2.998\times10^{8})^2}{10^6\times1.602\times10^{-19}}$ MeV

$= \boxed{938.4\text{ MeV}}$ i.e. fairly similar to the above answer.

So if $mc^2 = \frac{\hbar^2\pi^2}{2ma^2} \Rightarrow a^2 = \frac{\hbar^2\pi^2}{2m^2c^2}$

$$\Rightarrow \boxed{a = \frac{\hbar\pi}{\sqrt{2}mc}}$$

$$\Rightarrow a = \frac{\frac{6.626\times10^{-34}}{2\pi}\times\pi}{\sqrt{2}\times1.6726\times10^{-27}\times2.998\times10^{8}} = \boxed{0.467\times10^{-15}\text{ m}}$$

So perhaps this should be the a value!

(iii) $\int_{-\infty}^{\infty} |\psi|^2 dx = 1$ if $|\psi|^2$ is the probability density

Now the energy of the particle is $E = \frac{h^2}{2\lambda^2 M}$

$\left[\frac{p^2}{2M} \right.$ = total energy in box, since $V=0$, $p = \frac{h}{\lambda}$ de Broglie

$\therefore \quad E = \left. \frac{h^2}{2\lambda^2 M} \right]$

wave version

$\omega = \frac{2\pi v}{\lambda}$ where v is the 'particle speed'

$Mv = p$ so $v = \frac{h}{\lambda M}$ $\therefore \omega = \frac{2\pi}{\lambda}\frac{h}{\lambda M}$

$\therefore \omega = \frac{2\pi h}{\lambda^2 M}$

So $\frac{\omega}{2\pi} h = \frac{h^2}{2\lambda^2 M} = E \Rightarrow \boxed{\omega = E/\hbar}$

So in our case: $\boxed{\omega_n = \frac{\hbar \pi^2 n^2}{2ma^2}}$

Now $e^{-i\omega t} = \cos \omega t - i \sin \omega t$ (de Moivre's theorem)

So $\cos \omega t$ is the real part of $e^{-iEt/\hbar}$.

why do we need the complex part too?

well $\boxed{E\psi = i\hbar \frac{\partial \psi}{\partial t}}$ (Schrödinger eqn, assuming E is a constant).

So if $\psi = e^{-iEt/\hbar} \sin\left(\frac{n\pi x}{a}\right)$

$i\hbar \frac{\partial \psi}{\partial t} = -i^2 \frac{E\hbar}{\hbar} \psi = E\psi$ ✓

So ψ is actually complex, if S.E eqn is fundamental!

ie $\boxed{\psi(x,t) = A_n e^{-iE_n t/\hbar} \sin\left(\frac{n\pi x}{a}\right)}$ $\boxed{E_n = \frac{\hbar^2 \pi^2 n^2}{2ma^2}}$

So from Born: $\int_0^a |\psi|^2 dx = 1$ {Assume A_n is real}

$$|\psi|^2 = \psi\psi^* = A_n^2 e^{-iEt/\hbar} \sin\left(\frac{n\pi x}{a}\right) e^{iEt/\hbar} \sin\left(\frac{n\pi x}{a}\right)$$

$$= A_n^2 \sin^2\left(\frac{n\pi x}{a}\right)$$

$$\therefore \quad A_n = \frac{1}{\sqrt{\int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx}}$$

$\sin^2 z = \frac{1}{2}(1-\cos 2z)$ $\quad \therefore \int_0^a \sin^2\left(\frac{n\pi x}{a}\right) dx$

$$= \frac{1}{2}\int_0^a \left(1 - \cos\left(\frac{2n\pi x}{a}\right)\right) dx$$

$$= \frac{1}{2}\left[x - \frac{a}{2n\pi}\sin\left(\frac{2n\pi x}{a}\right)\right]_0^a$$

$$= \boxed{\frac{1}{2}a}$$

$$\therefore \quad \boxed{A_n = \sqrt{\frac{2}{a}}}$$

in summary: $\boxed{\psi(x,t) = \sqrt{\frac{2}{a}}\, e^{-iE_n t/\hbar} \sin\left(\frac{n\pi x}{a}\right)}$

in range $0 \leq x \leq a$

where $\boxed{E_n = \hbar^2\pi^2 n^2/2ma^2}$

(iv) For a particle in a box, the standing wave implies there is as much momentum in $\rightarrow$ direction as $\leftarrow$

