Photon energy $E = hf$. *Planck's constant* $h = 6.626\times10^{-34}\,\text{Js}$, f is the frequency of the photon in Hz.

de Broglie relationship $p = h/\lambda = \hbar k$. Momentum p , wavenumber $k = 2\pi/\lambda$, $\hbar = \frac{1}{2\pi}h$.

Wave equation for disturbance $\psi(x,t) = f(x - ct)$ travelling at speed c is: $\dfrac{\partial^2\psi}{\partial x^2} = \dfrac{1}{c^2}\dfrac{\partial^2\psi}{\partial t^2}$

Schrödinger Equation for *wavefunction* $\psi(x,t)$ of particle of mass m subject to some potential energy V is:

$-\dfrac{\hbar^2}{2m}\dfrac{\partial^2\psi}{\partial x^2} + V\psi = i\hbar\dfrac{\partial\psi}{\partial t}$. If there is no time dependence, and particle has total energy E : $-\dfrac{\hbar^2}{2m}\dfrac{\partial^2\psi}{\partial x^2} + V\psi = E\psi$

Born interpretation of the wavefunction is that $|\psi|^2\,dx$ is the probability of finding a particle in range $[x, x+dx]$.

Uncertainty principle: $\Delta x\Delta p \geq \frac{1}{2}\hbar$, $\Delta E\Delta t \geq \frac{1}{2}\hbar$. $\Delta x = \sqrt{\langle x^2\rangle - \langle x\rangle^2}$ i.e. the standard deviation.

Expected value of x^n is: $\langle x^n\rangle = \int_{-\infty}^{-\infty} x^n|\psi|^2 dx$.

The Copenhagen Interpretation of measurement in QM. (Niels Bohr, Werner Heisenberg 1925-1927). For the discussion below, ‘state’ might mean position, velocity, spin, polarization etc).

• The state of all physical systems can be represented mathematically by a wavefunction. This is a solution to the Schrödinger Equation (the time dependent version if one predicts a system will change with time). This is essentially a combination of the wave equation with the conservation of energy.

• The modulus squared of the wavefunction is related to the probability of a particular state existing. (Born interpretation).

• Prior to measurement, a wavefunction can always be written as a superposition of eigenstates (i.e. possible outputs) of the measurement device. When a measurement is made the wavefunction ‘collapses’ to one of the possible eigenstates associated with the measurement device.

Physical constants

Speed of light:	$c = 2.998\times10^8\,\text{ms}^{-1}$	Electron mass:	$m_e = 9.109\times10^{-31}\,\text{kg}$
Proton mass:	$m_p = 1.6726\times10^{-27}\,\text{kg}$	Neutron mass:	$m_n = 1.6749\times10^{-27}\,\text{kg}$
Electron charge:	$e = 1.602\times10^{-19}\,\text{C}$		

Question 1

(i) An electron is travelling at speed $\frac{1}{137}c$. Calculate its de-Broglie wavelength.

(ii) The de-Broglie wavelengths of an electron and neutron are the same. What is the ratio of their speeds, assuming classical Physics? What is the ratio of their kinetic energies?

(iii) The Uncertainty principle can be used to estimate properties of short-lived particles such as W and Z bosons in particle physics. Let $\Delta x\Delta p \approx \frac{1}{2}\hbar$, $\Delta E\Delta t \approx \frac{1}{2}\hbar$, $\Delta p \approx mc$, $\Delta E \approx mc^2$. If the mass m of the Z boson is $91.2\,\text{GeV/c}^2$, determine its approximate range Δx and lifetime Δt .

(iv) Show that $\psi(x,t) = A\sin(kx - \omega t)$ is a solution of the wave equation.

(v) Explain the idea of 'wavefunction collapse' using the example of electron diffraction using the double-slit experiment.

(vi) Show that the classical kinetic energy of a particle of mass m can be written as $E = p^2/2m$.

(vii) A beam of neutrons is scattered by an atomic lattice with spacing $s = 2.15 \times 10^{-10}\,\text{m}$. If the angular spacing between the first two maxima of the resulting diffraction pattern is 30°, calculate the de-Broglie wavelength of the neutrons, and hence their energy in eV, and speed in m/s (assuming classical physics).