$\therefore \quad \boxed{\langle p \rangle = 0}$

Now $\frac{p^2}{2m} = E \quad \therefore \quad p^2 = 2mE \quad \therefore \quad \langle p^2 \rangle = 2m\langle E \rangle$

$$E = \hbar^2\pi^2 n^2/2ma^2 \Rightarrow \boxed{\langle p^2 \rangle = \frac{\hbar^2\pi^2 n^2}{a^2}}$$

(v) $\langle x \rangle = \left(\sqrt{\frac{2}{a}}\right)^2 \int_0^a x \sin^2\left(\frac{n\pi x}{a}\right) dx$ $\quad$ ie $\int_0^a x|\psi|^2 dx$

$$\int x \sin^2 \alpha x \, dx = \frac{1}{4}x^2 - \frac{x \sin 2\alpha x}{4\alpha} - \frac{\cos 2\alpha x}{8\alpha^2} + C$$

So $$\langle x \rangle = \frac{2}{a}\left[\frac{1}{4}x^2 - \frac{x}{4n\pi/a}\sin\left(\frac{2n\pi x}{a}\right) - \frac{\cos\left(\frac{2n\pi x}{a}\right)}{8(n\pi/a)^2}\right]_0^a$$

$$= \frac{2}{a}\frac{1}{4}a^2$$

Note:

$$\boxed{\begin{array}{l}\sin(2n\pi) = 0 \\ \cos(2n\pi) = 1\end{array}}$$

$$\therefore \boxed{\langle x \rangle = \frac{1}{2}a}$$

$$\langle x^2 \rangle = \frac{2}{a}\int_0^a x^2 \sin^2\left(\frac{n\pi x}{a}\right) dx$$

$$\int x^2 \sin^2 \alpha x \, dx = \frac{1}{6}x^3 - \frac{x \cos 2\alpha x}{4\alpha^2} - \frac{(2\alpha^2 x^2 - 1)\sin 2\alpha x}{8\alpha^3} + C$$

$$\therefore \langle x^2 \rangle = \frac{2}{a}\left[\frac{1}{6}x^3 - \frac{x}{4(n\pi/a)^2}\cos\left(\frac{2n\pi x}{a}\right) - \frac{\left(2\left(\frac{n\pi}{a}\right)^2 x^2 - 1\right)\sin\left(\frac{2n\pi x}{a}\right)}{8(n\pi/a)^3}\right]_0^a$$

$$= \frac{2}{a}\left[\frac{1}{6}a^3 - \frac{a}{4\left(\frac{n\pi}{a}\right)^2}\right]$$

$$= \frac{a^2}{3} - \frac{1}{2}\frac{a^2}{n^2\pi^2}$$

$$\therefore \boxed{\langle x^2 \rangle = \frac{1}{3}a^2\left(1 - \frac{3}{2n^2\pi^2}\right)}$$

Now $$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{\frac{1}{3}a^2 - \frac{a^2}{2n^2\pi^2} - \frac{1}{4}a^2}$$

$$\Rightarrow \Delta x = \sqrt{\frac{a^2}{12} - \frac{6a^2}{12n^2\pi^2}} = \sqrt{\frac{a^2 n^2 \pi^2 - 6a^2}{12 n^2 \pi^2}}$$

$$\therefore \quad \Delta x = \frac{a}{n\pi}\sqrt{\frac{n^2\pi^2 - 6}{3\times 2^2}} = \frac{a}{2n\pi}\sqrt{\frac{n^2\pi^2}{3} - 2}$$