(viii) Silver has a work function of 4.3eV. The maximum kinetic energy of electrons produced when Silver absorbs UV light is 5.1eV. Calculate the wavelength (in nm) of the UV light. What is the kinetic energy (in eV) of an electron which has the same de-Broglie wavelength?

(ix) In order to resolve features of size Δx using a microscope, diffraction means the smallest features are $\approx \lambda$ i.e. the wavelength of the light used. Calculate the energy (in eV) of electrons in an *electron-microscope* that has a resolution limit of 0.42nm.

(x) A 5MeV electron is emitted by a nucleus undergoing beta-decay. Use the Uncertainty principle to estimate a lower limit for the interaction time Δt for this process, and also the spatial extent Δx.

Question 2 A particle of mass m is confined to an effectively infinite potential 'box' within range $0 \le x \le a$. The potential is zero within the box, infinite outside.

(i) Explain why standing waves of the form $\psi(x,t) = A_n \sin\left(\frac{n\pi x}{a}\right)\cos\omega t$ are a sensible choice for the *wavefunction* of the particle. What type of number is n?

(ii) Show that $\psi(x,t) = A_n \sin\left(\frac{n\pi x}{a}\right)\cos\omega t$ is a solution to the *time independent Schrödinger equation* and hence find an expression for particle energy $E = E(n,a)$. Evaluate this (in MeV) for a proton, with $a = 0.47 \times 10^{-15}\,\text{m}$ and compare it to the rest mass energy of a proton, $m_p c^2$.

(iii) Explain why $\cos\omega t$ should instead be $e^{-iEt/\hbar}$, and then use the *Born-interpretation* of the wavefunction to determine A_n.

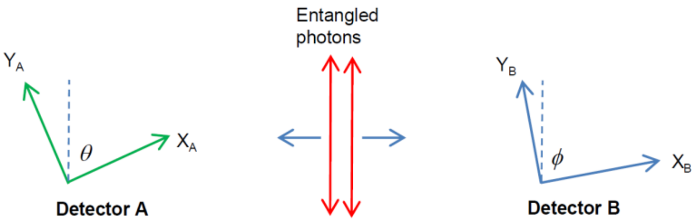
(iv) Explain why $\langle p \rangle = 0$, and show that $\langle p^2 \rangle = \dfrac{\hbar^2\pi^2 n^2}{a^2}$.

(v) Prove that $\langle x \rangle = \frac{1}{2}a$ and $\langle x^2 \rangle = \frac{1}{3}a^2\left(1 - \dfrac{3}{2n^2\pi^2}\right)$. You can quote the integrals:

$$\int x\sin^2\alpha x dx = \tfrac{1}{4}x^2 - \frac{x\sin 2\alpha x}{4\alpha} - \frac{\cos 2\alpha x}{8\alpha^2} + c; \quad \int x^2\sin^2\alpha x dx = \tfrac{1}{6}x^3 - \frac{x\cos 2\alpha x}{4\alpha^2} - \frac{\left(2\alpha^2x^2 - 1\right)\sin 2\alpha x}{8\alpha^3} + c$$

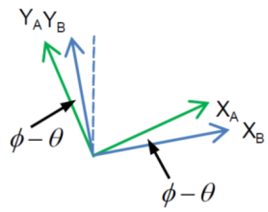
(vi) Hence show that $\Delta x \Delta p = \frac{1}{2}\hbar\sqrt{\dfrac{\pi^2 n^2}{3} - 2}$, and verify that the particle always satisfies the Uncertainty principle.

Question 3 Two vertically polarized 'entangled' photons are emitted and travel in opposite directions. They are received by detectors A and B respectively, which are aligned at angles θ, ϕ as shown. Using an idea from *Malus' law*, the probability of a detector yielding X polarization is proportional to the square of the projection of 'vertical' on the X direction. For example: $P(X_A) = \cos^2\theta$, $P(Y_B) = \sin^2\phi$.



Determine the probability of each detector yielding different X,Y outcomes if

$\theta = -45^\circ, \phi = 30^\circ$. Assume the measurements are independent.



Now repeat the calculation using the *Copenhagen Interpretation*, and assume A measures the polarization first (although it makes no difference if B does).