$$[\ 12 = 3\times 2^2\ ]$$

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \frac{\hbar \pi n}{a} \qquad \text{Since } \langle p^2\rangle = \frac{\hbar^2\pi^2 n^2}{a^2}$$

$$\therefore \quad \Delta p \Delta x = \frac{\hbar \pi n}{\cancel{a}} \times \frac{\cancel{a}}{2\cancel{n}\pi}\sqrt{\frac{n^2\pi^2}{3} - 2}$$

$$\therefore \quad \boxed{\Delta p \Delta x = \frac{\hbar}{2}\sqrt{\frac{n^2\pi^2}{3} - 2}}$$

Now $n = 1$ is the smallest n, so $\sqrt{\frac{\pi^2}{3} - 2}$ is the smallest multiple of $\hbar/2$ in the above expression.

$\sqrt{\frac{\pi^2}{3} - 2} = 1.14$, so $\Delta p \Delta x > \hbar/2$

and $\therefore$ satisfies the uncertainty principle $\boxed{\Delta p \Delta x \geq \hbar/2}$

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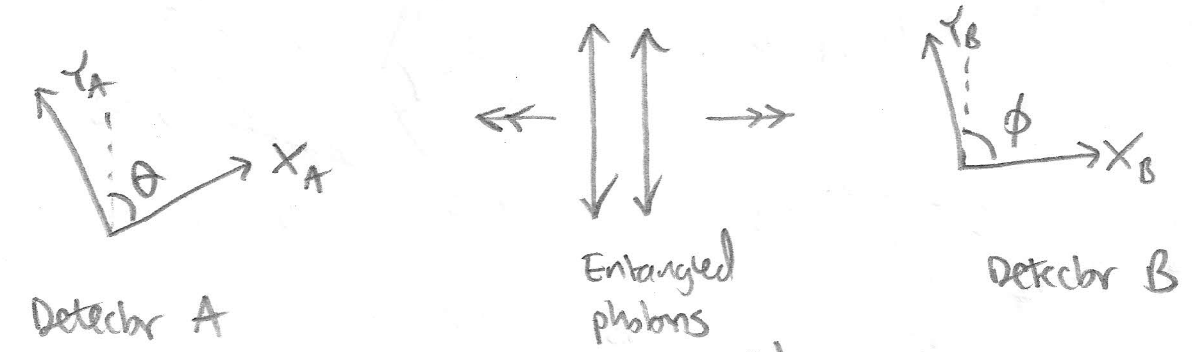
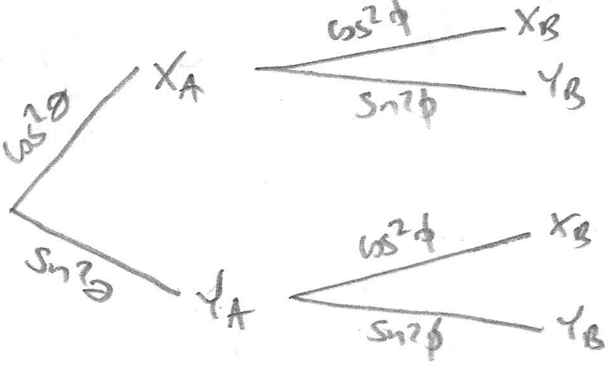


Classical outcome:
(i.e. A and B independent)



So $P(X_A, Y_B \text{ or } Y_A, X_B)$ (i.e. mismatch probability)

$$= \boxed{\cos^2\theta \sin^2\phi + \sin^2\theta \cos^2\phi}$$

let $\theta = -45°$, $\phi = 30°$

$$\therefore P(\text{mismatch}) = (\cos(-45°))^2 (\sin 30°)^2 + (\sin(-45°))^2 \cos(30°))^2$$

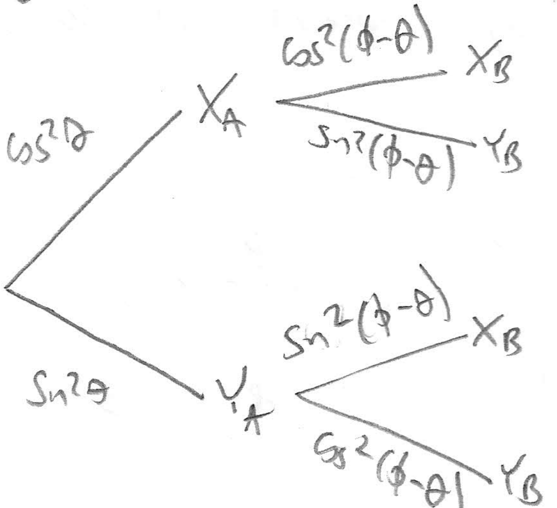
$$= \left(\frac{1}{\sqrt{2}}\right)^2 \left(\frac{1}{2}\right)^2 + \left(\frac{-1}{\sqrt{2}}\right)^2 \left(\frac{\sqrt{3}}{2}\right)^2$$

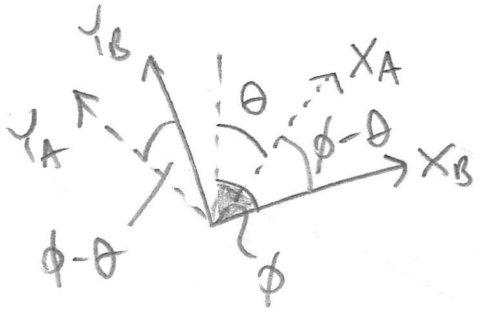
$$= \left(\frac{1}{2}\right)\left(\frac{1}{4}\right) + \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)$$

$$= \left(\frac{1}{2}\right)\left(\frac{1}{4} + \frac{3}{4}\right)$$

$$= \boxed{\frac{1}{2}}$$

Now use the **Copenhagen Interpretation** if A measures first* → then it sets the polarization to that measured by A. EVEN IF B IS A VERY LONG WAY AWAY!





So $P(\text{mismatch})$

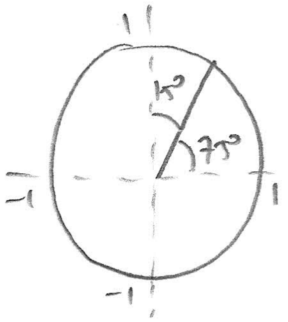
$$= \cos^2\theta \sin^2(\phi-\theta) + \sin^2\theta \sin^2(\phi-\theta)$$

$$= \boxed{\sin^2(\phi-\theta)}$$

* You get the same result if B measures first

$\therefore$ if $\theta = -45^\circ$, $\phi = 30^\circ$

$$P(\text{mismatch}) = \sin^2(30^\circ + 45^\circ)$$
$$= \sin^2(75^\circ)$$



$$\sin(30^\circ + 45^\circ) = \sin 30^\circ \cos 45^\circ + \cos 30^\circ \cos 45^\circ$$
$$= \left(\tfrac{1}{2}\right)\left(\tfrac{1}{\sqrt{2}}\right) + \left(\tfrac{\sqrt{3}}{2}\right)\left(\tfrac{1}{\sqrt{2}}\right)$$
$$= \frac{1}{\sqrt{2}}\left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)$$
$$= \boxed{\frac{1}{2\sqrt{2}}(1+\sqrt{3})}$$

So $P(\text{mismatch}) = \dfrac{(1+\sqrt{3})^2}{8} = \boxed{0.933}$

So a much higher probability than 0.50 as predicted classically. This asymmetry is the basis of Quantum cryptographic methods – in which via the reception statistics of polarized photons, you can tell whether the message has been eavesdropped.

$$\frac{P(\text{mismatch})_{\text{Quantum}}}{P(\text{mismatch})_{\text{classical}}} = \frac{(1+\sqrt{3})^2}{8} \times 2$$
$$= \frac{(1+\sqrt{3})^2}{4} = \boxed{1.87}$$